A NOTE ON THE IDENTITIES OF DEDEKIND RINGS

Ryûki Matsuda

It has been proved that, if a commutative ring R without zero-divisors satisfies the condition of Dedekind rings, R is the usual Dedekind ring with the identity. In this note, we extend the assertion and prove the following:

THEOREM. Let R be a commutative ring all elements of which are not zero-divisors. If every ideal of R is a product of prime ideals ($\neq R$), R has the identity.

PROOF. Let K be the total quotient ring of R . We denote $(R:R)_K$ by R^{-1} . If aR^{-1} coincides with R for all non-zero-divisors a of R , we have

$$R = a^2R^{-1} = aR.$$

Then we have $a = ab$ for some b of R . We see that b is the identity of R . We may suppose therefore that dR^{-1} is a proper subset of R for some non-zero-divisor d of R . Since dR^{-1} is an ideal of R , there exist prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ such that $dR^{-1} = \mathfrak{p}_1 \dots \mathfrak{p}_n$ ($n \geq 1$). There exist prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_m$ also such that $\mathfrak{p}_n R = \mathfrak{q}_1 \dots \mathfrak{q}_m$. Every $\mathfrak{q}_1$ contains $\mathfrak{p}_n$ and $\mathfrak{p}_n$ contains $\mathfrak{q}_j$ for some j , say $j = 1$. If $\mathfrak{p}_n R$ is a proper subset of $\mathfrak{p}_n$, we have

$$\mathfrak{p}_n R = \mathfrak{p}_n \mathfrak{q}_2 \dots \mathfrak{q}_m \quad (m \geq 2).$$

Multiplying $d^{-1}R \cdot \mathfrak{p}_1 \dots \mathfrak{p}_{n-1}$ on the both sides, we have

$$R^2 = R \mathfrak{q}_2 \dots \mathfrak{q}_m.$$

There arises the contradiction of $R \subset \mathfrak{q}_m$. Therefore $\mathfrak{p}_n R$ coincides with $\mathfrak{p}_n$. Therefore we have

$$dR^{-1} \cdot R = \mathfrak{p}_1 \dots \mathfrak{p}_n R = \mathfrak{p}_1 \dots \mathfrak{p}_n = dR^{-1}.$$

Therefore we see that $R^{-1}R$ coincides with R^{-1} . Since $R \supset R^{-1}R$ and $1 \in R^{-1}$ by the definition, we see that R contains the identity.

REMARK. There exist rings R which satisfy the conditions of the theorem and has zero-divisors. $\mathbb{Z}/(4)$ is one of the examples.

Ryūki Matsuda
Department of Mathematics,
Faculty of Science,
Ibaraki University, Mito.

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