On p -adic L -functions attached to elliptic curves with complex multiplication and the Riemann-Hurwitz genus formula

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§0. Introduction

Let K be a quadratic imaginary field and let p be a prime number which splits in K , say $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. Let K_∞ be the unique $\mathbb{Z}_p$ -extension of K unramified outside $\mathfrak{p}$. Let F be an abelian extension of K of order prime to p and $F_\infty = FK_\infty$. Let M_∞ be the maximal abelian $\mathbb{Z}_p$ -extension of F_∞ unramified outside $\mathfrak{p}$. Write X_∞ for the Galois group of M_∞ over F_∞ , endowed with its natural action of the Galois group $\text{Gal}(F_\infty/K)$. Let $\Gamma = \text{Gal}(F_\infty/F)$. It is well known that X_∞ is a finitely generated $\mathbb{Z}_p[[\Gamma]]$ -torsion $\mathbb{Z}_p[[\Gamma]]$ -module.

In [2] Gillard and in [7] Schneps proved that the μ -invariant of X_∞ is 0 inspired by the work of Sinnott [9] in the cyclotomic case. On the other hand in [6] Kida proved an analogue of the Riemann-Hurwitz genus formula by describing the λ -invariants in cyclotomic case. Kida's formula has three proves: the first one is in Kida [6] using genus theory, the second one is in Iwasawa [5] using Galois cohomology and the third one is in Grass [3,4] and Sinnott [8] using p -adic L -functions. We study here the elliptic analogue of Kida's formula to get our main theorem (Theorem 6). Our method is followed by the third one especially Sinnott's paper.

§1. Notations

Let K be an imaginary quadratic field with the integer ring $\mathfrak{o}$. Let p be a rational prime which splits in K , say $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. Let K_∞ be the unique $\mathbb{Z}_p$ -extension of K unramified outside $\mathfrak{p}$ and let K_n ($n = 0, 1, \dots$) be the unique subfield of K_∞ such that $[K_n : K] = p^n$. Let F be an abelian extension of K of order prime to p . Let $F_n = FK_n$ and $F_\infty = FK_\infty$. Let $\Gamma = \text{Gal}(F_\infty/F)$ be the Galois group of F_∞ over F . Let $M_{\infty, F}$ be the maximal p -extension of F_∞ unramified outside $\mathfrak{p}$. Write $X_{\infty, F} = \text{Gal}(M_{\infty, F}/F_\infty)$, endowed with its natural action of $\text{Gal}(F_\infty/K)$. Let $\mathfrak{f}$ be an integral ideal of K and let $K(\mathfrak{f})$ be the ray class field mod $\mathfrak{f}$ of K . Let $K(\mathfrak{f}p^\infty) = \bigcup_n K(\mathfrak{f}p^n)$ and $G(\mathfrak{f}) = \text{Gal}(K(\mathfrak{f}p^\infty)/K)$. Let χ be a finite order character from $\text{Gal}(F_\infty/K)$ to $\mathbb{C}_p^\times$. Fix a homomorphism

$$\kappa : \text{Gal}(F_\infty/K) \simeq \text{Gal}(F_\infty/K_\infty) \times \text{Gal}(K_\infty/K) \rightarrow \text{Gal}(K_\infty/K) \simeq (1 + p\mathbb{Z}_p) \quad \text{if } p \neq 2 \\ \simeq (1 + 4\mathbb{Z}_p) \quad \text{if } p = 2$$

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Suppose that f divides non p -part of the conductor of χ . We may assume that F is contained in $K(f)$. Let E be an elliptic curve defined over $K(f)$ with fixed Weierstrass model, satisfying the following two conditions. (1) E has complex multiplication by $\mathfrak{o}$ with grössencharacter ψ . Thus ψ is a homomorphism from the group of all fractional ideal of $K(f)$ relatively prime to some ideal into $\overline{\mathbf{Q}}$. (2) $\psi(\mathfrak{a}) = \kappa(\sigma_{N_{K(f)/K}\mathfrak{a}})$ where $\mathfrak{a}$ is an ideal relatively prime to $p f$ and $\sigma_{\mathfrak{a}} = (\mathfrak{a}, F_{\infty}/K)$ is the Artin symbol of $\mathfrak{a}$. Let L be the period lattice of E . Replacing E by one of its conjugates, if necessary, we assume

$$L = \Omega f, \quad \Omega \in \mathbf{C}^{\times}$$

Let Ω_p be a p -adic period of E . (See [1]) Let

$$L_{\infty, f}(\chi \kappa^{-k}, s) = \frac{(s+k)}{(2\pi)^{s+k}} \sum_{(\mathfrak{a}, f)=1} \frac{\chi \kappa^{-k}(\sigma_{\mathfrak{a}})}{N \mathfrak{a}^s}$$

where $\sigma_{\mathfrak{a}} = (\mathfrak{a}, F_{\infty}/K)$ be the Artin symbol of $\mathfrak{a}$ and $\Gamma(s)$ be the gamma function.

We follow the notation of [1] in reviewing the construction of the p -adic L -function $L_{p, f}(\chi, s)$,

$$\Omega_p^k L_{p, f}(\chi, k) = \Omega^k G(\chi \kappa^{-1})(1 - (\chi^{-1} \kappa^k(\sigma_{\mathfrak{q}}))/p) L_{\infty, f}(\chi \kappa^{-k}, 0) \quad 1 \leq k \in \mathbf{Z}$$

Here $G(\chi \kappa^{-k})$ is defined as follows. Let

$$S = \{\gamma \in \text{Gal}(K(f p^n \overline{p}^{\infty})/K) \mid \gamma|_{\text{Gal}(K(f \overline{p}^{\infty})/K)} = (p^n, K(f \overline{p}^{\infty})/K)\}$$

where n is the exact power of p dividing the conductor of $\chi \kappa^{-k}$ and ζ_n is a primitive p -th root of unity. Then

$$G(\chi \kappa^{-1}) = \frac{\kappa(p^n)}{p^n} \sum_{\gamma \in S} \chi(\gamma) (\zeta_n^{\gamma})^{-1}$$

It is known that there exists a measure μ_f on $G(f)$ such that

$$L_{p, f}(\chi, s) = \int_{G(f)} \chi^{-1} \kappa^s(\sigma) d\mu_f$$

and a power series $G_{p, f}(\chi, T) = \int_{G(f)} \chi^{-1}(\sigma) (1+T)^{\kappa(\sigma)} d\mu_f$ such that

$$L_{p, f}(\chi, s) = G_{p, f}(\chi; u^s - 1)$$

where $u = \kappa(\gamma_0)$ and γ_0 is a generator of $\text{Gal}(K_{\infty}/K) \simeq \text{Gal}(F_{\infty}/F)$. Let $\mathfrak{o}'$ be the ring of integers in a finite extension of $\mathbf{Q}_p$. Let $f(T) = a_0 + a_1 T + a_2 T^2 + \dots \in \Lambda = \mathfrak{o}'[[T]]$ be a non zero power series with coefficients in $\mathfrak{o}'$. Let $\mu(f) = \min\{\text{ord}_p a_i : i \geq 0\}$ be the μ -invariant of f and $\lambda(f) = \min\{i \geq 0 : \text{ord}_p a_i = \mu(f)\}$ be the λ -invariant of f .

Two Λ -modules are said to be pseudo-isomorphic if there is a map between them with finite kernel and cokernel. any finitely generated torsion Λ -module Y is pseudo-isomorphic to a module of the form $\bigoplus \Lambda/f_i\Lambda$ for certain $f_i \in \Lambda$, and the characteristic power series $(\prod f_i)$ is a well-defined invariant of Y which we will denote by $\text{char}(Y)$. It is well known that $X_{\infty,F}$ is a finitely generated torsion Λ -module, with the action $Tx = (\gamma_0 - 1)x$. Let $\mu_F = \mu(\text{char}(X_{\infty,F}))$ and let $\lambda_F = \lambda(\text{char}(X_{\infty,F}))$. Let $\mu_f(\chi) = \mu(G_{p,f}(\chi, T))$ and let $\lambda_f(\chi) = \lambda(G_{p,f}(\chi, T))$.

If f is the non p -part of the conductor χ , we omit the subscript f from our notations: thus $L_p(\chi, s)$, $G_p(\chi, s)$, $\mu(\chi)$, $\lambda(\chi)$. If G is a group $\hat{G}$ denotes the character group of G .

§2.

We use the next two important lemmas.

LEMMA 1.

$$\mu_F = 0$$

PROOF. See Gillard [2] or Schneps [7]

The next lemma is a link between $\mathbf{Z}_p$ -extension and L -function.

LEMMA 2.

$$\begin{aligned} \mu_F &= \sum_{\chi \in \widehat{\text{Gal}(F_{\infty}/K_{\infty})}} \mu(\chi) \\ \lambda_F &= \sum_{\chi \in \widehat{\text{Gal}(F_{\infty}/K_{\infty})}} \lambda(\chi) + 1 \end{aligned}$$

PROOF. See de Shalit[1]

The next lemma gives some information on $\lambda_f(\chi)$ when f is varied.

LEMMA 3. *Let χ be a finite order character of $\text{Gal}(F/K)$. Let f and f' be integral ideals of K which are divisible by the non- p -part of the conductor of χ , and suppose that f is divisible by f' . Then*

$$\lambda_{f'}(\chi) = \lambda_f(\chi) + \sum g(q)$$

where the summation is taken over primes q which divides $f'f^{-1}$ in K such that $\chi(\sigma_q)$ has p -power order and $g(q)$ denotes the number of places of K_{∞} lying above q .

PROOF. Define a p -adic integer $t(q)$ by $\sigma_q \equiv \gamma_0^{t(q)} \pmod{\text{Gal}(F/K)}$ where γ_0 is a generator of $\text{Gal}(K_{\infty}/K)$. Write

$$-t(q) = p^a u \quad a \geq 0, \quad u \in \mathbf{Z}_p^{\times}.$$

From the definition,

$$L_{p,f'}(\chi, s) = L_{p,f}(\chi, s) \prod (1 - \chi \kappa^{-s}(\mathfrak{q}))$$

$$G_{p,f'}(\chi, T) = G_{p,f}(\chi, T) \prod E_{\mathfrak{q}}(T)$$

where the product is taken over primes $\mathfrak{q}$ which divide $f'f^{-1}$ in K and

$$E_{\mathfrak{q}}(T) = (1 - \chi(\sigma_{\mathfrak{q}})(1 + T)^{-t(\mathfrak{q})})$$

Then

$$\begin{aligned} E_{\mathfrak{q}}(T) &\equiv 1 - \chi(\sigma_{\mathfrak{q}})(1 + T^{p^a})^u \pmod{p\mathfrak{o}'[[T]]} \\ &\equiv 1 - \chi(\sigma_{\mathfrak{q}}) - \chi(\sigma_{\mathfrak{q}})uT^{p^a} \pmod{(p, T^{p^a+1})\mathfrak{o}'[[T]]} \end{aligned}$$

It follows that

$$\begin{aligned} \mu(E_{\mathfrak{q}}(T)) &= 0 \\ \lambda(E_{\mathfrak{q}}(T)) &= p \quad \text{if } \chi(\sigma_{\mathfrak{q}}) \text{ is a } p\text{-power root of unity} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Let $D_{\mathfrak{q}}$ (resp. $I_{\mathfrak{q}}$) be the decomposition (resp. inertia) group of $\mathfrak{q}$ for the extension K_{∞}/K . Then $D_{\mathfrak{q}}/I_{\mathfrak{q}}$ is generated by

$$\gamma_0 \pmod{\text{Gal}(F/K)}$$

It follows that $g(\mathfrak{q})$ is finite and equal to p^a q.e.d.

The next lemma gives some information on $\lambda_f(\chi)$ when χ is varied.

LEMMA 4. *Let χ be a finite order character of $\text{Gal}(F_{\infty}/K)$ and let ψ be a finite order character of $\text{Gal}(F_{\infty}/K)$ of p -power order. Then*

$$\lambda_f(\chi) = \lambda_f(\chi\psi)$$

PROOF. Let $\mathfrak{o}_{\chi, \psi}$ be the ring of integers in a finite extension of $\mathbf{Q}_p$ containing the values of both χ and ψ and let π be a local parameter in $\mathfrak{o}_{\chi, \psi}$

Then since ψ has a p -power order,

$$G_{p,f}(\chi\psi, T) - G_{p,f}(\chi, T) = \int (\psi^{-1}(\sigma) - 1)\chi^{-1}(\sigma)(1 + T)^{\kappa(\sigma)} d\mu_f \in \pi\mathfrak{o}_{\chi, \psi}[[T]]$$

Since $\mu_f(\chi) = \mu_f(\chi\psi) = 0$, the result is obtained. q.e.d.

LEMMA 5. *Let χ be a finite order character of $\text{Gal}(F_{\infty}/K)$ and let ψ be a finite order character of $\text{Gal}(F_{\infty}/K)$ of p -power order. Suppose that the order*

of χ be prime to p . Let L be the extension of K corresponding to χ and let $L_\infty = LK_\infty$. Then

$$\lambda(\chi\psi) = \lambda(\chi) + N$$

where N is the number of places v on K_∞ such that (1) v doesn't lie above p and $v|_K$ is ramified for ψ . (2) v splits completely in L_∞ .

PROOF. Let f (resp. f') be the non p -part of the conductor of χ (resp. $\chi\psi$). Since χ and ψ have relatively prime orders, f' is divisible by f . By Lemma 3 and Lemma 4,

$$\lambda(\chi\psi) = \lambda_{f'}(\chi\psi) = \lambda_{f'}(\chi) = \lambda(\chi) + M$$

where $M = \sum g(q)$, the summation taken over places q , which divides $f'f^{-1}$ for which $\chi(\sigma_q)$ has p -power.

Since χ has order prime to p , $\chi(\sigma_q)$ is p -power order if and only if $\chi(\sigma_q) = 1$, that is q splits completely in L . Then from the definition of $g(q)$, M is the number of places v on K_∞ which split completely in L_∞ and $v|_K = q$ is prime number which divides $f'f^{-1}$. Such v satisfy (1) and (2). Conversely in L and from the condition that order of χ is prime to order of ψ , $v|_K$ divides $f'f^{-1}$ q.e.d.

The next result is the main theorem.

THEOREM 6. Notations are as usual as in §1. Let $H \supset F \supset K$ be a tower of abelian extensions such that $\text{Gal}(H/F)$ is p -power order and the order of $\text{Gal}(F/K)$ is prime to p . Let λ_F and λ_H be the λ -invariant of $X_{\infty, F}$ and $X_{\infty, H}$, respectively. Then

$$\lambda_H - 1 = [H_\infty : F_\infty](\lambda_F - 1) + \sum_w (e(w/v) - 1)$$

where the summation is taken over all places w on H_∞ which don't lie above p and $v = w|_{F_\infty}$ and $e(w/v)$ denotes the ramification index of w over v .

PROOF. If $H \cap F_\infty$ doesn't contain F , take the subfield H'' of H containing F such that $H'' \cap F_\infty = F$, $H''_\infty = H_\infty$ and we may assume $H \cap F_\infty = F$.

We prove only the case when $[H : F] = p$ and $[F : K] = q$ (p, q : prime numbers of $p \neq q$). The other cases are the same way to prove.

Define H' the subfield of H such that $H \cong H' \times F$. We have a factorization

$$\prod_{\theta \in \widehat{\text{Gal}(H/K)}} L_p(\theta, s) = \prod_{\psi \in \widehat{\text{Gal}(H'/K)}} \prod_{\chi \in \widehat{\text{Gal}(F/K)}} L_p(\chi\psi, s)$$

and

$$G_p(\theta, T) = \prod \prod G_p(\chi\psi, T)$$

then

$$\sum \lambda(\theta) = \sum \sum \lambda(\chi\psi)$$

Let N be the number of places v of K_∞ such that (1) v doesn't lie above p and $v|_K$ is ramified for ψ , where ψ denotes a non-trivial character of $\text{Gal}(H'/K)$.

Let N' be the number of places v of K such that (1) and (2) v splits completely in F . Then from the lemmas,

$$\begin{aligned}
 \sum_{\theta \in \widehat{\text{Gal}(H/K)}} \lambda(\theta) &= \sum_{1 \neq \chi \in \widehat{\text{Gal}(F/K)}} (p\lambda(\chi) + (p-1)N') + p\lambda(1) + (p-1)N \\
 &= p \sum_{\chi \in \widehat{\text{Gal}(F/K)}} \lambda(\chi) + (p-1)((q-1)N' + N) \\
 &= p \sum \lambda(\chi) + \sum_w (e(w/v) - 1)
 \end{aligned}$$

where the second summation is taken over the places w of H_∞ which don't lie above p . From Lemma 2, we obtain the result. q.e.d.

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