ON A THEOREM OF SCHMIDT

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In [3], Schmidt defined the concept of a Roth system for linear forms with real coefficients in the n variables $\underline{x} = (x_1, \dots, x_n)$. In the present paper, we extend this to the case of complex coefficients and we prove the result analogous to that of [3].

1. INTRODUCTION. We consider a linear form $L(\underline{x})$ with real or complex coefficients in $\underline{x} = (x_1, \dots, x_n)$. We say such a linear form to be real if all its coefficients are reals, otherwise complex. We say a system of linear forms $L_1, \dots, L_t$ to be symmetric, if for each linear form L_j , its complex conjugate $\bar{L}_j$ occurs always in it. Let n be a natural number, and let s, t be non-negative integers with $n = s + t$. Suppose that $L_1, \dots, L_t$ is a symmetric system of linear forms with algebraic coefficients in $\underline{x} = (x_1, \dots, x_n)$. We call $L_1, \dots, L_t$ to be a Roth system, if for each $\delta > 0$, the inequalities

$$|L_j(\underline{x})| < |\underline{x}|^{-s/t-\delta} \quad (1 \leq j \leq t)$$

have only finitely many solutions in integer points $\underline{x}$. We say that $L_1, \dots, L_t$ have rank r on a subspace S of C^n , if the restrictions of $L_1, \dots, L_t$ to S have rank r . Then, we have the following theorem analogous to theorem 1 of [3].

THEOREM 1. Suppose that $L_1, \dots, L_t$ is a symmetric system of linear forms with algebraic coefficients. Then, a necessary and sufficient condition for $L_1, \dots, L_t$ to be a Roth system is that on every rational subspace S^d of dimension d with $1 \leq d \leq n$, the forms $L_1, \dots, L_t$ have rank

r with

$$r \geq dt/n. \quad (1)$$

The necessity is easy to see. Suppose that there is some rational subspace S^d such that the forms $L_1, \dots, L_t$ on it have rank r with $r < dt/n \leq d$. By reordering the variables if necessary, we may assume that the d linear forms $L_1, \dots, L_r, x_{r+1}, \dots, x_d$ are linearly independent on S^d . By Minkowski's theorem on linear forms (see e.g., [2, p. 39]), there is for every $Q > 0$ an integer point $\underline{x} \neq \underline{0}$ in S^d with

$$|L_j(\underline{x})| < Q^{r-d} \quad (1 \leq j \leq r), \quad |x_i| < Q^r \quad (r+1 \leq i \leq d),$$

where the constants in $<$ depend only on $L_1, \dots, L_t$.

Setting $|\underline{x}| = \max(|x_1|, \dots, |x_n|)$, we obtain

$$|L_j(\underline{x})| < Q^{r-d} \quad (1 \leq j \leq t), \quad |\underline{x}| < Q^r, \quad (2)$$

and hence

$$|L_j(\underline{x})| < |\underline{x}|^{-(d-r)/r} = |\underline{x}|^{1-d/r} \quad (1 \leq j \leq t). \quad (3)$$

Since Q in (2) may be taken arbitrarily large, the inequalities (3) have infinitely many solutions in integer points $\underline{x} \neq \underline{0}$ in S^d . Since $r < dt/n$, we can find a $\delta > 0$ such that $r = dtn^{-1}(1+2\delta)^{-1}$. Then, we have

$$|L_j(\underline{x})| < |\underline{x}|^{1-n(1+2\delta)/t} < |\underline{x}|^{-s/t-\delta} \quad (1 \leq j \leq t),$$

and hence $L_1, \dots, L_t$ is not a Roth system.

From theorem 1, we have at once the following

COROLLARY 1. A necessary and sufficient condition for $L_1, \dots, L_t$ to be a Roth system is that there are only finitely many integer points $\underline{x} \neq \underline{0}$ with

$$|L_1(\underline{x}) \dots L_t(\underline{x})| \leq |\underline{x}|^{t-n}.$$

Taking $t = n$ in particular, we have the following

COROLLARY 2. A necessary and sufficient condition for $L_1, \dots, L_n$ to be a Roth system is that there are only finitely many integer solutions $\underline{x} \neq \underline{0}$ with

$$L_1(\underline{x}) \dots L_n(\underline{x}) = 1.$$

2. PROOF OF THEOREM 1. Let $L_1, \dots, L_n$ be a symmetric system of linear forms in $\underline{x} = (x_1, \dots, x_n)$ with algebraic coefficients. Let $c_1, \dots, c_n$ be reals with

$$c_1 + \dots + c_n = 0. \quad (4)$$

We call that $(L_1, \dots, L_n; c_1, \dots, c_n)$ is a generalized Roth system, if for each $\delta > 0$ there is a positive number $Q_1 = Q_1(\delta; L_1, \dots, L_n; c_1, \dots, c_n)$ such that the inequalities

$$|L_j(\underline{x})| \leq Q_1^{c_j - \delta} \quad (1 \leq j \leq n) \quad (5)$$

have no integer solutions $\underline{x} \neq \underline{0}$ when $Q > Q_1$. (We assume $c_1 = c_j$ for a pair of complex conjugates $L_1(\underline{x})$ and $L_j(\underline{x})$).

From now on, we assume

$$c_1 \leq c_2 \leq \dots \leq c_n. \quad (6)$$

Now let S^d be a subspace of C^n of dimension $d > 0$, and suppose that $L_1, \dots, L_n$ have rank r on S^d . If $r = d$, let $i(1)$ be the smallest integer such that $L_{i(1)} \neq 0$ on S^d , $i(2)$ the smallest integer such that $L_{i(1)}, L_{i(2)}$ have rank 2 on S^d and so on. In this way, we obtain d integers $i(1), \dots, i(d)$. Put

$$c(S^d) = c_{i(1)} + \dots + c_{i(d)}.$$

If $r < d$, put $c(S^d) = +\infty$. Then, we obtain the following theorem analogous to theorem 2 of [3].

THEOREM 2. Suppose that $L_1, \dots, L_n$ is a symmetric system of linear forms with algebraic coefficients, and $c_1, \dots, c_n$ are constants subject to (4) and (6). Then, a necessary and sufficient condition for $(L_1, \dots, L_n; c_1, \dots, c_n)$ to be a generalized Roth system is

$$c(S^d) \leq 0$$

for every rational subspace $S^d \neq \underline{0}$.

Using theorem 2, we will conclude the proof of theorem 1.

PROOF OF THEOREM 1 (continued). The necessity has been proved before. We will prove the sufficiency. The inequality (1), with $d = n$, implies that $L_1, \dots, L_t$ are linearly independent. By reordering the variables if necessary, we

may assume that the n forms $L_1, \dots, L_t, x_1, \dots, x_s$ are linearly independent. (Note that the variables $x_1, \dots, x_s$ do not occur when $t = n$). Let us consider the system

$$(L_1, \dots, L_t, x_1, \dots, x_s; c_1 = -s, \dots, c_t = -s, c_{t+1} = t, \dots, c_n = t). \quad (7)$$

(When $t = n$, we obtain $c_1 = \dots = c_n = 0$). Since $L_1, \dots, L_t, x_1, \dots, x_s$ have rank n , they have rank d on every subspace of dimension d . Let S^d be a rational subspace of dimension d , and let r be the rank of $L_1, \dots, L_t$ on S^d . Then, we have

$$i(1) < i(2) < \dots < i(r) \leq t < i(r+1) < \dots < i(d),$$

and hence

$$\begin{aligned} c(S^d) &= c_{i(1)} + \dots + c_{i(r)} + c_{i(r+1)} + \dots + c_{i(d)} \\ &= r(-s) + (d-r)t = td - nr. \end{aligned}$$

(When $t = n$, we have $r = d$ and $c(S^d) = 0 + \dots + 0 = 0$). The inequality (1) implies $c(S^d) \leq 0$, and hence (7) is a generalized Roth system by theorem 2. Thus for any $\epsilon > 0$, the inequalities

$$|L_j(\underline{x})| \leq Q^{-s-\epsilon} \quad (1 \leq j \leq t), \quad |x_i| \leq Q^{t-\epsilon} \quad (1 \leq i \leq s)$$

have no integer solutions $\underline{x} \neq \underline{0}$ if Q is large. This implies that $L_1, \dots, L_t$ is a Roth system.

3. PROOF OF THEOREM 2. A set M of positive numbers will be called unbounded, if there is no K with $z \leq K$ for every z in M . For the proof of theorem 2, we require

LEMMA. Let $L_1, \dots, L_n$ be independent linear forms in $\underline{x} = (x_1, \dots, x_n)$ with real algebraic coefficients, and let $c_1, \dots, c_n$ be constants with (4). For every $Q > 0$ the inequalities

$$|L_j(\underline{x})| \leq Q^{c_j} \quad (1 \leq j \leq n)$$

determine a parallelepiped $\Pi = \Pi(Q)$. Denote the successive minima of Π by $\lambda_1 = \lambda_1(Q), \dots, \lambda_n = \lambda_n(Q)$. Suppose that there are a $\delta > 0$, a number d with $1 \leq d \leq n-1$ and an unbounded set M of positive numbers such that

$$\lambda_d < \lambda_{d+1} Q^{-\delta} \quad (8)$$

for every Q in M . Then, there are a fixed rational sub-
space S^d of dimension d and an unbounded subset M' of
 M such that for every Q in M' the first d successive
minima of $\prod(Q)$ are attained by points $w_1, \dots, w_d$ in S^d .

PROOF. This is theorem 3 of [3].

PROOF OF THEOREM 2. Suppose first that $c(S^d) > 0$ for some rational subspace $S^d \neq \underline{0}$. If $L_1, \dots, L_n$ have rank $r < d$ on S^d , then for each $\epsilon > 0$ there is an integer point $\underline{x} \neq \underline{0}$ on S^d with $|L_j(\underline{x})| < \epsilon$ ($1 \leq j \leq n$), and hence for arbitrary $Q > 0$, $\delta > 0$ there is an $\underline{x} \neq \underline{0}$ with (5). Thus in this case, we do not have a generalized Roth system. Hence we may assume that $L_1, \dots, L_n$ have rank $r = d$ on S^d . Let δ be the positive number with $3d\delta = c(S^d)$. By applying Minkowski's linear forms theorem (see e.g., [2, p. 39]) to the lattice of integer points on S^d , there is for every $Q > 0$ an integer point $\underline{x} \neq \underline{0}$ in S^d with

$$|L_{i(k)}(\underline{x})| < Q^{c_{i(k)} - 3\delta} \quad (1 \leq k \leq d).$$

By (6) and our choice of $i(1), \dots, i(d)$, this implies

$$|L_j(\underline{x})| < Q^{c_j - 3\delta} \quad (1 \leq j \leq n),$$

and hence it implies (5), if Q is large enough. Thus again $(L_1, \dots, L_n; c_1, \dots, c_n)$ is not a generalized Roth system.

Suppose next that $(L_1, \dots, L_n; c_1, \dots, c_n)$ is not a generalized Roth system. Then, we have to show $c(S^d) > 0$ for some rational subspace $S^d \neq \underline{0}$. When $L_1, \dots, L_n$ are linearly dependent, this is true for the case $S^d = R^n$. Hence we may assume that $L_1, \dots, L_n$ are linearly independent. We will now construct real linear forms $M_1, \dots, M_n$ as follows which is due to Schmidt (see [4]). If L_j is real, we put $M_j = L_j$. If L_j and L_{j+1} are complex conjugates each other, we have $L_j = L_j^R + iL_j^I$, $L_{j+1} = L_{j+1}^R + iL_{j+1}^I$, where L_j^R, L_{j+1}^R and L_j^I, L_{j+1}^I are the real and imaginary parts of L_j and L_{j+1} , respectively. Now by our assumption, there are an $\epsilon > 0$ and an unbounded set N of posi-

tive numbers such that for every Q in N , the inequalities

$$|L_j(\underline{x})| \leq Q^{c_j - \varepsilon} \quad (1 \leq j \leq n) \quad (9)$$

have an integer point $\underline{x} = \underline{x}(Q) \neq \underline{0}$. For every Q in N ,

$\underline{x} = \underline{x}(Q)$ satisfies either $|L_j^R(\underline{x})| \geq |L_j^I(\underline{x})|$ or $|L_j^R(\underline{x})| \leq |L_j^I(\underline{x})|$. Hence without loss of generality, we may assume that for every Q in N , $\underline{x} = \underline{x}(Q)$ satisfies

$$|L_j^R(\underline{x})| \geq |L_j^I(\underline{x})|.$$

Further, by the similar arguments, we may assume that

$\underline{x} = \underline{x}(Q)$ satisfies

$$L_j^R(\underline{x})L_j^I(\underline{x}) \geq 0.$$

Now if L_j and L_{j+1} are complex conjugates each other, we

put $M_j = L_j^R$ and $M_{j+1} = L_j^R + L_j^I$. Then for every $\underline{x} = \underline{x}(Q)$ ($Q \in N$), we have

$$|L_j(\underline{x})| = |L_{j+1}(\underline{x})| \gg \ll |M_j(\underline{x})| \gg \ll |M_{j+1}(\underline{x})|.$$

From this and (9) we have for every $\underline{x} = \underline{x}(Q)$ ($Q \in N$)

$$|M_j(\underline{x})| \ll Q^{c_j - \varepsilon} \quad (1 \leq j \leq n).$$

Put $\varepsilon_0 = \varepsilon/2$. Then for every Q in some unbounded subset N' of N , $\underline{x} = \underline{x}(Q)$ satisfies

$$|M_j(\underline{x})| \leq Q^{c_j - \varepsilon_0} \quad (1 \leq j \leq n).$$

For every Q in N' , let $\lambda_1, \dots, \lambda_n$ be the successive minima of the parallelepiped $\Pi = \Pi(Q)$ defined by the inequalities

$$|M_j(\underline{x})| \leq Q^{c_j} \quad (1 \leq j \leq n).$$

Then, by the definition of the successive minima, we have

$$\lambda_1 \leq Q^{-\varepsilon_0}.$$

Further, since

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \quad \text{and} \quad 1 \ll \lambda_1 \lambda_2 \dots \lambda_n \ll 1 \quad (10)$$

(see e.g., [1, chapter VIII]), we have

$$\lambda_d \ll \lambda_{d+1} Q^{-\varepsilon_0/(n-1)}$$

for some d in $1 \leq d < n$, where the constants in $\ll$ depend only on $L_1, \dots, L_n$. For large Q , this implies (8) with $\delta = \varepsilon_0/n$. Thus there are a fixed d in $1 \leq d < n$ and a fixed $\delta > 0$ such that (8) holds for every Q in some unbounded set M . Hence we may apply lemma.

The arguments of the rest of this paper is now all the same with [3]. Let S^d be the subspace in the conclusion of lemma, and put

$$\Pi'(Q) = \Pi(Q) \cap S^d.$$

Then, $\Pi'(Q)$ is a convex and symmetric set in the space S^d . Let

$$\lambda'_1 = \lambda'_1(Q), \dots, \lambda'_d = \lambda'_d(Q)$$

be the successive minima of $\Pi'(Q)$ with respect to the lattice of integer points in S^d , and let $V' = V'(Q)$ be the $(d - \text{dimensional})$ volume of $\Pi'(Q)$. By Minkowski's theorem on successive minima, we have

$$1 \ll \lambda'_1 \dots \lambda'_d V' \ll 1, \quad (11)$$

with the constants in $\ll$ depending only on S^d . On the other hand, by lemma, there is an unbounded subset M' of M such that for every Q in M' the first d minima of $\Pi(Q)$ are attained by points in S^d . For these values of Q , we have

$$\lambda'_1 = \lambda_1, \dots, \lambda'_d = \lambda_d.$$

Hence by (8) and (10), we have

$$\begin{aligned} \lambda'_1 \dots \lambda'_d &= \lambda_1 \dots \lambda_d = (\lambda_1 \dots \lambda_d)^{d/n} (\lambda_1 \dots \lambda_d)^{(n-d)/n} \\ &< (\lambda_1 \dots \lambda_d)^{d/n} (\lambda_{d+1} \dots \lambda_n)^{d/n} Q^{-\delta d(n-d)/n} \\ &\ll Q^{-\delta d(n-d)/n} = Q^{-\tau}, \end{aligned}$$

where $\tau = \delta d(n-d)/n$. From this and (11), we have

$$V'(Q) \gg Q^\tau, \quad (12)$$

and this inequality holds for every Q in M' .

Choose $t(1), \dots, t(d)$ as before, i.e., let $t(1)$ be the smallest integer such that $M_{t(1)} \neq 0$ on S^d , $t(2)$ the smallest integer such that $M_{t(1)}, M_{t(2)}$ have rank 2 on S^d

and so on. The set $\Pi'(Q)$ is contained in the parallelepiped $\Pi''(Q)$ of points $\underline{x}$ of S^d with

$$|M_{t(j)}(\underline{x})| \leq Q^{c_{t(j)}} \quad (1 \leq j \leq d).$$

Let $V'' = V''(Q)$ be the $(d - \text{dimensional})$ volume of $\Pi''(Q)$. Then, we have

$$V'' \leq Q^{c_{t(1)} + \dots + c_{t(d)}}. \quad (13)$$

Further, from the construction of $M_1, \dots, M_n$ we have

$$c(S^d) = c_{t(1)} + \dots + c_{t(d)}.$$

Hence by (12), (13) and since $V' \leq V''$, we obtain

$$c(S^d) \geq \tau > 0.$$

This completes the proof of theorem 2.

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