On the Maintenance of Oscillations under the Effect of a Small Nonlinear Damping

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1. Introduction. An important question in the qualitative theory of the equation

$$(1) \quad x^{(n)} + Q(t, x, x', \dots, x^{(n-1)}) = 0, \quad x_1 Q(t, x_1, x_2, \dots, x_n) \geq 0$$

is whether we can maintain the oscillation of all of its solutions by adding a small nonlinear damping, i. e., whether we can extend the oscillation criteria of (1) to the equation

$$(2) \quad x^{(n)} + P(t, x, x', \dots, x^{(n-1)})x^{(n-1)} + Q(t, x, x', \dots, x^{(n-1)}) = 0,$$

where the function P is small in some sense.

This problem goes back to Howard [2] who considered the linear equation

$$(3) \quad x^{(n)} + P_{n-1}(t)x^{(n-1)}(t) + \dots + P_1(t)x'(t) + P(t)x(t) = 0, \quad n \text{ is even}$$

but did not give any "a priori" conditions on the functions P_i which would guarantee the oscillation of its solutions. It was Bobisud [1] who first considered this problem for the equation (2) with $n=2$, and gave two results in this direction.

Our aim here is to show that Bobisud's results do hold equally well in the case of the general equation (2). Moreover, we extend to the above case some known results concerning the "no damping" case (1). Thus, our theorems contain as special cases ($p \equiv 0$) results of Kartsatos [3], [4], Onose [5], [6] and Ryder, Wend [7].

2. A solution of (2) defined on $[T_x, +\infty)$ ($T_x \geq 0$ depends on the

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particular solution $x(t)$) and not identically zero is said to be *oscillatory* if it has a zero on $[t, +\infty)$ for every $t \geq T_*$. Equation (2) itself is said to be oscillatory if every solution $x(t) \neq 0, t \in [T_*, +\infty)$ is oscillatory. By $\mathbf{F}$ we denote the family of all solutions of (2) which are indefinitely extendable to the right.

The following conditions are considered in the results of this paper:

(i) $Q(t, x_1, x_2, \dots, x_n)$ and $P(t, x_1, x_2, \dots, x_n)$ are continuous on $S = [0, +\infty) \times R^n$, $R = (-\infty, +\infty)$ and $x_1 Q(t, x_1, x_2, \dots, x_n) > 0$ for every $x_1 \neq 0$;

(ii) there exist continuous nonnegative functions k, m such that $-k(t) \leq P(t, x_1, x_2, \dots, x_n) \leq m(t)$ for every $t \in [0, +\infty)$;

(iii) $a(t)\phi(x_1) \leq Q(t, x_1, x_2, \dots, x_n)$ for $x_1 > 0$, and $Q(t, x_1, x_2, \dots, x_n) \leq b(t)\phi(x_1)$ for $x_1 < 0$, $(t, x_1, \dots, x_n) \in S$, where $a(t)$ and $b(t)$ are nonnegative and locally integrable on $[0, +\infty)$; moreover, $\phi(x)$ and $\psi(x)$ are nondecreasing and such that $\phi(x) > 0$ for $x > 0$ and $\phi(x) < 0$ for $x < 0$;

(iv) for some $\alpha > 0$

$$\int_a^{+\infty} \frac{du}{\phi(u)} < +\infty \quad \text{and} \quad \int_{-a}^{-\infty} \frac{du}{\psi(u)} < +\infty.$$

The following lemma is needed in most of the results of this paper:

LEMMA. *In addition to (i), (ii), assume that for any $t > 0$*

$$\lim_{t \rightarrow +\infty} \int_t^t \exp \left[- \int_t^s m(u) du \right] ds = +\infty :$$

then every nonoscillatory solution $x \in \mathbf{F}$ is such that $x(x)x^{(n-1)}(t) > 0$ and

$$|x^{(n-1)}(t)| \leq |x^{(n-1)}(t_1)| \exp \left[\int_{t_1}^t k(s) ds \right], \quad t \in [t_1, +\infty)$$

for some $t_1 \geq T_$.*

PROOF. Let $x(t)$, $t \in [T_*, +\infty)$ be a nonoscillatory solution in $\mathbf{F}$. Without any loss of generality we assume that $x(t) > 0, t \geq t_1 \geq$

T_x . We show first that $x^{(n-1)}(t)$ is eventually of one sign. In fact, assume the contrary and let t be a zero of $x^{(n-1)}(t)$. Then from (2) we obtain

$$(4) \quad x^{(n)}(t) = -Q(t, x(t), x'(t), \dots, x^{(n-1)}(t)) < 0.$$

Thus, $x^{(n)}(\bar{t}) < 0$ for every zero of $x^{(n-1)}(t)$; this means that $x^{(n-1)}(t)$ can not have more than one zero on $[t_1, +\infty)$. Assume now that $x^{(n-1)}(\bar{t}) < 0$ for some $\bar{t} \in [t_1, +\infty)$. Then $x^{(n-1)}(t) < 0$ for every $t \geq \bar{t}$. Thus, from (2) and for every $t > \bar{t}$ we obtain

$$x^{(n)}(t) + P(t, x(t), x'(t), \dots, x^{(n-1)}(t))x^{(n-1)}(t) < 0.$$

Divide this inequality by $x^{(n-1)}(t) < 0$, integrate from $\bar{t}$ to t and multiply by $x^{(n-1)}(t)$ to obtain

$$x^{(n-1)}(t)/x^{(n-1)}(\bar{t}) > \exp\left[-\int_{\bar{t}}^t P(u, x(u), x'(u), \dots, x^{(n-1)}(u))du\right],$$

which yields

$$x^{(n-1)}(t)x^{(n-1)}(\bar{t}) > \exp\left[-\int_{\bar{t}}^t m(u)du\right] < 0.$$

Integration of this inequality leads to

$$x^{(n-2)}(t) \leq x^{(n-2)}(\bar{t}) + x^{(n-1)}(\bar{t}) \int_{\bar{t}}^t \exp\left[-\int_{\bar{t}}^s m(u)du\right]ds$$

which implies the contradiction $\lim_{t \rightarrow \infty} x(t) = -\infty$, for the right-hand side tends to $-\infty$ as $t \rightarrow +\infty$. Hence $x^{(n-1)}(t) \geq 0$ on $[t_1, +\infty)$. Suppose now that $x^{(n-1)}(\bar{t}) = 0$ for some $\bar{t} \in [t_1, +\infty)$. Then (4) holds, i. e., a contradiction to the fact that $x^{(n-1)}(t)$ is nonnegative on $[t_1, +\infty)$. Consequently, $x^{(n-1)}(t) > 0$, $t \in [t_1, +\infty)$. From (1) we obtain

$$x^{(n-1)}(t) \leq x^{(n-1)}(t_1) + \int_{t_1}^t k(s)x^{(n-1)}(s)ds,$$

which, applying Gronwall's inequality proves the lemma.

The theorem which follows concerns itself with the bounded solutions of (2) and extends a theorem of Kartsatos [3, Th. 1].

THEOREM 1. *In addition to the hypotheses of the lemma, assume that $Q = Q_0(t)G(x_1, x_2, \dots, x_n)$, where $Q_0(t) > 0$, G is continuous, $x_1 G(x_1, x_2, \dots, x_n) > 0$ for $x_1 \neq 0$ and moreover*

$$\int_0^{+\infty} t^{n-1} k(t) dt < +\infty, \quad \int_0^{+\infty} t^{n-1} Q_0(t) dt = +\infty;$$

then if n is even every bounded solution $x \in \mathbf{F}$ is oscillatory, while if n is odd every bounded solution $x \in \mathbf{F}$ is oscillatory or tends monotonically to zero along with its first $n-2$ derivatives as $t \rightarrow +\infty$.

PROOF. Let $x(t)$, $t \in [T_x, +\infty)$ be a bounded nonoscillatory solution of (2) such that $x(t) > 0$, $t \in [t_1, +\infty)$: then it follows from the lemma that

$$0 < x^{(n-1)}(t) \leq x^{(n-1)}(t_1) \exp \left[\int_{t_1}^{+\infty} k(s) ds \right] = N_x, \quad t \in [t_1, +\infty).$$

Since $x(t)$ is bounded, we must have

for $n = \text{even}$: $(-1)^{i+1} x^{(i)}(t) > 0$, $t \in [t_1, +\infty)$, $i = 1, 2, \dots, n-1$,

for $n = \text{odd}$: $(-1)^{i+1} x^{(i)} < 0$, $t \in [t_1, +\infty)$, $i = 0, 1, 2, \dots, n-1$

(see Lemmas 1, 2, in Ryder, Wend [7]). Let $n = \text{even}$; then the above inequalities imply that $\lim_{t \rightarrow +\infty} x^{(i)}(t) = 0$, $i = 1, 2, \dots, n-2$, i. e., $x^{(i)}(t)$, $i = 1, 2, \dots, n-1$ are all bounded on the interval $[t_1, +\infty)$.

Since $x'(t) > 0$, $x(t)$ is also bounded below by $x(t_1) > 0$. It follows from the continuity of the function G that there exist two positive constants L, M such that

$$L < G(x(t), x'(t), \dots, x^{(n-1)}(t)) < M, \quad t \in [t_1, +\infty).$$

Multiply (2) by t^{n-1} and integrate between t_1 , $t \geq t_1$ to obtain

$$\begin{aligned} t^{n-1} x^{(n-1)}(t) - \int_{t_1}^t s^{n-2} x^{(n-1)}(s) ds \\ \leq C - \int_{t_1}^t s^{n-1} P(s, x(s), x'(s), \dots, x^{(n-1)}(s)) x^{(n-1)}(s) ds \end{aligned}$$

$$(5) \quad \leq \int_{t_1}^t s^{n-1} Q_0(s) G(x(s), x'(s), \dots, x^{(n-1)}(s)) ds$$

$$\leq C + N_x \int_{t_1}^t s^{n-1} k(s) ds - L \int_{t_1}^t s^{n-1} Q_0(s) ds$$

where $C = \text{constant}$. Our hypotheses imply that the right-hand side of (5) tends to $-\infty$ as $t \rightarrow +\infty$ and this yields

$$(6) \quad \int_{t_1}^{+\infty} s^{n-2} x^{(n-1)}(s) ds = +\infty.$$

The above estimate coincides with (6) of Th. 1 in [3], and the proof follows as therein. Thus, the case $n = \text{even}$ is disposed of and the same steps can be followed in order to prove the case $n = \text{odd}$, given that the function G is also bounded below by a positive constant if $x \in \mathbf{F}$ is such that $x'(t) < 0$ and $\lim_{t \rightarrow +\infty} x(t) = \alpha > 0$.

The following theorem extends the first of Bobisud's results in [1].

THEOREM 2. *In addition to (i), (ii) assume that*

(a) *given $\delta > 0$, $M > 0$, there exist $t_{\delta, M} > 0$ and a function $g(t) = g_{\delta, M}(t)$ defined on $[t_{\delta, M}, +\infty)$ with*

$$\int_{t_{\delta, M}}^{+\infty} g(u) du = +\infty$$

and such that $|x_1| \geq \delta$, $x_n \leq M$ and $x_1 x_n > 0$ imply $|Q(t, x_1, x_2, \dots, x_n)| \geq g(t)$;

(b) *for any $i > 0$,*

$$\int_0^{+\infty} k(s) ds < +\infty, \quad \int_i^{+\infty} \exp\left[-\int_i^s m(u) du\right] ds = +\infty;$$

then if n is even, every $x \in \mathbf{F}$ is oscillatory while if n is odd, every $x \in \mathbf{F}$ is either oscillatory or tends to zero monotonically along with its first $n-2$ derivatives as $t \rightarrow +\infty$

PROOF. Assume that $x \in \mathbf{F}$ is such that $x(t) > 0$ and $0 < x^{(n-1)}(t) \leq C$ for every $t \in [t_1, +\infty)$; then from (2) and for $n = \text{even}$ we obtain

$$x^{(n-1)}(t) = x^{(n-1)}(T) - \int_T^t P x^{(n-1)}(s) ds - \int_T^t Q ds$$

$$\begin{aligned}
 (7) \quad & \leq x^{(n-1)}(T) + C \int_T^t k(s) ds - \int_T^t g_{x(t_1), c}(s) ds \\
 & \rightarrow -\infty \text{ as } t \rightarrow +\infty
 \end{aligned}$$

where $T = \max\{t_{x(t_1)}, t_1\}$ ($t_{x(t_1)} = t_\delta$, $x(t) \geq x(t_1) > 0$).

It is evident that (7) implies a contradiction to the positivity of the function $x(t)$. Consequently, the case $n = \text{even}$ is settled and the case $n = \text{odd}$ does not require a separate proof since in this case we also have $x(t) > \alpha - \varepsilon$ (for some ε such that $0 < \varepsilon < \alpha < +\infty$) for any solution $x(t) \rightarrow \alpha$ as $t \rightarrow +\infty$, and for any $t \geq t_{\alpha-\varepsilon} \equiv t_\delta$.

This completes the proof of the theorem.

The analog to Bobisud's Th. 2 is given below:

THEOREM 3. *In addition to (i), (ii) assume that*

(a) *given $\delta > 0$ there exists $t_\delta > 0$ and a function $g(t) = g_\delta(t)$ defined on $[t_\delta, +\infty)$ with*

$$\frac{1}{t} \int_\delta^t (t-u)g(u)du \rightarrow +\infty \text{ as } t \rightarrow +\infty$$

and such that $|x_1| \geq \delta$ and $x_1 x_n > 0$ imply $|q(t, x_1, x_2, \dots, x_n)| \geq g(t)$;

(b) *for any t_1, t_2 with $t_1 \geq t_2 > 0$ the expression*

$$\frac{1}{t} \int_{t_1}^t (t-u)k(u) \exp\left[-\int_{t_2}^u k(s)ds\right]du$$

is bounded while

$$\lim_{t \rightarrow +\infty} \int_{t_1}^t \exp\left[-\int_{t_1}^s m(u)du\right]ds = +\infty:$$

then if n is even every $x \in \mathbf{F}$ is oscillatory, while if n is odd every $x \in \mathbf{F}$ is oscillatory or tends monotonically to zero.

PROOF. As in Th. 2, assume that $x(t) > 0$, $x^{(n-1)}(t) > 0$ on $[t_1, +\infty)$. Then integrate (2) twice and use the lemma to obtain

$$x^{(n-2)}(t) \leq x^{(n-2)}(T) + x^{(n-1)}(T)(T-t_1)$$

$$(8) \quad +x^{(n-1)}(T_1) \int_{T_1}^t (t-u)k(u) \exp \left[\int_{t_1}^u k(s)ds \right] du \\ - \int_{T_1}^t (t-u)g(u)du,$$

where $T = \max\{t_{x(t_1)}, t_1\}$. Divide (8) by t and take the limit of the right-hand side as $t \rightarrow +\infty$ to obtain $\lim_{t \rightarrow +\infty} x^{(n-2)}(t)/t = -\infty$, which implies a contradiction to the positivity of $x(t)$. Thus, the proof is complete.

3. The theorems of this section extend results of Kartsatos [3], [4], Ryder, Wend [7] and Onose [6], [5].

THEOREM 4. *In addition to (i)–(iv) assume that*

$$\int_0^{+\infty} t^{n-1}a(t)dt = \int_0^{+\infty} t^{n-1}b(t)dt = +\infty, \\ \int_0^{+\infty} t^{n-1}k(t)dt < +\infty,$$

and for any $t > 0$,

$$\int_t^{+\infty} \exp \left[- \int_t^s m(u)du \right] ds = +\infty;$$

then if n is even every $x \in \mathbf{F}$ is oscillatory, while if n is odd every $x \in \mathbf{F}$ is oscillatory or tends monotonically to zero along with its first $n-2$ derivatives as $t \rightarrow +\infty$.

PROOF. For $x \in \mathbf{F}$ bounded the theorem follows from Theorem 1. Assume now that $x(t) > 0$, $x^{(n-1)}(t) > 0$ for $t \in [t_1, +\infty)$ and $\lim_{t \rightarrow +\infty} x(t) = +\infty$; then multiply (2) by $t^{n-1}/\phi(x(t))$ and integrate from t_1 to $t \geq t_1$ to obtain

$$t^{n-1}x^{(n-1)}(t)/\phi(x(t)) - (n-1) \int_{t_1}^t [s^{n-2}x^{(n-1)}(s)/\phi(x(s))] ds \\ \leq C + \int_{t_1}^t s^{n-1}x^{(n-1)}(s)d[1/\phi(x(s))]$$

$$\begin{aligned}
(9) \quad & + \int_{t_1}^t [s^{n-1}k(s)x^{(n-1)}(s)/\phi(x(s))]ds - \int_{t_1}^t s^{n-1}a(s)ds \\
& \leq [N/\phi(x(s))] \int_{t_1}^t s^{n-1}k(s)ds - \int_{t_1}^t s^{n-1}a(s)ds
\end{aligned}$$

where we have used the fact that $\int_{t_1}^t s^{n-1}x^{(n-1)}(s)d[1/\phi(x(s))] \leq 0$ (considered in the Riemann-Stieltjes sense) and $x(t) \leq N$ for $t \in [t_1, +\infty)$. Taking the limit as $t \rightarrow +\infty$ we arrive at

$$\int_{t_1}^{+\infty} [s^{n-2}x^{(n-1)}(s)/\phi(x(s))]ds = +\infty$$

which is identical to (16) of Th. 2 in [3] (for n =even), and the proof follows as therein, with attention to the use of Riemann-Stieltjes integrals. For n =odd the same proof holds (see also Ryder, Wend [7]), given that in this case $x(t)$ is also bounded below by $x(t_1)$ for every $t \geq t_1$. This completes the proof of the theorem.

THEOREM 5. *In addition to (i)–(ii) assume that $P \geq 0$,*

$$\lim_{t \rightarrow +\infty} \int_t^s \exp \left[- \int_t^s m(u)du \right] ds = +\infty \quad \text{and}$$

(a) *there exist positive constants λ_0, M, N and constants β, γ with $0 \leq \beta \leq 1, 0 \leq \gamma \leq 1$ such that*

$$\begin{aligned}
\phi(\lambda y) &\geq M\lambda^\beta \phi(y), & y > 0 \\
\phi(\lambda y) &\leq N\lambda^\gamma \phi(y), & y < 0
\end{aligned}
\quad \lambda \geq \lambda_0 > 0$$

and

$$\int_0^{+\infty} t^{(n-1)\beta} a(t)dt = \int_0^{+\infty} t^{(n-1)\gamma} b(t)dt = +\infty;$$

then if n is even every $x \in \mathbf{F}$ is oscillatory, while if n is odd every $x \in \mathbf{F}$ is oscillatory or tends monotonically to zero together with its first $n-1$ derivatives.

PROOF. Suppose that there exists $x \in \mathbf{F}$ such that $x(t) > 0, x^{(n-1)}(t) > 0, t \in [t_1, +\infty)$. Then from (2) we obtain

$$(10) \quad x^{(n)}(t) + a(t)\phi(x(t)) \leq 0,$$

and the rest of the proof follows as in Th. 2 of Ryder, Wend [7].

REMARKS. It is evident that Th 4 also holds if we replace the integral condition on p by the condition $p \geq 0$, in which case we arrive again at the relation (10).

We conjecture here that Th. 5 holds if $P \geq 0$ is replaced by

$$\int_0^{+\infty} t^{(n-1)\varepsilon} k(t) dt < +\infty,$$

where $\varepsilon = \max\{\beta, \gamma\}$. Theorem 1 of Ryder, Wend contains Th. 2 of Kartsatos [3], and Th. 2 of the same authors is actually due to Kartsatos [4].

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