On Locally Inner Automorphisms of Certain C^* -algebras

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The purpose of this note is to extend, in case T is a locally compact Hausdorff space, the results of Smith [1] concerning locally inner automorphisms of $C(T, L(H))$, C^* -algebra consisting of continuous $L(H)$ -valued functions on a compact Hausdorff space T .

1. An automorphism of a C^* -algebra B means a $*$ -isomorphism of B . The set of all automorphisms of B is denoted by $\text{Aut}(B)$. $\text{Aut}(B)$ is a group in a natural way, the group operation being a composition of mappings. Moreover, $\text{Aut}(B)$ is a topological group with uniform topology, that is

$$\|\alpha\| = \sup\{\|\alpha(x)\| : x \in B, \|x\| \leq 1\}.$$

Let $\text{CF}(B)$ be the group of all center-fixing automorphisms of B (i. e. $\alpha \in \text{Aut}(B)$, $\alpha(x) = x$ for all $x \in Z(B)$ (the center of B)). Let T be a locally compact Hausdorff space, $\mathfrak{B} = ((B(t))_{t \in T}, \theta)$ the continuous field of C^* -algebras on T . Let B be the C^* -algebra defined by $\mathfrak{B}$. If $t \in T$ we denote by

$$e_t: B \rightarrow B(t)$$

the mapping given by $e_t(x) = x(t)$.

LEMMA 1. 1. If $\alpha \in \text{CF}(B)$ and $t \in T$, then $\alpha(\text{Ker } e_t) \subset \text{Ker } e_t$.

PROOF. Let M_t be the maximal ideal of $C_0(T)$, the algebra of all continuous functions on T which vanish at infinity, determined by $t \in T$, i. e. $M_t = \{f \in C_0(T) : f(t) = 0\}$. We assert that

$\text{Ker } e_t =$ the uniform closure of $\{f \cdot x : f \in M_t, x \in B\}$. If $x \in \text{Ker } e_t$,

then for any $\varepsilon > 0$, there is a compact neighbourhood U_t of t such that $\|x(s)\| \leq \varepsilon/2$ for any $s \in U_t$. Let K be a compact subset of T such that $K \supset U_t$ and $\|x(s)\| \leq \varepsilon/2$ for any $s \in T \setminus K$. Let $f \in M_t$ be a function on T such that $0 \leq f(K \setminus U_t) = 1$. Then we have

$$\begin{aligned} \|f \cdot x - x\| &= \sup\{\|f(s)x(s) - x(s)\| : s \in T \setminus K\} \\ &\quad + \sup\{\|f(s) - 1\| \|x(s)\| : s \in U_t\} \leq \varepsilon, \end{aligned}$$

and hence $x \in \overline{M_t \cdot B}$ where $M_t \cdot B = \{f \cdot x : f \in M_t, x \in B\}$.

Conversely, a limit point of $M_t \cdot B$ is the element of $\text{Ker } e_t$, since the evaluation mapping $e_t : B \rightarrow B(t)$ is continuous.

If $f \in M_t$ and $x \in B$, then

$$\alpha(f \cdot x) = \alpha((f \cdot 1) \cdot x) = \alpha(f \cdot 1)\alpha(x) = f \cdot \alpha(x).$$

Thus $\alpha(f \cdot x) \in \text{Ker } e_t$. Therefore we have $\alpha(\text{Ker } e_t) \subset \text{Ker } e_t$ since α is uniformly continuous.

Let $\alpha \in \text{CF}(B)$, $a \in B(t)$ and $t \in T$. Then there is $x \in B$ with $x(t) = a$ and we define a function $\tilde{\alpha}_t : B(t) \rightarrow B(t)$ by

$$\tilde{\alpha}_t(a) = e_t(\alpha(x)).$$

This is well defined by Lemma 1. 1.

LEMMA 1. 2. If $\alpha, \beta \in \text{CF}(B)$ and $t \in T$, then $(\alpha \beta)_t = \tilde{\beta}_t \tilde{\alpha}_t$.

PROOF. For any $x \in B$, we have

$$\begin{aligned} (\alpha \beta)_t(e(x)) &= e_t(\alpha(\beta(x))) = \tilde{\alpha}_t(e_t(\beta(x))) \\ &= \tilde{\alpha}_t \tilde{\beta}_t(e_t(x)). \end{aligned}$$

PROPOSITION 1. 3. If $\alpha \in \text{CF}(B)$ and $t \in T$, then $\tilde{\alpha}_t \in \text{Aut}(B(t))$.

PROOF. We can easily show that $\tilde{\alpha}_t$ is the $*$ -homomorphism of $B(t)$. By Lemma 1. 2 it follows that $(\tilde{\alpha}_t^{-1})_t = (\tilde{\alpha}_t)^{-1}$. Then $\tilde{\alpha}_t$ is the automorphism of $B(t)$.

Let $\Pi_{t \in T} \text{Aut}(B(t))$ be a product of topological groups of automorphisms of C^* -algebras. It is the group under the pointwise multi-

plication. $\Pi_{t \in T} \text{Aut}(B(t))$ becomes a topological group with the topology determined by the system of neighbourhoods of points φ in $\Pi_{t \in T} \text{Aut}(B(t))$ having the form,

$$U(\varphi, \varepsilon) = \{\varphi' \in \Pi_{t \in T} \text{Aut}(B(t)) : \sup_{t \in T} \|\varphi_t - \varphi'_t\| < \varepsilon\}$$

where $\varepsilon > 0$. We shall regard $\tilde{\alpha}$ as an element of $\Pi_{t \in T} \text{Aut}(B(t))$.

PROPOSITION 1. 4. *A map $\alpha \rightarrow \tilde{\alpha}$ is the isomorphism of the topological group $\text{CF}(B)$ into the topological group $\Pi_{t \in T} \text{Aut}(B(t))$ and the image $\text{CF}(B)^\sim$ is closed.*

PROOF. By Lemma 1. 2 it follows that $\alpha \rightarrow \tilde{\alpha}$ is the homomorphism of the group $\text{CF}(B)$ into the group $\Pi_{t \in T} \text{Aut}(B(t))$. Let $\alpha, \beta \in \text{CF}(B)$. We have

$$\begin{aligned} \sup_{t \in T} \|\tilde{\alpha}_t - \tilde{\beta}_t\| &\geq \sup_{t \in T} \sup_{\|x\| \leq 1} \|e_t(\alpha - \beta)(x)\| \\ &= \|\alpha - \beta\|. \end{aligned}$$

Then $\alpha \rightarrow \tilde{\alpha}$ is open and injective.

Let $x \in B$ with $\|x(t)\| \leq 1$ where $t \in T$. Then there is a compact neighbourhood U_t of t such that $\|x_s\| \leq 2$ on U_t . Let $f \in C_0(T)$ such that $0 \leq f \leq 1$, $f(T \setminus U_t) = 0$ and $f(t) = 1$, so that $e_t(f \cdot x) = x(t)$ and $\|f \cdot x\| \leq 2$. Then we have

$$\begin{aligned} \sup_{t \in T} \|\tilde{\alpha}_t - \tilde{\beta}_t\| &\leq \sup_{t \in T} \sup_{\|x\| \leq 2} \|e_t(\alpha - \beta)(x)\| \\ &\leq 2 \|\alpha - \beta\| \end{aligned}$$

Thus $\alpha \rightarrow \tilde{\alpha}$ is continuous.

Finally, suppose that a net $\{\tilde{\alpha}^i\} \subset \text{CF}(B)^\sim$ converges to $\varphi \in \Pi_{t \in T} \text{Aut}(B(t))$. Then, for any $\varepsilon > 0$, there exists λ_0 such that $\sup_{t \in T} \|\tilde{\alpha}^i_t - \varphi_t\| < \varepsilon$ for every $i \geq \lambda_0$. We define $\alpha(x)(t) = \varphi_t(x(t))$ for any $t \in T$, $x \in B$. A simple computation implies that $\alpha \in \text{CF}(B)$ and $\tilde{\alpha} = \varphi$ and so $\text{CF}(B)^\sim$ is closed.

2. Let S be a compact subset of the locally compact Hausdorff space T and $\mathfrak{B}|_S$ a continuous field induced by $\mathfrak{B}$ on S where $\mathfrak{B} =$

$((B(t))_{t \in T}, \theta)$ is a continuous field of C^* -algebras on T [2]. Let $B|_S$ be the C^* -algebra defined by $\mathfrak{B}|_S$ where B is the C^* -algebra defined by $\mathfrak{B}$. By Corollaire 11. 5 in [2], the following restriction map ρ is the $*$ -homomorphism of B onto $B|_S$,

$$\rho(X) = x \text{ where } X \in B \text{ and } x = X|_S = (X(t))_{t \in S}.$$

DEFINITION. If $\alpha \in \text{CF}(B)$ and S is a compact subset of T , then we define $\alpha|_S: B|_S \rightarrow B|_S$ by $\alpha|_S(x) = \alpha(X)|_S$ where $X \in B$ is any extension of x . This is well defined by Lemma 1. 1.

The following diagram is commutative where $\alpha \in \text{CF}(B)$:

$$\begin{array}{ccc} & \alpha & \\ B & \longrightarrow & B \\ \rho \downarrow & & \downarrow \rho \\ B|_S & \xrightarrow{\alpha|_S} & B|_S \end{array}$$

DEFINITION. If $\alpha \in \text{CF}(B)$, then α is called *locally inner* iff there exists a relatively compact open covering $(T_i)_{i \in I}$ on T such that $\alpha|_{\bar{T}_i}$ is inner for all $i \in I$ where $\bar{T}_i$ is the closure of T_i . The set of all locally inner automorphisms is denoted by $\text{loc-Inn}(B)$.

We note that if $\mathfrak{B}$ is a constant field on a compact Hausdorff space T , then the above definition for locally inner automorphisms is nothing but the definition of Smith [1] (p 631).

PROPOSITION 2. 1. *The $\text{loc-Inn}(B)$ is a subgroup of $\text{Aut}(B)$.*

PROOF. We obtain easily the result above, applying the proof of Proposition 3. 4 in [1]

3. Let T be a locally compact Hausdorff space and H a Hilbert space. Let $L(H)$ be the algebra of all continuous linear operators on H . Let $B = C_0(T, L(H))$, the algebra of all continuous functions $x: T \rightarrow L(H)$ which vanish at infinity. Then B is the C^* -algebra defined by the constant field $\mathfrak{B}$ on T defined by $L(H)$.

LEMMA 3. 1. *Let T , B , and H be as above. Let S be a compact subset of T . Then if $\alpha \in \text{CF}(B)$, then $\alpha|_S \in \text{CF}(B|_S)$ and $(\alpha|_S)^\sim = \tilde{\alpha}|_S$.*

PROOF. Let a be a center element of $B|_S$. Note that $B|_S = C(S, L(H))$, the algebra of all continuous functions $a: S \rightarrow L(H)$. Then there exists $f \in C(S)$ such that $a = f \cdot 1$ where 1 is the identity of $L(H)$. Let $f_0 \in C_0(T)$ with $f = f_0|_S$, so that $f_0 \cdot 1$ is the center element of B . Hence,

$$\alpha|_S(a) = \alpha(f_0 \cdot 1)|_S = (f_0 \cdot 1)|_S = a.$$

By the diagram of the section 2, we see that $(\alpha|_S)^\sim = \tilde{\alpha}|_S$.

THEOREM 3. 2. *If $\alpha \in \text{CF}(B)$ then $\tilde{\alpha}: T \rightarrow \text{Aut}(L(H))$ is continuous in the uniform operator topology iff $\alpha \in \text{loc-Inn}(B)$.*

PROOF. Suppose that $\alpha \in \text{loc-Inn}(B)$ and let $(T_i)_{i \in I}$ be a relatively compact open covers of T such that $\alpha|_{\bar{T}_i}$ is inner for all $i \in I$. By the proposition 3. 16 in [1] and Lemma 3. 1, $\tilde{\alpha}|_{\bar{T}_i}: \bar{T}_i \rightarrow \text{Aut}(L(H))$ is continuous. Therefore $\tilde{\alpha}$ is continuous.

Conversely, suppose that $\tilde{\alpha}: T \rightarrow \text{Aut}(L(H))$ is continuous. Let $(T_i)_{i \in I}$ be a relatively compact open covers of T . By Lemma 3. 1 $(\alpha|_{\bar{T}_i})^\sim$ is continuous map of $\bar{T}_i$ into $\text{Aut}(L(H))$. Then we have $\alpha|_{\bar{T}_i} \in \text{loc-Inn}(B|_{\bar{T}_i})$ for all $i \in I$ by the theorem 3. 12 in [1]. Therefore $\alpha \in \text{loc-Inn}(B)$ by the note in the section 2.

We can see that $C(T, \text{Aut}(L(H)))$, the group of all continuous functions of T into $\text{Aut}(L(H))$, is the topological subgroup of $\prod_{t \in T} \text{Aut}(L(H))$.

THEOREM 3. 3. $\sim: \text{loc-Inn}(B) \rightarrow C(T, \text{Aut}(L(H)))$ is an isomorphism of topological groups.

PROOF. We shall use the technique in the proof of the theorem 3. 5 in [1]. Let $\varphi \in C(T, \text{Aut}(L(H)))$. For any $x \in B$ we define $\hat{\varphi}(x)(t) = \varphi(t)(x(t))$ for any $t \in T$.

Note that

$$\|\varphi(t)(x(t)) - \varphi(s)(x(s))\| \leq \|x(t) - x(s)\| + \|\varphi(t) - \varphi(s)\| \|x(s)\|$$

for $x \in B$, and $t, s \in T$. Then $\hat{\varphi}(x)$ is a continuous map of T into $L(H)$ and we can easily show that $\hat{\varphi}(x)$ vanishes at infinity. Hence,

$\hat{\varphi}$ defines an automorphism of B . Note that the center of $B = C_0(T) \cdot 1$. Then $\hat{\varphi} \in \text{CF}(B)$ by the definition of $\wedge$. For any $x \in B$ and $t \in T$ we have

$$\hat{\varphi}^{\sim}(t)(x(t)) = \hat{\varphi}(x)(t) = \varphi(t)(x(t)).$$

Then $\hat{\varphi}^{\sim} = \varphi$, so that $\hat{\varphi} \in \text{loc-Inn}(B)$ by Theorem 3. 2.

On the other hand, let $\alpha \in \text{loc-Inn}(B)$. For any $x \in B$ and $t \in T$, $(\tilde{\alpha})^{\wedge}(x)(t) = (\tilde{\alpha})(t)(x(t)) = \alpha(x)(t)$. Then $(\tilde{\alpha}) = \alpha$. Therefore $\sim$ and $\wedge$ are inverse isomorphism of abstract groups and the result follows from Proposition 1. 4.

COROLLARY 3. 4. *If T is a separable locally compact Hausdorff space, then $\text{CF}(B) = \text{loc-Inn}(B)$.*

PROOF. Let $\alpha \in \text{CF}(B)$ and $a \in L(H)$. Let $t \in T$ and U_t a compact neighbourhood of t . There is $x \in B$ such that $x(s) = a$ for all $s \in U_t$. Then we have

$$\|\tilde{\alpha}_t(a) - \tilde{\alpha}_s(a)\| = \|\alpha(x)(t) - \alpha(x)(s)\|$$

for all $s \in U_t$. Then $\tilde{\alpha}: T \rightarrow \text{Aut}(L(H))$ is strongly continuous. Applying Kallmann's theorem (Theorem F in [1]) we can show our corollary (cf. Proof of Theorem 3. 11 in [1]).

References

- [1] M. B. Smith, On automorphism groups of C^* -algebras, Trans. Amer. Math. Soc. **152** (1970), 623-648.
- [2] J. Dixmier, Les C^* -algèbre et leur représentations, Gauthier-Villars, Paris, 1964.