Extending Bounded Holomorphic Functions from Certain Subvarieties of a Strongly Pseudoconvex Domain

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1. Introduction. G. M. Henkin ([2]) proved that any bounded holomorphic function defined on an analytic closed submanifold in general position in a strongly pseudoconvex domain can be continued to a bounded holomorphic function in the entire domain. In this paper we extend G. M. Henkin's results to certain subvarieties, which possess sufficiently restricted singular points, in a strongly pseudoconvex domain. Our theorem will be proved by applying the integral formula constructed by E. L. Stout ([3]) for certain hypersurfaces in a strongly pseudoconvex domain and the method of G. M. Henkin ([2]) which was used to prove the extension theorem for analytic submanifolds in a strongly convex domain.

2. Let Ω be an arbitrary subset in C^{N+1} . We let $O(\Omega)$ designate the space of holomorphic functions in Ω and we let $H^\infty(\Omega)$ designate the space of bounded holomorphic functions in Ω . Let $D, \tilde{D}$ ($\bar{D} \subset \tilde{D}$) be strongly pseudoconvex domains in C^{N+1} with C^∞ -boundaries. We fix a function $F \in O(\tilde{D})$, $F \neq 0$. We set $Z(F) = \{z: F(z)=0\}$, $\Delta = Z(F) \cap D$ and $\tilde{\Delta} = Z(F) \cap \tilde{D}$. Then D can be written as $D = \{z \in \tilde{D}: \rho(z) < 0\}$, where ρ is a C^∞ -strongly plurisubharmonic function on $\tilde{D}$ with $d\rho \neq 0$ on ∂D . Let $D_\nu = \{z \in D: \rho(z) < -\varepsilon_\nu\}$ and $\Delta_\nu = Z(F) \cap D_\nu$, where $\{\varepsilon_\nu\}$ is a sequence of sufficiently small strictly decreasing positive numbers which converges 0. Now we make the following assumptions:

- a) Δ is a non-empty connected set.
- b) $dF \neq 0$ on $Z(F) \cap \partial D$. (A)
- c) $Z(F)$ meets ∂D transversally.

In this setting, we have the following formula.

THEOREM 1. *If $f \in H^\infty(\Delta)$, then the following formula holds for all $z \in \Delta$.*

$$f(z) = \lim_{\nu \rightarrow \infty} \int_{\partial \Delta_\nu} f(\zeta) \frac{\bar{\Psi}_z \bar{\omega}_F}{G(z, \zeta)^N \|\text{grad } F(\zeta)\|} = \lim_{\nu \rightarrow \infty} \int_{\partial \Delta_\nu} f(\zeta) \tilde{K}(z, \zeta), \text{ say,}$$

where $G(z, \zeta)$ is a C^∞ -function in a neighbourhood $U \times V$ of $\bar{D} \times \partial D$ such that for

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each $\zeta \in V$, $G(\cdot, \zeta)$ is holomorphic in U and such that $G(z, \zeta) = 0$ implies $z = \zeta$. Moreover, G admits a division

$$G(z, \zeta) = \sum_{j=1}^{N+1} (z_j - \zeta_j) g_j(z, \zeta)$$

with $g_j: U \times V \rightarrow \mathbb{C}$ of class C^∞ and holomorphic in the first variable. $\tilde{\Psi}_z(\zeta)$ is a $C^\infty(0, N-1)$ form in a neighbourhood of $\bar{D} \times \partial D$ and, for each ζ near ∂D , coefficients of $\tilde{\Psi}_z(\zeta)$ is holomorphic in $\bar{D}$. $\tilde{\omega}_F$ is given by

$$\tilde{\omega}_F = \sum_{j=1}^{N+1} (-1)^{j-1} \bar{F}_j d\zeta_1 \wedge \cdots \wedge \widehat{d\zeta_j} \wedge \cdots \wedge d\zeta_{N+1},$$

where $F_j = \partial F / \partial \zeta_j$, $j = 1, \dots, N+1$, and $\widehat{}$ means to be omitted.

PROOF. By theorem II.1 of E. L. Stout ([3]), it follows that

$$f(z) = \int_{\partial \Delta_v} f(\zeta) \tilde{K}(z, \zeta) \quad \text{for } z \in \Delta_v.$$

Therefore theorem 1 follows.

THEOREM 2. *Let*

$$H_v(z) = \int_{\partial \Delta_v} f(\zeta) \tilde{K}(z, \zeta)$$

and

$$L(f)(z) = H(z) = \lim_{v \rightarrow \infty} H_v(z). \quad (1)$$

Then $H(z)$ is holomorphic on $\bar{D} - \partial \Delta$ and $H(z) = f(z)$ for all $z \in \Delta$.

PROOF. For $z \in W \in D_{v_0}$, and $v > \mu \geq v_0$, we have

$$H_v(z) - H_\mu(z) = \int_{\partial(\Delta_v - \Delta_\mu)} f(\zeta) \tilde{K}(z, \zeta) = \int_{\Delta_v - \Delta_\mu} f(\zeta) \bar{\partial}_\zeta \tilde{K}(z, \zeta).$$

Since the form $\bar{\partial}_\zeta K(z, \zeta)$ is bounded for $\zeta \in \Delta_v - \Delta_\mu$ and $z \in W$, there exists a constant K with

$$|H_v(z) - H_\mu(z)| \leq K \sup_{\zeta \in \Delta} |f(\zeta)| \cdot \text{Vol}(\Delta_v - \Delta_\mu).$$

Hence $H_v(z)$ converges locally uniformly on D . Therefore $H(z)$ is holomorphic in D . For $z \in \partial D - \partial \Delta$, $H_v(z)$ is holomorphic in z for sufficiently large v . Since $H_v(z)$ converges uniformly in a neighborhood of z , together with theorem 1, theorem 2 follows.

3. Since the problem is local, we may assume that $F = z_{N+1}$. Then $H_v(z)$ can be written as

$$\begin{aligned}
 H_v(z) &= \int_{\partial \Delta_v} f(\zeta) \frac{\sum_{m=1}^N \Psi_m(z, \zeta)}{G(z, \zeta)^N} d\zeta_1 \wedge \cdots \wedge \widehat{d\zeta_m} \wedge \cdots \wedge d\zeta_N \wedge \omega(\zeta) \\
 &= \int_{\partial \Delta_v} f(\zeta) \frac{K(z, \zeta)}{G(z, \zeta)^N}, \text{ say, where } \omega(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_N.
 \end{aligned}$$

LEMMA 1. Let $f(\zeta) \in H^\infty(D)$. Then any $z \in (\bar{D} - \partial D)$ and any vector $w = (w_1, \dots, w_{N+1}) \neq 0$ satisfy the equation

$$\begin{aligned}
 \frac{dH(z + \lambda w)}{d\lambda} \Big|_{\lambda=0} &= \lim_{v \rightarrow \infty} \frac{dH_v(z + \lambda w)}{d\lambda} \Big|_{\lambda=0} = \lim_{v \rightarrow \infty} \int_{\partial \Delta_v} \frac{f(\zeta) \sum_{j=1}^{N+1} \frac{\partial}{\partial z_j} K(z, \zeta) w_j}{G(z, \zeta)^N} \\
 &\quad - \lim_{v \rightarrow \infty} \int_{\partial \Delta_v} f(\zeta) \frac{N \sum_{j=1}^{N+1} \frac{\partial}{\partial z_j} G(z, \zeta) w_j K(z, \zeta)}{G(z, \zeta)^{N+1}}. \quad (2)
 \end{aligned}$$

PROOF. This follows immediately from the formula (1).

4. Let $z^0 \in \partial D$. Let i_0 be an index such that for some $\gamma_0 > 0$

$$\left| \frac{\partial \rho}{\partial z_{i_0}}(z^0) \right| \geq \gamma_0.$$

Let

$$S_{z^0, \sigma} = \{\zeta \in C^{N+1} : |\zeta - z^0| < \sigma\},$$

Let $z \in S_{z^0, \sigma}$. We consider the system of equations for $\zeta^0 = (\zeta_1^0, \dots, \zeta_{N+1}^0)$ of the following form:

$$\begin{aligned}
 \sum_{i=1}^{N+1} \frac{\partial \rho}{\partial \zeta_i}(\zeta^0) (\zeta_i^0 - z_i) &= 0, \\
 \zeta_i^0 &= z_i \quad (i=1, 2, \dots, N, i \neq i^0), \\
 \zeta_{N+1}^0 &= 0.
 \end{aligned} \quad (3)$$

LEMMA 2 (G. M. Henkin [2]). There exist positive constants σ_1, γ_1 and γ_2, γ_3 ($\gamma_1 \leq 1$), depending only on D and M , such that for any $\sigma \leq \sigma_1$ and any $z \in S_{z^0, \gamma_1 \sigma}$ there exists a unique solution $\zeta^0 = \zeta^0(z)$ of the system (3) which belongs to the set $S_{z^0, \sigma} \cap \bar{D}$. Here the point $\zeta^0 = \zeta^0(z)$ has the following properties;

$$\begin{aligned}
 |z - \zeta^0(z)|^2 &\leq \frac{1}{\gamma_2} [\rho(z) - \rho(\zeta^0(z))], \\
 |z - \zeta^0(z)|^2 &\geq |z_{N+1}|^2 \geq \gamma_3 [\rho(z) - \rho(\zeta^0(z))], \\
 \zeta^0(z) &= z \quad \text{for any } z \in S_{z^0, \gamma_1 \sigma} \cap \bar{D}.
 \end{aligned}$$

PROOF. Since D is locally biholomorphic to a convex domain in C^{N+1} , we can assume S_{z^0, σ_1} is convex. Then lemma follows from lemma 6 of G. M. Henkin ([2]).

LEMMA 3. Let $f(\zeta)$ be a bounded holomorphic function on Δ . Then any point $z^0 \in \partial\Delta$ and any point $z \in S_{z^0, \gamma_1 \sigma_1}$ satisfy the inequality

$$\left| \frac{dH(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right|_{\lambda=1} \leq \gamma_4 \sup_{\zeta \in \Delta} |f(\zeta)|,$$

where $\zeta^0 = \zeta^0(z)$, γ_1, σ_1 are from lemma 2, and the γ_4 depends only on D and Δ .

PROOF. We set $\varepsilon = |z_{N+1}|$, where $z = (z_1, \dots, z_{N+1}) \in S_{z^0, \gamma_1 \sigma_1}$. Then lemma 2 implies the inequalities

$$\varepsilon \leq |\zeta^0(z) - z| \leq \left\{ \frac{\rho(z) - \rho(\zeta^0(z))}{\gamma_2} \right\}^{1/2} \leq \frac{\varepsilon}{(\gamma_2 \gamma_3)^{1/2}}.$$

Since $\sum_{i=1}^{N+1} \frac{\partial \rho}{\partial \zeta_i}(\zeta^0)(\zeta_i^0 - z_i) = 0$, it follows that

$$\begin{aligned} \left| \sum_{i=1}^{N+1} \frac{\partial G(z, \zeta)}{\partial z_i}(\zeta_i^0 - z_i) \right| &= \left| \sum_{i=1}^{N+1} \left(\frac{\partial G(z, \zeta)}{\partial z_i} + 2 \frac{\partial \rho}{\partial \zeta_i}(\zeta^0) \right)(\zeta_i^0 - z_i) \right| \\ &= \left| \sum_{i=1}^{N+1} \left(\frac{\partial G(z, \zeta)}{\partial z_i} - \frac{\partial G(z, \zeta^0)}{\partial z_i} + O(|\zeta^0 - z|) \right)(\zeta_i^0 - z_i) \right| \end{aligned}$$

$\leq \gamma_5 \varepsilon (|\zeta^0 - z| + \varepsilon)$. Here we have used the following equation (H. Grauert-I. Lieb [1]);

$$\frac{\partial G(z, \zeta^0)}{\partial z_i} = 2 \frac{\partial \rho}{\partial \zeta_i}(\zeta^0) + O(|\zeta^0 - z|).$$

By the equation (2), we have the following inequalities:

$$\begin{aligned} \left| \frac{dH(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right|_{\lambda=1} &= \left| \frac{dH(z + \lambda(z - \zeta^0))}{d\lambda} \right|_{\lambda=0} \leq \\ \lim_{\gamma \rightarrow \infty} \left\{ \gamma_6 \int_{\partial \Delta \cap S_{z^0, \sigma_1}} \frac{|f(\zeta)| |z - \zeta^0| d\lambda}{|G(z, \zeta)|^N} + \gamma_7 \int_{\partial \Delta \cap S_{z^0, \sigma_1}} \frac{|f(\zeta)| \varepsilon (|\zeta - z| + \varepsilon) d\lambda}{|G(z, \zeta)|^{N+1}} \right\} \\ &+ \gamma_8 \sup_{\zeta \in \Delta} |f(\zeta)|. \end{aligned}$$

By a smooth change of coordinates, we have (H. Grauert-I. Lieb [1])

$$|G(z, \zeta)| \geq A(R^2 + x_1''), \quad \text{for } \rho(\zeta) \geq \rho(z),$$

where $z_j - \zeta_j = x_j' + ix_j''$, $j = 1, \dots, N+1$, and $|z - \zeta| = R$. If we set

$$x_1''^2 + x_2''^2 + \dots + x_N''^2 = r^2, \text{ and } \varepsilon^2 = x_{N+1}'^2 + x_{N+1}''^2,$$

then to prove the lemma 3, it suffices to show the following inequalities:

$$\begin{aligned} I_1: \int_0^\pi \int_0^1 \frac{r^{2N-2} \sin^{2N-3} \phi d\phi dr}{((r^2 + \varepsilon^2)^2 + r^2 \cos^2 \phi)^{N+1/2}} &\leq \gamma_8 \frac{1}{\varepsilon}, \\ I_2: \int_0^\pi \int_0^1 \frac{r^{2N-2} (r^2 + \varepsilon^2)^{1/2} \sin^{2N-3} \phi d\phi dr}{((r^2 + \varepsilon^2)^2 + r^2 \cos^2 \phi)^{N+1/2}} &\leq \gamma_9 \frac{1}{\varepsilon}, \end{aligned}$$

$$I_3: \int_0^\pi \int_0^1 \frac{r^{2N-2} \sin^{2N-3} \phi d\phi dr}{((r^2 + \varepsilon^2)^2 + r^2 \cos^2 \phi)^{N+1/2}} \leq \gamma_{10} \frac{1}{\varepsilon^2}.$$

I_1 and I_3 are consequences of I_2 . But I_2 follows immediately by simple calculations. Therefore lemma 3 is proved.

THEOREM 3. *Under the hypothesis (A), formula (1) defines a bounded linear operator $L: H^\infty(D) \rightarrow H^\infty(D)$.*

PROOF. We let $(\partial\Delta)_\sigma$ designate the σ -neighbourhood of the manifold $\partial\Delta$. By G. M. Henkin ([2]), it suffices to show that

$$\sup_{z \in [(\partial\Delta)_\sigma - \partial\Delta] \cap \partial D_\nu} |H(z)| \leq \gamma_{11} \sup_{\zeta \in D} |f(\zeta)|,$$

where the constant γ_{11} depends only on D and Δ . We now fix $z \in [(\partial\Delta)_\sigma - \partial\Delta] \cap \partial D_\nu$. We set $\varepsilon = |z_{N+1}|$. Then $\varepsilon \leq \sigma$. Let the point $z^0 \in \partial\Delta$ be such that $z \in S_{z^0, \sigma}$. Let $\sigma < \frac{1}{2} \gamma_1^2 \sigma_1$. Then by lemma 2 there exists a solution $\zeta^0 = \zeta^0(z)$ of the system (3) belonging to the set $(S_{z^0, \gamma_1 \sigma_1 / 2}) \cap \tilde{D}$ and satisfying the inequality

$$\gamma_3 \varepsilon^2 \leq \rho(z) - \rho(\zeta^0(z)) \leq \frac{1}{\gamma_2} \varepsilon^2. \quad (4)$$

Let

$$T_\nu = \{\lambda \in C: Z(\lambda) = \zeta^0 + \lambda(z - \zeta^0) \in D_\nu \cap S_{z^0, \sigma_1}\}.$$

T_ν is a convex domain containing $\lambda=0$, since $D_\nu \cap S_{z^0, \sigma_1}$ is convex. For any λ we have

$$\sum_{i=1}^{N+1} \frac{\partial \rho}{\partial \zeta_i}(\zeta^0)(\zeta_i^0 - z_i(\lambda)) = 0. \quad (5)$$

From equation (5), we have

$$|\lambda(z - \zeta^0)|^2 = |z(\lambda) - \zeta^0|^2 \leq \frac{1}{\gamma_3} \{\rho(z(\lambda)) - \rho(\zeta^0)\}.$$

Hence for $\lambda \in \partial T_\nu \cap \partial D_\nu$ we obtain

$$|z(\lambda) - z^0| \leq |z(\lambda) - \zeta^0| + |\zeta^0 - z^0| \leq \frac{\sigma}{(\gamma_2 \gamma_3)^{1/2}} + \frac{1}{2} \sigma_1 \gamma_1.$$

We impose one further restriction on the constant σ : $\sigma \leq \frac{1}{2} \sigma_1 \gamma_1 (\gamma_2 \gamma_3)^{1/2}$. Then by the uniqueness of a solution of the system (3), it follows that $\zeta^0(z(\lambda)) = \zeta^0(z) = \zeta^0$ for any $\lambda \in \partial T_\nu \cap \partial D_\nu$. We obtain the following inequalities from lemma 2:

$$|\lambda| \frac{\varepsilon}{(\gamma_2 \gamma_3)^{1/2}} > |\lambda| |z - \zeta^0| = |z(\lambda) - \zeta^0| \geq (\gamma_2 \gamma_3)^{1/2} \varepsilon.$$

Consequently

$$|\lambda| \geq \gamma_2 \gamma_3 \quad \text{for any } \lambda \in \partial T_v \cap \partial D_v.$$

If $\lambda \in \partial T_v - \partial D_v$, then there exist γ_{12} such that $|\lambda| \geq \gamma_{12}$. Let $\gamma_{13} = \min(\gamma_2 \gamma_3, \gamma_{12})$. Then

$$|\lambda| \geq \gamma_{13} \tag{6}$$

for any $\lambda \in \partial T_v$. By lemma 3, we have

$$\left| \frac{dH_v(\zeta^0 + t(z(\lambda) - \zeta^0))}{dt} \right|_{t=1} \leq \gamma_4 \sup_{\zeta \in \Delta} |f(\zeta)| \tag{7}$$

for any $\lambda \in \partial T_v$. We note that

$$\frac{dH_v(\zeta^0 + t(z(\lambda) - \zeta^0))}{dt} \Big|_{t=1} = \lambda \frac{dH_v(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda}.$$

From (6) and (7) we have

$$\left| \frac{dH_v(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right| \leq \frac{\gamma_7}{|\lambda|} \sup_{\zeta \in \Delta} |f(\zeta)|$$

for any $\lambda \in \partial T_v$. Since the function $dH_v(\zeta^0 + \lambda(z - \zeta^0))/d\lambda$ is holomorphic in λ for all $\lambda \in \bar{T}_v$, it follows that

$$\sup_{\lambda \in \bar{T}_v} \left| \frac{dH_v(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right| \leq \frac{\gamma_7}{\gamma_{13}} \sup_{\zeta \in \Delta} |f(\zeta)|.$$

Consequently

$$|H_v(z) - H_v(\zeta^0)| = \left| \int_0^1 \frac{d}{d\lambda} H_v(\zeta^0 + \lambda(z - \zeta^0)) d\lambda \right| \leq \frac{\gamma_7}{\gamma_{13}} \sup_{\zeta \in \Delta} |f(\zeta)|.$$

It follows from (4) that $\zeta^0 \in \Delta_v$. Since $H_v(\zeta^0) = f(\zeta^0)$, we obtain the following inequality

$$|H_v(z)| \leq \left(\frac{\gamma_7}{\gamma_{13}} + 1 \right) \sup_{\zeta \in \Delta} |f(\zeta)|.$$

Therefore

$$|H(z)| \leq \left(\frac{\gamma_7}{\gamma_{13}} + 1 \right) \sup_{\zeta \in \Delta} |f(\zeta)|.$$

Therefore theorem 3 is proved.

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