Infinite Group Rings II

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In this paper a ring is a commutative ring with the identity. Let X be an indeterminate, G a totally ordered commutative group (additively written and nonzero). We consider the group ring $A[X; G] = A_G[X]$ of G over a ring A . $A_G[X]$ is the ring containing A of elements $\sum a_\alpha X^\alpha$, $a_\alpha \in A$, $\alpha \in G$ almost all a_α are zero. Gilmer-Parker [9], [10] and the author [16] considered $A_G[X]$. This paper is a continuation of [16]. Let I be any nonempty set, and $\{X_i; I\}$ a set of indeterminates with the index set of I . We may define the group ring $A_G[X_i; I]$ of G over a ring A with the indeterminates $\{X_i; I\}$. $A_G[X_i; I]$ is the ring of elements $\sum a(X_{i_1}^{e_1} \cdots X_{i_n}^{e_n}) X_{i_1}^{e_1} \cdots X_{i_n}^{e_n}$, $a(X_{i_1}^{e_1} \cdots X_{i_n}^{e_n}) \in A$; $e_1, \dots, e_n \in G$; $i_1, \dots, i_n \in I$; $n < \infty$, almost all coefficients $a(X_{i_1}^{e_1} \cdots X_{i_n}^{e_n})$ are zero. We denote an integral domain (with 1) by D and the quotient field of D by K . The quotient field of $D_G[X_i; I]$ is denoted by $D_G(X_i; I)$. For an element $f = \sum a_\alpha X^\alpha$ of $A_G[X]$, the support $\{\alpha; a_\alpha \neq 0\}$ is written by $\text{supp}(f)$. The degree of f is $\sup(\text{supp}(f))$ and denoted by $\deg(f) = d(f)$, and $\inf(\text{supp}(f))$ is the order of f and denoted by $\text{ord}(f) = o(f)$. $a_{o(f)}$ is the initial or the leading coefficient and $a_{d(f)}$ is the terminal coefficient. If $a_{d(f)}$ is the identity, f is said monic or unitary. This paper consists of nine sections. §1 is on the irreducible elements, §2 is on the pseudo-prüfer domains, §3 is on the radicals, §4 is on the finite character properties, §5 is on the noetherian properties, §6 is on the integral elements, §7 is on the rational functions rings, §8 is on the Kronecker function rings and §9 is on the others. Throughout the paper, A is a ring, X is an indeterminate, G a nonzero totally ordered commutative additive group, I any nonempty set (finite or infinite), $\{X_i; I\}$ a set of indeterminates with the index set of I , D an integral domain, K the quotient field of D .

§1. The irreducible elements.

In the first section, we consider some properties on the irreducibility.

PROPOSITION 1. *Let p be a prime element of an integral domain D , $f = \sum a_\alpha X^\alpha$ is an element of $D_G[X]$. If f satisfies the following (1) or (2), then f is an irreducible element of $D_G[X]$:*

Received February 16, 1976.

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(1) p does not divide the terminal coefficient, p divides the other coefficients and p^2 does not divide the initial coefficient.

(2) p does not divide the initial coefficient, p divides the other coefficients and p^2 does not divide the terminal coefficient.

COROLLARY 2. Let p be a prime element of a domain D , $f = a_0 + a_1X + \cdots + a_nX^n$ an element of $D[X]$. If f satisfies the following (1) or (2), then f is an irreducible element of $D[X]$:

(1) $a_n \not\equiv 0(p)$; $a_0, a_1, \dots, a_{n-1} \equiv 0(p)$; $a_0 \not\equiv 0(p^2)$.

(2) $a_0 \not\equiv 0(p)$; $a_1, a_2, \dots, a_n \equiv 0(p)$; $a_n \not\equiv 0(p^2)$.

PROPOSITION 3. Let K be a field of characteristic $p > 0$, $f = \sum_0^n a_j X^{a_j}$ is an irreducible monic element of $K_G[X]$. If all the a_j belong to K^p , then $f(X^p)$ is a p -th power of an irreducible element. If some a_j does not belong to K^p , then $f(X^p)$ is irreducible in $K_G[X]$.

PROOF. Let θ be the canonical isomorphism of K onto $K^{1/p}$. θ canonically induces an isomorphism of $K_G[X]$ onto $K_G^{1/p}[X]$. Since $\sum a_j X^{a_j}$ is irreducible in $K_G[X]$, $\sum a_j^{1/p} X^{a_j}$ is irreducible in $K_G^{1/p}[X]$. If all the a_j belong to K^p , $\sum a_j^{1/p} X^{a_j}$ is irreducible in $K_G[X]$ and we have $\sum a_j X^{pa_j} = (\sum a_j^{1/p} X^{a_j})^p$. Suppose that some a_j does not belong to K^p and $\sum a_j X^{pa_j}$ is reducible in $K_G[X]$. A proper factor of $\sum a_j X^{pa_j}$ is of the form $(\sum a_j^{1/p} X^{a_j})^k$. Take a, b such that $ka + pb = 1$. We have

$$\sum a_j^{1/p} X^{a_j} = (\sum a_j^{1/p} X^{a_j})^{ka} (\sum a_j^{1/p} X^{a_j})^{pb} \in K_G[X].$$

Therefore all the $a_j^{1/p}$ belong to K .

PROPOSITION 4. Let A be a ring, $\mathfrak{p}$ a prime ideal. Let $f = a_0 + a_1X^{\alpha_1} + \cdots + a_nX^{\alpha_n}$ be an element of $A_G[X]$, where $o(f) = 0$, $0 < \alpha_1 < \cdots < \alpha_n$ such that $a_0 \equiv a_1 \equiv \cdots \equiv a_{k-1} \equiv 0(\mathfrak{p})$, $a_k \not\equiv 0(\mathfrak{p})$, $a_0 \not\equiv 0(\mathfrak{p}^2)$. If $f = f_1 \cdots f_m$ is an irreducible decomposition, where all f_j are irreducible and $o(f_j) = 0$, then we have $d(f_j) \geq \alpha_k$ for some j .

$o(\)$ denotes the order and $d(\)$ denotes the degree.

THEOREM 5. Let D be a factorial domain, π a prime element of D , S a factorial domain containing D which is integral over D , K the quotient field of D , L the quotient field of S . Suppose that $[L:K]$ is a prime number p distinct from the characteristic and there is an element T such that $L = K(T)$, $T^p \in D$. Then we have one of the following (1), (2), (3):

(1) π is irreducible in S .

(2) There are an irreducible element π' and a unit U of S such that $\pi = \pi'^p U$.

(3) There are $g(X) \in K[X]$ and a unit U of S such that $g(T)$ is irreducible in S and $\pi = U \prod_{j=0}^{p-1} g(\zeta^j T)$, where ζ is a primitive p -th root of 1.

We can prove the Theorem as [16] 7. From Theorem 5, we have the following:

PROPOSITION 6. Let D be a factorial domain, G a finitely generated totally ordered abelian group, H an ordered subgroup of G such that $(G:H)=p$ is a prime number, $f(X)$ an irreducible element of $D_H[X]$ such that the initial coefficient a is a nonunit of D , $a = u\pi_1^{e_1} \dots \pi_k^{e_k}$ is the irreducible decomposition, where $\pi_1, \dots, \pi_k$ are mutually nonassociated irreducibles, u is a unit, and let

$$q = \text{GCD}(e_1, \dots, e_k) \times \text{characteristic exponent}.$$

Then, if q is relatively prime with p , $f(X)$ is irreducible in $D_G[X]$.

COROLLARY 7. Let D be a factorial domain, $f(X)$ is an irreducible element of $D_G[X]$, a_i the leading coefficient of f , a_t the terminal coefficient of f , $a_i = u\pi_1^{e_1} \dots \pi_k^{e_k}$, $a_t = u'\pi_1^{e'_1} \dots \pi_k^{e'_k}$ the irreducible decompositions, $\pi_1, \dots, \pi_k$ are mutually nonassociated irreducibles, $\pi'_1, \dots, \pi'_k$ are mutually nonassociated irreducibles, u, u' are units and let $q = \text{GCD}(e_1, \dots, e_k) \times \text{ch. exp.}$, $q' = \text{GCD}(e'_1, \dots, e'_k) \times \text{ch. exp.}$. Then, if $Z \ni n > 0$ satisfies (1) or (2) of the following, $f(X^n)$ is irreducible in $D_G[X]$:

$$(1) \quad k > 0, (n, q) = 1.$$

$$(2) \quad k' > 0, (n, q') = 1.$$

From the Corollary, it follows:

REMARK 8. Let D be a factorial domain, $G_0 = \{\alpha \in G; \alpha \geq 0\}$, $f(X)$ an irreducible element of $D_{G_0}[X]$ of order 0, a_i the leading coefficient of $f(X)$, a_t the terminal coefficient of $f(X)$, $a_i = u\pi_1^{e_1} \dots \pi_k^{e_k}$, $a_t = u'\pi_1^{e'_1} \dots \pi_k^{e'_k}$, $\pi_1, \dots, \pi_k$ mutually nonassociated irreducibles, $\pi'_1, \dots, \pi'_k$ mutually nonassociated irreducibles, u, u' units, $q = \text{GCD}(e_1, \dots, e_k) \times \text{ch. exp.}$, $q' = \text{GCD}(e'_1, \dots, e'_k) \times \text{ch. exp.}$. If $Z \ni n > 0$ satisfies the following (1) or (2), then $f(X^n)$ is an irreducible element of $D_{G_0}[X]$:

$$(1) \quad k > 0, (n, q) = 1.$$

$$(2) \quad k' > 0, (n, q') = 1.$$

$D_{G_0}[X]$ is the set of $f \in D_G[X]$ such that $o(f) \geq 0$. The case $G = \mathbb{Z}$ in the Remark is [16] Th. 7.

§2. The pseudo-prüfer domains.

Let D be a domain, K its quotient field. If $x/y \in D$ for $x, y \in K$, y divides x . If any finite subset of $K - 0$ has the greatest common divisor, the least common multiple, D is the pseudo-bezout domain (or GCD-domain). If a subset of $K - 0$ which has a common divisor, a common multiple has the greatest common divisor, the least common multiple, D is the pseudo-principal domain. If any finitely generated ideal of D is principal, D is called the bezout domain. If the monoid of divisors of finite type is a group, D is the pseudo-prüfer domain (or the prüfer v -multiplication ring). This section concerns with these properties.

PROPOSITION 1. *Let D be a pseudo-bezout domain, $f, g \in D_G[X]$, $c(f)$, $c(g)$ are the GCD's of the coefficients of f and g respectively. Then, f divides g in $D_G[X]$ if and only if f divides g in $K_G[X]$ and $c(f)$ divides $c(g)$ in D .*

PROOF. By §6, Prop. 1.

We have the definition of the type $(0, 0, 0, \dots)$ on G ([7] chap. 7, §42).

PROPOSITION 2. *If $D_G[X]$ is pseudo-principal, then D is pseudo-principal. If D is pseudo-principal, G is of type $(0, 0, 0, \dots)$, then $D_G[X_i; I]$ is pseudo-principal.*

PROOF. If D is pseudo-principal and G is of type $(0, 0, 0, \dots)$, then $K_G[X_i; I]$ is factorial (§4, Th. 7). Therefore, $D_G[X_i; I]$ is pseudo-principal by Prop. 1.

REMARK 3. (1) Let D be a pseudo-bezout domain, $a_1, \dots, a_n \in D - 0$, $d = \text{GCD}(a_1, \dots, a_n)$. Then, d is the GCD of $a_1, \dots, a_n$ in $D_G[X]$.

(2) Let D be a pseudo-principal domain, $\{a_j; J\}$ a subset of $D - 0$, $d = \text{GCD}(a_j; J)$. Then d is the GCD of $\{a_j; J\}$ in $D_G[X]$.

THEOREM 4. *$G_G[X_i; I]$ is pseudo-bezout if and only if D is pseudo-bezout.*

PROOF. By [9] 6, [16] 1.

PROPOSITION 5. *Let D be a GCD-domain, $f, f' \in K_G[X]$. Then we have $c(f)c(f') = c(ff')$.*

PROOF. D becomes an integrally closed domain. The assertion follows from §6, Prop. 1.

THEOREM 6. *Let D be a GCD-domain. Then, the group of divisibility of $D_G[X_i; I]$ is canonically isomorphic to the direct product of those of D and $K_G[X_i; I]$.*

PROOF. We use Prop. 5. The isomorphism is

$$(f/g) \mapsto ((c(f)/c(g)), (f/g)).$$

PROPOSITION 7. *Let D be a bezout domain, f an irreducible element of $D_G[X_i; I]$. Then, f is also irreducible in $K_G[X_i; I]$.*

PROOF. Reduce to the case where I is finite.

PROPOSITION 8. *Let D be a GCD-domain, $f \in D_G[X_i; I] - D$ is irreducible in $D_G[X_i; I]$. Then, f is prime in $K_G[X_i; I]$ and in $D_G[X_i; I]$.*

PROOF. Reduce to [8] (34.10).

PROPOSITION 9. *Let D be a factorial domain, $B = D_G[X_i; I]$, $f_1, \dots, f_r \in B$. Then, in order $\Sigma B f_j = B$ it is necessary and sufficient that, there are $g_1, \dots, g_r \in D_G[X_i; I]$, $d \in D - 0$ such that the GCD of $\{\text{coefficients of } g_j; 1 \leq j \leq r\}$ is 1, $\Sigma g_j f_j = d$ and for all irreducible factors p of d ,*

$$\Sigma (D/pD)_G[X_i; I] f_j = (D/pD)_G[X_i; I].$$

PROOF. The sufficiency: Let $d = p_1^{e_1} \dots p_m^{e_m} \times \text{unit}$ be the irreducible decomposition of d . We have

$$\begin{aligned} \Sigma h_{kj} f_j &\equiv 1(p_k); \quad -1 + \Sigma h_{kj} f_j = p_k h_k; \\ \prod_k (-1 + \Sigma h_{kj} f_j)^{e_k} &= \prod_k (p_k h_k)^{e_k}; \quad \Sigma h'_{kj} f_j \equiv 1(d). \end{aligned}$$

Since $d \in \Sigma B f_j$, we have $1 \in B f_j$.

PROPOSITION 10. *If D is a pseudo-prüfer domain, then $D_G[X]$ is also a pseudo-prüfer domain.*

PROOF. D is regularly integrally closed. Therefore we have the natural homomorphism (cf. § 6, Th. 4)

$$D_f(D_G[X]) \longrightarrow D_f(K_G[X]) \times G_f(D).$$

The inverse image of $(0, 0)$ consists of only 0. Let $f_1, \dots, f_n \in D_G[X]$. We will show that $-\text{div}(f_1, \dots, f_n)$ belongs to $D_f(D_G[X])$. There are $y_1, \dots, y_m \in K$ such that

$$\text{div}'(\text{coefficients of } f_j; 1 \leq j \leq n) + \text{div}'(y_1, \dots, y_m) = \text{div}' D,$$

where $'$ denotes the divisors on D . Let $f \in D_G[X]$ be GCD of $f_1, \dots, f_n$ in $K_G[X]$.

$$\text{div}(f_1, \dots, f_n) f^{-1} \cdot (\text{coeff. of } f)(y_1, \dots, y_m) \mapsto (0, 0).$$

Therefore we have

$$-\text{div}(f_1, \dots, f_n) = \text{div } f^{-1} \cdot (\text{coeff. of } f)(y_1, \dots, y_m) \in D_f(D_G[X]).$$

THEOREM 11. *Let D be a domain, K its quotient field. Then, D is a prüfer*

domain if and only if for any $f, g \in K_G[X_i; I]$ we have $A_f A_g = A_{fg}$.

A_f denotes the A -module generated by the coefficients of f .

PROOF. The necessity follows from §9, Th. 8. The sufficiency: Suppose that $y \in K$ satisfies a monic polynomial $f \in D[X]$. There is $g \in K[X]$ such that $f(X) = (X - y)g(X)$. For $G \ni \alpha > 0$, we have $f(X^\alpha) = (X^\alpha - y)g(X^\alpha)$.

$$D = A_{f(X^\alpha)} = A_{(X^\alpha - y)} A_{g(X^\alpha)} = (1, y) A_{g(X^\alpha)} \supset (1, y) \ni y.$$

Therefore, D is integrally closed. For any $a, b \in D$, we have

$$(aX^\alpha + b)(aX^\alpha - b) = a^2 X^{2\alpha} - b^2; \quad (a, b)^2 = (a^2, b^2).$$

By [8] Th. 24.3, D is a prüfer domain.

§3. The radicals.

The set of nilpotents of A is the nilradical of A . The intersection of all maximal ideals of A is the Jacobson radical. This section concerns with the radicals.

PROPOSITION 1. Let A be a ring with the nilradical 0, $f = \sum a_\alpha X^\alpha$, $g = \sum b_\beta X^\beta \in A_G[X]$, $\deg(fg) = \delta$. Then if $\alpha + \beta > \delta$, we have $a_\alpha b_\beta = 0$.

PROOF. Set $f = \sum_0^n a(i) X^{\alpha(i)}$, $\alpha(0) < \alpha(1) < \dots < \alpha(n)$, $g = \sum_0^m b(j) X^{\beta(j)}$, $\beta(0) < \beta(1) < \dots < \beta(m)$, $\deg f = \alpha(n)$, $\deg g = \beta(m)$. We may suppose $\delta < \alpha(n) + \beta(m)$. At first, we have $a(n)b(m) = 0$. Suppose that $\alpha(n-1) + \beta(m) = \alpha(n) + \beta(m-1) > \delta$, then we have

$$a(n-1)^2 b(m)^2 = -a(n-1)b(m)a(n)b(m-1) = 0; \quad a(n-1)b(m) = 0.$$

Similarly we have $a(n)b(m-1) = 0$; hence $a(n-1)b(m) + a(n)b(m-1) = 0$. The rests are similar.

PROPOSITION 2. Let A be a ring, $f \in A_G[X]$ the leading coefficient of which is a unit. Then, f is a unit of $A_G[X]$ if and only if all the other coefficients are nilpotents.

PROOF. We may suppose that $o(f) = 0$, the leading coefficient is 1. The sufficiency: We have $f = 1 + g$, $g^n = 0$ for some n . Then, $1 - g + g^2 - \dots \pm g^{n-1}$ is the inverse of f . The necessity: Set $f = 1 + a(1)X^{\alpha(1)} + \dots + a(n)X^{\alpha(n)}$. There is $g = 1 + b(1)X^{\beta(1)} + \dots + b(m)X^{\beta(m)}$ such that $fg = 1$; $b(m) + \sum_{\alpha(i) + \beta(j) = \beta(m)} a(i)b(j) = 0$. By Prop. 1, $b(m)$ is a nilpotent.

THEOREM 3. The nilradical of $A_G[X_i; I]$ consists of the elements coefficients of which are all nilpotents. The Jacobson radical of $A_G[X_i; I]$ coincides

with the nilradical.

PROOF. Suppose $0 \neq f \in A_G[X]$ belongs to the Jacobson radical of $A_G[X]$. There is $\alpha \in G$ such that $0 < o(X^\alpha f)$. $1 + X^\alpha f$ is invertible. Therefore, the coefficients of f are nilpotents by Prop. 2.

PROPOSITION 4. Let A be a ring, n the nilradical of A , $f = \sum a_\alpha X^\alpha$, $g = \sum b_\beta X^\beta \in A_G[X]$.

(1) If all coefficients of fg are nilpotents, $a_\alpha b_\beta$ are nilpotents for all α, β .

(2) If $\bar{0} \neq \bar{f}g \in (A/n)_G[X]$, let $\deg \bar{f}g = \delta$, $o(\bar{f}g) = \delta'$. Then,

(i) For α, β such that $\alpha + \beta > \delta$, $a_\alpha b_\beta$ are nilpotents.

(ii) For α', β' such that $\alpha' + \beta' < \delta'$, $a_{\alpha'} b_{\beta'}$ are nilpotents.

$\bar{f}g$ is the element of $(A/n)_G[X]$ obtained from fg by reducing the coefficients mod. n . Prop. 1 becomes a special case of Prop. 4.

We say the intersection of all nonzero prime ideals of A the pseudoradical of A . A domain D with nonzero pseudoradical is said G -domain.

REMARK 5. $A_G[X; I]$ is never a G -domain.

PROPOSITION 6. The pseudoradical of $A_G[X; I]$ coincides with the nilradical.

PROOF. By Th. 3.

If every nonnil ideal of A contains a nonzero idempotent, A is said a Zorn ring [2] § 6, exerc. 13).

REMARK 7. $A_G[X]$ is never a Zorn ring.

PROOF. Suppose that $A_G[X]$ is a Zorn ring. Let n be the nilradical of A , $A/n = B$, $\mathfrak{A}$ a nonnil ideal of $B_G[X]$. There is $f \in A_G[X]$ such that $\bar{0} \neq \bar{f} \in \mathfrak{A}$ and $\bar{f}$ is not nilpotent. Since f is not nilpotent, there is $e \neq 0$ such that $(f) \ni e^2 = e$. If $\bar{e} = \bar{0}$, we have $e^n = 0$ for some n ; a contradiction. Therefore $B_G[X]$ is a Zorn ring. Therefore we may suppose $n = 0$. Take $0 < \alpha \in G$. There is e such that $(1 - X^\alpha) \ni e^2 = e \neq 0$. Let

$$e = a_0 X^{\alpha_0} + \dots + a_n X^{\alpha_n}, \quad a_j \neq 0, \alpha_0 < \dots < \alpha_n.$$

If $\alpha_n > 0$, we have $a_n^2 = 0$; a contradiction. If $\alpha_0 < 0$, we have $a_0^2 = 0$; a contradiction. Therefore $e \in A$. We have $e = (1 - X^\alpha)f$ for some f . Obviously, this is impossible.

If A has only a finite number of maximal ideals, A is said semi-local ring [3] chap. 2, § 3.

REMARK 8. $A_G[X]$ is never a semi-local ring.

PROOF. Suppose that $A_G[X]$ is a semi-local ring. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be the set of maximal ideals of $A_G[X]$, $\mathfrak{n}$ the nilradical of A . By Th. 3 we have

$$(A/\mathfrak{n})_G[X] \cong \frac{A_G[X]}{\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n} \cong A_G[X]/\mathfrak{m}_1 \oplus \dots \oplus A_G[X]/\mathfrak{m}_n.$$

Therefore, $(A/\mathfrak{n})_G[X]$ satisfies the descending chain condition. By [6] 3, Th. 1, G becomes a finite group; a contradiction.

PROPOSITION 9. Let $f, g \in A_G[X_i; I]$. We have

$$\sqrt{A_{fg}} = \sqrt{A_f} \cap \sqrt{A_g}.$$

PROOF. By [19] Chap. 2, 2, exerc. 1.

§4. The finite characters.

Let D be a domain, K its quotient field. Valuation rings of K containing D are said valuation overrings of D . If there is a family $\{V_\lambda; A\}$ of valuation overrings of D satisfying the following (1), (2), we say that D is of finite character [8] (43.1):

- (1) $D = \bigcap_A V_\lambda$.
- (2) For every $x \in K - 0$, x is a unit of almost all V_λ .

This section concerns with the finite characters.

$D_G(X_i; I)$ denotes the quotient field of $D_G[X_i; I]$.

PROPOSITION 1. Let v be a valuation of a field K . Then the map

$$K_G[X_i; I] \ni \sum a(X_{i_1}^{e_1} \cdots X_{i_n}^{e_n}) X_{i_1}^{f_1} \cdots X_{i_n}^{f_n} \mapsto \inf v(a(X_{i_1}^{e_1} \cdots X_{i_n}^{e_n}))$$

induces a valuation of $K_G(X_i; I)$ extending v .

PROOF. Similar with [16] 3.2.

The valuation on $K_G(X_i; I)$ of Prop. 1 is said the trivial extension of v to $K_G(X_i; I)$. Let D be the valuation ring of v . The valuation ring of the trivial extension of v to $K_G(X_i; I)$ is said the trivially extended valuation ring of D to $K_G(X_i; I)$. The trivially extended valuation ring is $D_G(X_i; I)_{\text{loc}}$ (§7). The value groups of D and $D_G(X_i; I)_{\text{loc}}$ are isomorphic.

PROPOSITION 2. Let K be a field, v a valuation of K , $n \in \mathbb{Z}$. Then there is one and only one valuation w of $K_G(X)$ such that for every $f = \sum a_j X^{a_j} \in K_G[X]$,

$$w(f) = \inf (v(a_j) + jn).$$

The proof is as [4] chap. 6, §10, n°1, Lem. 1.

REMARK 3. Let v be a valuation of a field K . Then, there is one and only one valuation w of $K_G(X)$ such that for every $f = \sum a_j X^{a_j} \in K_G[X]$, $w(f) = \inf(v(a_j))$.

THEOREM 4. $D_G[X_i; I]$ is factorial if and only if D is factorial and G is of type $(0, 0, 0, \dots)$.

PROOF. Reduce to [9] Th. 7.13.

THEOREM 5. $D_G[X_i; I]$ is a Krull domain if and only if D is a Krull domain, G is of type $(0, 0, 0, \dots)$.

PROOF. As [16] 3.3. Use Prop. 1, Th. 4.

The divisor class group of a Krull domain D is denoted by $C(D)$.

THEOREM 6. Let D be a Krull domain, G of type $(0, 0, 0, \dots)$. Then, $C(D_G[X_i; I])$ and $C(D)$ are canonically isomorphic.

PROOF. $D_G[X_i; I]$ is a Krull domain by Th. 5. There is the natural map $C(D) \rightarrow C(D_G[X_i; I])$ by §9, Prop. 1. The assertion follows as [16] 5.3 by Th. 4.

If there is a family $\{V_\lambda; A\}$ of valuation overrings of height one of D such that $D = \bigcap V_\lambda$ and for every $x \in K - 0$, x is a unit in almost all V_λ , we say that D is of finite character of real type.

PROPOSITION 7. (1) If $D_G[X]$ is of finite character of real type, then so is D .

(2) If D is of finite character of real type and G is of type $(0, 0, 0, \dots)$, then $D_G[X_i; I]$ is of finite character of real type.

PROOF. (2): Suppose D is of finite character of real type under $\{V_\lambda; A\}$. Let W_λ be the trivial extension of V_λ to $K_G(X_i; I)$, $\{f_\sigma; \Sigma\}$ be a maximal subset of mutually nonassociated irreducibles of $K_G[X_i; I]$ and W'_σ the f_σ -adic valuation of $K_G(X_i; I)$. Then, $\{W_\lambda, W'_\sigma; A, \Sigma\}$ defines the finite character of real type of $D_G[X_i; I]$.

If A has only one maximal ideal, A is said the local ring.

THEOREM 8. Let D be a local Krull domain, G of type $(0, 0, 0, \dots)$. Then $C(D_G(X_i; I)_{\text{loc}})$ and $C(D)$ are canonically isomorphic.

PROOF. $D_G(X_i; I)_{\text{loc}}$ is a Krull domain by Th. 5. We may suppose $I = \text{one point}$. Since $C(D) \cong C(D_G[X])$ by Th. 6, the canonical map $C(D) \rightarrow C(D_G(X)_{\text{loc}})$ is a surjection. Let α be a divisorial ideal of D such that $\alpha D_G(X)_{\text{loc}} = f D_G(X)_{\text{loc}}$ is principal. We will show α is principal. We may suppose $\alpha \neq D$. Set $f = \sum a_\alpha X^\alpha$, $a_\alpha \in \alpha$; may suppose $\alpha(f) = 0$. We have $a_\alpha = fh/g$, $g \in m_{D_G[X]}$; in the maximal ideal of D . If we may take $h \in m_{D_G[X]}$ for some α , we have $\alpha D_G(X)_{\text{loc}} = a_\alpha D_G(X)_{\text{loc}}$. If h belongs to $m_{D_G[X]}$ for every α such that $a_\alpha \neq 0$, we have $A_f \subset m_{A_f}$. Then A_f

vanishes by the Nakayama's lemma; a contradiction. Therefore we may suppose $f \in \mathfrak{a}$; hence $\mathfrak{a} = fD$. Therefore $C(D) \rightarrow C(D_G(X)_{\text{loc}})$ is an injection.

PROPOSITION 9. *Let $\{V_\lambda; A\}$ be a family of valuation overrings of D such that $D = \bigcap V_\lambda$, G of type $(0, 0, 0, \dots)$, V'_λ the trivial extension of V_λ to $K_G(X)$, $\{f_\sigma; \Sigma\}$ a maximal subset of mutually nonassociated irreducibles of $K_G[X]$, W_σ the f_σ -adic valuation ring of $K_G(X)$.*

(1) *Every W_σ is rank one discrete and is essential for $D_G[X]$ and $D_G[X] = \cap \{V'_\lambda, W_\sigma; A, \Sigma\}$.*

(2) *If D is of finite character under $\{V_\lambda; A\}$, then $D_G[X]$ is of finite character under $\{V'_\lambda, W_\sigma; A, \Sigma\}$.*

(3) *If D is a generalized Krull domain, then $D_G[X]$ is a generalized Krull domain.*

We say that a domain D is an almost Dedekind domain if $D_{\mathfrak{m}}$ is a noetherian valuation ring for each maximal ideal $\mathfrak{m}$ of D . Denote the G -powered Kronecker function ring of the b -operation with respect to $\{X_i; I\}$ over an integrally closed domain D by D_b (§8).

THEOREM 10. *Let D be an integrally closed domain which is not a field. Then the following (1), (2), (3) are equivalent:*

- (1) *D is almost Dedekind.*
- (2) *$D_G(X_i; I)_{\text{loc}}$ is almost Dedekind.*
- (3) *D_b is almost Dedekind.*

PROOF. (2) $\rightarrow$ (3): Since $D_G(X_i; I)_{\text{loc}} = D_b$ by §8, Th. 14, D_b is almost Dedekind. (1) $\leftrightarrow$ (3): As we see in the proof of §8, Prop. 9, there is a one-to-one correspondence between the set of valuation overrings of D and the set of valuation overrings of D_b ; this correspondence preserves rank and discreteness. Therefore the assertion follows from [8] (36.2). (1) $\rightarrow$ (2): We have $D_G(X_i; I)_{\text{loc}} = D_b$ by §8, Th. 14. Since (1) implies (3), $D_G(X_i; I)_{\text{loc}}$ is almost Dedekind.

§5. The noetherian properties.

$A_G[X]$ is noetherian if and only if A is noetherian and G is finitely generated ([6] 3, Th. 2). This section concerns with the properties which reduce to the noetherian properties.

We shall say that a ring A is an *RM-ring* if $A/\mathfrak{a}$ satisfies the minimum condition for each nonzero ideal $\mathfrak{a}$ of A .

REMARK 1. $A_G[X]$ is never an *RM-ring*.

PROOF. Suppose that $A_G[X]$ is an *RM-ring*. We see as [8] (40.8) that

$A_G[X]$ satisfies the maximum condition. Therefore $A_G[X]$ is of the form $A_z[X_1, \dots, X_n]$.

$$A_z[X_1, \dots, X_n]/(1 - X_n) \cong A_z[X_1, \dots, X_{n-1}] \cong (A_z[X_1, \dots, X_{n-2}])_z[X_{n-1}].$$

Therefore $(A_z[X_1, \dots, X_{n-2}])_z[X_{n-1}]$ satisfies the minimum condition; a contradiction with [6] 3, Th. 1.

We say that a noetherian ring A is regular if for every prime ideal $\mathfrak{p}$ of A , $A_{\mathfrak{p}}$ is regular [12] (0, 17.3.6).

PROPOSITION 2. $A_G[X]$ is a regular ring if and only if A is regular and G is finitely generated.

PROOF. Assume $A_G[X]$ is a regular ring. At first, G is of the form $Z\alpha_1 \oplus \dots \oplus Z\alpha_n$. Set $X^{\alpha_i} = X_i$; $A_G[X] = A_z[X_1, \dots, X_n]$. To say that A is regular, we may assume $n=1$. Therefore, assume $A_z[X]$ is regular. Let $\mathfrak{p}$ be a prime ideal of A and $B = A_{\mathfrak{p}}$, $\mathfrak{m} = \mathfrak{p}A_{\mathfrak{p}}$.

$$A_{\mathfrak{p}} \subset A_z[X]_{\mathfrak{p}A_z[X]} = B_z[X]_{\mathfrak{p}B_z[X]} = B[X]_{\mathfrak{m}B[X]} = B(X).$$

Since $A_z[X]_{\mathfrak{p}A_z[X]}$ is a regular local domain, B is a noetherian local domain. We have $\dim B(X) = \dim B$ by [18] (9.10). Set $\dim B = m$. Since the vector space $\frac{\mathfrak{m}B(X)}{\mathfrak{m}^2B(X)}$ over a field $\frac{B(X)}{\mathfrak{m}B(X)}$ is generated by $\{x + \mathfrak{m}^2B(X); x \in \mathfrak{m}\}$, there is a basis $\{x_j + \mathfrak{m}^2B(X); x_j \in \mathfrak{m}, 1 \leq j \leq m\}$. We have $\mathfrak{m}B(X) = (x_1, \dots, x_m)B(X)$ by [21] chap. 8, §11, Th. 25. Every $x \in \mathfrak{m}$ is of the form $\sum x_i f_i/g$; hence $x \in (x_1, \dots, x_m)B$; hence $\mathfrak{m} = (x_1, \dots, x_m)B$. Therefore B is regular.

The next is on the Cohen-Macaulay ring [12] (0, 16.5.13) (Macaulay ring of [15]).

PROPOSITION 3. $A_G[X]$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and G is finitely generated.

PROOF. If $A_G[X]$ is Cohen-Macaulay, $A_G[X]$ is of the form $A_z[X_1, \dots, X_n]$. Since $1 - X_n$ is not a zerodivisor and

$$\frac{(A_z[X_1, \dots, X_{n-1}])_z[X_n]}{1 - X_n} \cong A_z[X_1, \dots, X_{n-1}],$$

$A_z[X_1, \dots, X_{n-1}]$ is Cohen-Macaulay. Therefore A is Cohen-Macaulay.

The next is on the universally catenary ring [13] (5.6.2).

PROPOSITION 4. $A_G[X]$ is universally catenary if and only if A is universally catenary and G is finitely generated.

The next is on the excellent ring [13] (7.8.2).

PROPOSITION 5. $A_G[X]$ is excellent if and only if A is excellent and G is

finitely generated.

PROOF. Assume $A_G[X]$ is excellent. By Prop. 4, A is universally catenary. Remark that A is a homomorphic image of $A_G[X]$. For a prime ideal $\mathfrak{p}$ of A , we have $A_G[X]/\mathfrak{p} \cong A/\mathfrak{p}$; hence A is $J-2$.

The next is on the Nagata ring [17] (31.A).

PROPOSITION 6. $A_G[X]$ is a Nagata ring if and only if A is a Nagata ring and G is finitely generated.

The next is on Matsumura's G -ring [17] (39.A).

PROPOSITION 7. $A_G[X]$ is a Matsumura's G -ring if and only if A is a Matsumura's G -ring and G is finitely generated.

The next is on the integrally noetherian domain [4] chap. 5, §3, exerc., 6).

THEOREM 8. $D_G[X]$ is integrally noetherian if and only if D is integrally noetherian and G is finitely generated.

PROOF. Suppose that $D_G[X]$ is integrally noetherian, $t_1, \dots, t_n$ elements of an extension field of K , J the integral closure of $D[t_1, \dots, t_n]$. Then, $J_G[X]$ is the integral closure of $(D_G[X])[t_1, \dots, t_n]$ (§6, Th. 6). We have

$$J_G[X] = \sum_1^m g_j \cdot (D[t_1, \dots, t_n])_G[X]; \quad g_j = \sum b_{jk} X^{\alpha(k)};$$

$$J = \sum b_{kj} \cdot D[t_1, \dots, t_n].$$

Therefore D is integrally noetherian.

§6. The integral elements.

PROPOSITION 1. Let D be an integrally closed domain.

(1) Let $f = \sum a_\alpha X^\alpha$, $g = \sum b_\beta X^\beta \in D_G[X]$, $c \in D$. Then if $A_{fg} \subset cD$, we have $a_\alpha b_\beta \in cD$ for all α, β .

(2) Let $f, g \in K_G[X]$ be monic. If $fg \in D_G[X]$, then $f, g \in D_G[X]$.

(3) Let $f, g \in K_G[X_i; I]$. Then we have

$$\operatorname{div} A_f + \operatorname{div} A_g = \operatorname{div} A_{fg}.$$

(4) Let $f, g \in K_G[X_i; I]$. If A_{fg} is divisorial, we have $A_f A_g = A_{fg}$.

(5) If D is a GCD-domain and $f, g \in K_G[X_i; I]$. Then we have $c(f)c(g) = c(fg)$.

PROOF. (1): We may suppose that D is a valuation ring. Let v be the valuation of D , w the trivial extension of v to $K_G(X)$ and let $w(f) = v(a_{\alpha_0})$.

$$v(a_\alpha b_\beta) = v(a_\alpha) + v(b_\beta) \geq v(a_{\alpha_0}) + v(b_\beta) = v(a_{\alpha_0} b_\beta) \geq w(a_{\alpha_0} g)$$

$$= w(a_{\alpha_0}) + w(g) = w(fg) \geq v(c).$$

Therefore, $a_\alpha b_\beta \in cD$ for all α, β . (2): Take $c=1$ in (1). (3): By (1). (4): We have $\text{div } A_{fg} = \text{div } A_f A_g$ by (3); $A_f A_g \subset A_{fg}$. Therefore $A_f A_g = A_{fg}$. (5): By (3).

THEOREM 2. $D_G[X_i; I]$ is completely integrally closed if and only if D is completely integrally closed.

PROOF. By [16] 4.3.

THEOREM 3. $D_G[X_i; I]$ is integrally closed if and only if D is integrally closed.

PROOF. By [16] 4.4.

THEOREM 4. If D is regularly integrally closed, then $D_G[X]$ is regularly integrally closed.

PROOF. In the notations of [5] § 1, exerc. 30). We define a mapping of $D_f(D_G[X]) \rightarrow D_f(K_G[X]) \times G_f(D)$ as follows:

$$\text{div}((f_1, \dots, f_n)g^{-1}) \mapsto (\text{Div}((f_1, \dots, f_n)g^{-1}), \text{div}'(A_{f_1}, \dots, A_{f_n}) - \text{div}' A_g).$$

By Prop. 1, this becomes certainly a mapping and a homomorphism. Let $\text{div } \mathfrak{A}$ be a divisor on $D_G[X]$ of finite type; may suppose that $\mathfrak{A}$ is divisorial. Assume $\mathfrak{A}(D_G[X]: \mathfrak{A}) \subset D_G[X]\xi^{-1}$, $\xi \in K_G(X)$. If $D_G[X]\zeta \supset \mathfrak{A}$, $\zeta \in K_G(X)$, it follows $\zeta^{-1} \in D_G[X]: \mathfrak{A}$, $\zeta^{-1}\mathfrak{A} \subset \mathfrak{A}(D_G[X]: \mathfrak{A}) \subset D_G[X]\xi^{-1}$, $\xi\mathfrak{A} \subset D_G[X]\zeta$; hence $\xi\mathfrak{A} \subset \mathfrak{A}$; hence $(\xi D_G[X] + D_G[X])\mathfrak{A} = \mathfrak{A}$. Set $\xi = f'/g'$, $f', g' \in D_G[X]$, $\text{div } \mathfrak{A} = \text{div}(f_1, \dots, f_n)g^{-1}$. We have from the above homomorphism:

$$\begin{aligned} & (\text{Div}(f_1, \dots, f_n)g^{-1}, \text{div}'(A_{f_1}, \dots, A_{f_n}) - \text{div}' A_g) \\ &= (\text{Div}(f_1, \dots, f_n)g^{-1}, \text{div}'(A_{f_1}, \dots, A_{f_n}) - \text{div}' A_g) \\ &+ (\text{Div}(\xi, 1), \text{div}'(A_{f'}, A_{g'}) - \text{div}' A_{g'}). \end{aligned}$$

By the comparison of first components, we have $\xi \in K_G[X]$. Therefore we may assume $g' \in D$. By the comparison of second components, we have $\text{div}'(A_{f'}, A_{g'}) = \text{div}' A_{g'}$; hence $A_{f'} \subset g'D$; hence $\xi \in D_G[X]$; hence $\text{div } \mathfrak{A}(D_G[X]: \mathfrak{A}) = \text{div } D_G[X]$; hence $\text{div } \mathfrak{A}$ is invertible; hence $D_G[X]$ is regularly integrally closed.

PROPOSITION 5. Let D be a regularly integrally closed domain.

(1) Let $D_G[X] \ni f, f', g, g'$ and $f/g = f'/g'$. If $\text{div } A_f \geq \text{div } A_g$, then $\text{div } A_{f'} \geq \text{div } A_{g'}$.

(2) Set $B = \{f/g \in K_G(X), \text{div } A_f \geq \text{div } A_g\}$. Then B is a ring and $B \cap K = D$.

(3) B is a bezout domain

(4) If D is a Krull domain, B is a principal ideal domain.

PROOF. (3): Let $\mathcal{B}$ be a finitely generated ideal of B ; $\mathcal{B}=(f_1/g, \dots, f_n/g)$, $f_i, g \in D_G[X]$, $o(f_i), o(g) > 0$. Set $\deg(f_i) = \alpha_i$,

$$f = f_1 + f_2 X^{\alpha_1} + f_3 X^{\alpha_1 + \alpha_2} + \dots + f_n X^{\alpha_1 + \dots + \alpha_{n-1}}.$$

Then, $\operatorname{div} A_{f_i} \geq \operatorname{div} A_f$; $f_i \in fB$; hence $(f_1, \dots, f_n) = fB$; hence $\mathcal{B} = (f/g)$. (4): By (3).

THEOREM 6. Let R be an A -algebra, A' the integral closure of A in R . Then $A'_G[X_i; I]$ is the integral closure of $A_G[X_i; I]$ in $R_G[X_i; I]$.

PROOF. Reduce to [4] chap. 5, § 1, Prop. 12.

THEOREM 7. The integral closure of A in $A_G[X_i; I]$ is the set of elements coefficients of which are nilpotents except those of constant terms. This integral closure coincides with the complete integral closure of A in $A_G[X_i; I]$.

The constant term of $\sum a(X_{i_1}^{e_1} \dots X_{i_n}^{e_n}) X_{i_1}^{e_1} \dots X_{i_n}^{e_n}$ is one for $e_1 = \dots = e_n = 0$. We use § 3, Th. 3 for the proof and reduce to the case of $I = \text{one point}$.

THEOREM 8. Let B be an extension ring of A , A_0 the complete integral closure of A in B . Then $(A_0)_G[X_i; I]$ is the complete integral closure of $A_G[X_i; I]$ in $B_G[X_i; I]$.

For the proof, we reduce to the case of $I = \text{one point}$.

PROPOSITION 9. If a ring B is integral over A , then $B_G(X_i; I)_{\text{loc}}$ is integral over $A_G(X_i; I)_{\text{loc}}$.

PROOF. Reduce to [18] chap. I, 10, exerc. 5.

PROPOSITION 10. Let D be an integrally closed domain, $0 \neq f \in D_G[X_i; I]$, $\alpha = D$; $A_f, \mathfrak{A} = \alpha D_G[X_i; I]$. Then, we have

$$f \cdot K_G[X_i; I] \cap D_G[X_i; I] = f\mathfrak{A}.$$

PROOF. Let $g \in K_G[X_i; I]$. Then, $fg \in D_G[X_i; I] \Leftrightarrow A_{fg} \subset D \Leftrightarrow 0 \leq \operatorname{div} A_{fg} = \operatorname{div} A_f + \operatorname{div} A_g$ by Prop. 1 $\Leftrightarrow D \supset A_f A_g \Leftrightarrow A_g \subset \alpha \Leftrightarrow g \in \mathfrak{A}$.

We say that a domain D is $N-1$ if the integral closure of D is a finite D -module.

PROPOSITION 11. $D_G[X_i; I]$ is $N-1$ if and only if D is $N-1$.

PROOF. Let J be the integral closure of D . If $D_G[X_i; I]$ is $N-1$, we have $J_G[X_i; I] = \sum_1^n D_G[X_i; I] \cdot f_j$ by Th. 6. Then it follows $J = A_{f_1} + \dots + A_{f_n}$. Therefore D is $N-1$.

PROPOSITION 12. Let D be an integrally closed local domain, $f, g \in D_G[X]$. If A_{fg} is principal, then A_f, A_g are principal.

PROOF. Set $A_{fg} = cD$, $f = \sum a_i X^{a_i}$, $g = \sum b_j X^{a_j}$. We have $A_f A_g = cD$ by Prop. 1; $c = \sum x_{ij} a_i b_j$, $a_i b_j = c y_{ij}$, where $x_{ij}, y_{ij} \in D$; $1 = \sum x_{ij} y_{ij}$; hence some y_{ij} is a unit. For any i' , we have

$$a_{i'} b_j = c x_{i'j} = a_i b_j y_{ij}^{-1} x_{i'j}; \quad a_{i'} = a_i y_{ij}^{-1} x_{i'j} \in (a_i); \quad \text{hence } A_f = (a_i).$$

§7. The rational function rings.

PROPOSITION 1. The subset $S = \{f; A_f = A\}$ of $A_G[X_i; I]$ is multiplicatively closed. The quotient ring $S^{-1} \cdot A_G[X_i; I] = A_G(X_i; I)_{\text{loc}}$ is a faithfully flat A -module.

PROOF. Suppose that $A_{fg} \neq A$ for $f, g \in S$. Let $\mathfrak{m}$ be a maximal ideal containing A_{fg} , ψ the canonical map of $A_G[X_i; I]$ to $(A/\mathfrak{m})_G[X_i; I]$. Since $0 = \psi(fg) = \psi(f)\psi(g)$, we have $\psi(f) = 0$ or $\psi(g) = 0$. This contradicts with $A_f = A_g = A$. The latter follows from §9, Prop. 1.

PROPOSITION 2. Let $\mathfrak{a}$ be an ideal of A , then $\mathfrak{a} \cdot A_G(X_i; I)_{\text{loc}} \cap A = \mathfrak{a}$. If $\mathfrak{q}$ is $\mathfrak{p}$ -primary in A , then $\mathfrak{q} \cdot A_G(X_i; I)_{\text{loc}}$ is $\mathfrak{p} \cdot A_G(X_i; I)_{\text{loc}}$ -primary.

PROOF. By §9, Prop. 7, $\mathfrak{q} \cdot A_G[X_i; I]$ is $\mathfrak{p} \cdot A_G[X_i; I]$ -primary. Taking quotients, we see that $\mathfrak{q} \cdot A_G(X_i; I)_{\text{loc}}$ is $\mathfrak{p} \cdot A_G(X_i; I)_{\text{loc}}$ -primary.

THEOREM 3. Let B be an extension ring of A , A_0 the integral closure of A in B . Then, $(A_0)_G(X_i; I)_{\text{loc}}$ is the integral closure of $A_G(X_i; I)_{\text{loc}}$ in $B_G(X_i; I)_{\text{loc}}$.

PROOF. By §6, Prop. 9, $(A_0)_G(X_i; I)_{\text{loc}}$ is integral over $A_G(X_i; I)_{\text{loc}}$. Let $\xi/\zeta \in B_G(X_i; I)_{\text{loc}}$ be integral over $(A_0)_G(X_i; I)_{\text{loc}}$. We will show that $\xi/\zeta \in (A_0)_G(X_i; I)_{\text{loc}}$. We have

$$(\xi/\zeta)^n + f_{n-1}/g_{n-1}(\xi/\zeta)^{n-1} + \dots + f_0/g_0 = 0, \quad f_j/g_j \in (A_0)_G(X_i; I)_{\text{loc}}.$$

We assume $I = \text{one point}$. Let H be the subgroup of G generated by the powers appearing in $\xi, \zeta, f_{n-1}, g_{n-1}, \dots, f_0, g_0$; $H = \langle X_1 \oplus \dots \oplus X_m \rangle$. Set $X^{a_i} = X_i$. $A_0(X_1, \dots, X_m)_{\text{loc}}$ is integrally closed in $B(X_1, \dots, X_m)_{\text{loc}}$ ([8] V, 33, exerc, 2)). Therefore the assertion follows.

THEOREM 4. If D is an integrally closed domain, then $D_G(X_i; I)_{\text{loc}}$ is an integrally closed domain.

PROOF. $D_G[X_i; I]$ is integrally closed by §6, Th. 3. Therefore, so is its quotient ring $D_G(X_i; I)_{\text{loc}}$.

PROPOSITION 5. (1) Let $\mathfrak{a}$ be an ideal of A . Then we have

$$A_G(X_i; I)_{\text{loc}} / \mathfrak{a} \cdot A_G(X_i; I)_{\text{loc}} \cong (A/\mathfrak{a})_G(X_i; I)_{\text{loc}}.$$

(2) Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be ideals of A . Then we have

$$(\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n)A_G(X_i; I)_{\text{loc}} = \mathfrak{a}_1 A_G(X_i; I)_{\text{loc}} \cap \dots \cap \mathfrak{a}_n A_G(X_i; I)_{\text{loc}}.$$

PROOF. (2): Reduce to [18] (6.17).

PROPOSITION 6. Let $\mathfrak{m}'$ be an ideal of $A_G(X_i; I)_{\text{loc}}$. Then, $\mathfrak{m}'$ is a maximal ideal if and only if $\mathfrak{m}' = \mathfrak{m} \cdot A_G(X_i; I)_{\text{loc}}$ for a maximal ideal $\mathfrak{m}$ of A .

PROOF. Let $\mathfrak{m}$ be a maximal ideal of A . Since $(A/\mathfrak{m})_G(X_i; I)_{\text{loc}}$ is a field, $\mathfrak{m}A_G(X_i; I)_{\text{loc}}$ is a maximal ideal of $A_G(X_i; I)_{\text{loc}}$ by Prop. 5. Conversely, let $\mathfrak{m}'$ be a maximal ideal of $A_G(X_i; I)_{\text{loc}}$. Then the set $\{\text{coefficients of } f; f \in \mathfrak{m}' \cap A_G[X_i; I]\} = \mathfrak{m}$ is an ideal of A ; $\mathfrak{m} \neq A$; $\mathfrak{m}' \subset \mathfrak{m} \cdot A_G(X_i; I)_{\text{loc}} \cong A_G(X_i; I)_{\text{loc}}$; hence $\mathfrak{m}' = \mathfrak{m} \cdot A_G(X_i; I)_{\text{loc}}$; hence $\mathfrak{m}$ is maximal by Prop. 2.

We see by Prop. 6 that, if A is local, then $A_G(X_i; I)_{\text{loc}}$ is local; if A is semi-local, $A_G(X_i; I)_{\text{loc}}$ is semi-local.

PROPOSITION 7. (1) Let D be a domain, $f, g \in D_G[X_i; I]$ such that $A_f \subset A_g$. If A_g is invertible, then $f/g \in D_G(X_i; I)_{\text{loc}}$.

(2) If D is prüfer, then $D_G(X_i; I)_{\text{loc}}$ is bezout.

PROOF. (1): Since $A_g(D: A_g) = D$, $D: A_g$ is finitely generated, hence $A_h = D: A_g$ for $h \in K_G[X_i; I]$. We have

$$A_{gh} = A_g A_h = D$$

by § 9, Th. 8; hence $f/g = (fh)/(gh) \in D_G(X_i; I)_{\text{loc}}$. (2): Let $f, g \in D_G[X_i; I]$. Take a large $\alpha \in G$ and fix $i \in I$. Then, $A_f, A_g \subset A_{f+g} X_i^\alpha$. By (1), $\frac{f}{f+gX_i^\alpha}, \frac{g}{f+gX_i^\alpha} \in D_G(X_i; I)_{\text{loc}}$; hence $(f, g)D_G(X_i; I)_{\text{loc}} = (f+gX_i^\alpha)D_G(X_i; I)_{\text{loc}}$.

THEOREM 8. $A_G(X_i; I)_{\text{loc}}$ dominates ([18] 5) A .

PROOF. By Prop. 6 and Prop. 2.

A ring A is said a Kaplansky's valuation ring if for any a and b in A , either a divides b or b divides a .

PROPOSITION 9. If A is a Kaplansky's valuation ring, then so is $A_G(X_i; I)_{\text{loc}}$.

PROOF. Let $f = a_0 X^{\alpha_0} + a_1 X^{\alpha_1} + \dots + a_n X^{\alpha_n} \in A_G[X_i; I]$, $a_j \neq 0$, $f \neq 0$. There is k such that $A_f \subset a_k A$. Therefore f is associated with a_k in $A_G(X_i; I)_{\text{loc}}$. Therefore for any $\xi, \eta \in A_G(X_i; I)_{\text{loc}}$, we have $\xi|\eta$ or $\eta|\xi$ in $A_G(X_i; I)_{\text{loc}}$.

LEMMA 10. Let $\{\mathfrak{m}_\lambda; A\}$ be the set of maximal ideals of A , $\mathfrak{m}'_\lambda$ the ideal of $A_G[X_i; I]$ generated by $\mathfrak{m}_\lambda$, $S = \{f \in A_G[X_i; I]; A_f = A\}$. Then

(1) Each $\mathfrak{m}'_\lambda$ is a proper prime ideal of $A_G[X_i; I]$.

- (2) If $\lambda_1 \neq \lambda_2$, it follows $m'_{\lambda_1} \not\subset m'_{\lambda_2}$.
 (3) Every ideal $\mathfrak{A}$ of $A_G[X_i; I]$ contained in the set-theoretical union $\bigcup_A m'_\lambda$ is contained in some m'_λ .
 (4) $S = A_G[X_i; I] - \bigcup_A m'_\lambda$.

PROOF. (3): The set $\{ \text{coefficients of } f; f \in \mathfrak{A} \} = \mathfrak{a}$ is an ideal of A . Since $\mathfrak{A} \subset \bigcup_A m'_\lambda$, it follows $1 \notin \mathfrak{a}$; hence $\mathfrak{a} \subset m_\lambda$ for some λ ; hence $\mathfrak{A} \subset m'_\lambda$. (4): Let $f \in A_G[X_i; I]$. Then, $A_f = A \Leftrightarrow A_f \not\subset m_\lambda$ for each $\lambda \Leftrightarrow f \notin m'_\lambda$ for each λ ; hence $S = A_G[X_i; I] - \bigcup_A m'_\lambda$.

PROPOSITION 11. Let $\{m_\lambda; A\}$ be the set of maximal ideals of a domain D . Then we have

$D_G(X_i; I)_{\text{loc}} = \bigcap_A (D_{m_\lambda})_G(X_i; I)_{\text{loc}}$. If D' is the integral closure of D , then $D_G(X_i; I)_{\text{loc}} \subset D'_b$ (cf. § 8).

PROOF. We have by Lem. 10 and [8] (5.4)

$$D_G(X_i; I)_{\text{loc}} = \bigcap_A D_G[X_i; I]_{m_\lambda}.$$

§8. The Kronecker function rings.

We develop in this section the theory [8] V of the Kronecker function rings in our $A_G[X_i; I]$. Almost all notations are of [8].

LEMMA 1. Let $*$ be an e.a.b. $*$ -operation on a domain D . Then, for $f, g \in D_G[X_i; I] - 0$, we have $(A_{fg})^* = (A_f A_g)^*$.

PROOF. There is m such that $A_f^{m+1} A_g = A_f^m A_{fg}$ by § 9, Th. 8; hence $(A_f^{m+1} A_g)^* = (A_f^m A_{fg})^*$; hence $(A_f A_g)^* = (A_{fg})^*$ by e.a.b..

THEOREM 2. Let $*$ be an e.a.b. $*$ -operation on a domain D . Set

$$D_* = \{0\} \cup \{f/g; f, g \in D_G[X_i; I], A_f^* \subset A_g^*\}.$$

Then,

- (1) D_* is a domain. The quotient field of D_* is $K_G(X_i; I)$, and $D_* \cap K = D$.
 (2) D_* is a bezout domain.
 (3) If $\mathfrak{a}$ is a finitely generated nonzero fractional ideal of D , then $\mathfrak{a} D_* \cap K = \mathfrak{a}^*$.

PROOF. (1): Similar with the proof of [8] (32.7), (a). Use Lem. 1. (2): We will show that $\xi D_* + \eta D_*$ is principal for $\xi, \eta \in D_* - 0$. Set $\xi = f/h, \eta = g/h; f, g, h \in D_G[X_i; I] - 0$ and fix $i \in I$. Let $\alpha \in G$ be greater than any power appearing in f, g, h and set $\zeta = \xi + \eta X_i^\alpha$. We have $(\zeta) \subset (\xi, \eta)$; $\xi/\zeta = f/(f + g X_i^\alpha) \in D_*$; $\xi \in (\zeta)$; Similarly $\eta \in (\zeta)$; hence $(\xi, \eta) = (\zeta)$. (3): Let $\mathfrak{a} = (a_0, a_1, \dots, a_n)$ and fix $i \in I, 0 < \alpha$

$\in G$. By the proof of (2), $\alpha D_* = (a_0 + a_1 X_i^\alpha + a_2 X_i^{2\alpha} + \cdots + a_n X_i^{n\alpha}) D_*$. Let $d \in K - 0$. $d \in \alpha D_* \Leftrightarrow d/(a_0 + a_1 X_i^\alpha + \cdots + a_n X_i^{n\alpha}) \in D_* \Leftrightarrow (d) = (d)^* \subset (a_0, a_1, \dots, a_n)^* = \alpha^*$. From this follows our assertion.

PROPOSITION 3. *Let $*_1, *_2$ be two e.a.b. $*$ -operations on D . Then, $*_1, *_2$ are equivalent if and only if $D_{*1} = D_{*2}$.*

PROOF. Suppose that $*_1, *_2$ are equivalent. Let $f, g \in D_G[X_i; I] - 0$. $A_{f^*}^1 \subset A_{g^*}^1 \Leftrightarrow A_{f^*}^2 \subset A_{g^*}^2$. Hence $D_{*1} = D_{*2}$. Conversely assume $D_{*1} = D_{*2}$. Let α be a finitely generated nonzero fractional ideal of D . By Th. 2, (3), we have

$$\alpha^{*1} = \alpha D_{*1} \cap K = \alpha D_{*2} \cap K = \alpha^{*2}.$$

PROPOSITION 4. *Let $*$ be an e.a.b. $*$ -operation on D , V^* a valuation overring of D^* . Then, V^* is the trivial extension of $V^* \cap K$ to $K_G(X_i; I)$.*

PROOF. Let v^*, v be the valuations of $K_G(X_i; I)$, K of V^* , $V^* \cap K$ respectively and let

$$f = \sum a(X_{i_1}^{\epsilon_1} \cdots X_{i_n}^{\epsilon_n}) X_{i_1}^{\epsilon_1} \cdots X_{i_n}^{\epsilon_n} \in K_G[X_i; I] - 0.$$

Since $X_i^\alpha, X_i^{-\alpha} \in D_* \subset V^*$ for every $\alpha \in G, i \in I$, it follows $v^*(X_i^\alpha) = 0$; hence $v^*(f) \geq \inf v(a(X_{i_1}^{\epsilon_1} \cdots X_{i_n}^{\epsilon_n}))$; $a(X_{i_1}^{\epsilon_1} \cdots X_{i_n}^{\epsilon_n})/f \in D_*$; $v(a(X_{i_1}^{\epsilon_1} \cdots X_{i_n}^{\epsilon_n})) \geq v^*(f)$; hence $v^*(f) = \inf v(a(X_{i_1}^{\epsilon_1} \cdots X_{i_n}^{\epsilon_n}))$.

THEOREM 5. *Let $\{V_\lambda; A\}$ be a family of valuation overrings of a domain D such that $D = \bigcap V_\lambda$, $*$ the w -operation on D induced by $\{V_\lambda; A\}$, V_λ^* the trivial extension of V_λ to $K_G(X_i; I)$. Then, we have $D_* = \bigcap_A V_\lambda^*$.*

PROOF. Similar with [8] (32.11).

Let D be an integrally closed domain, $*$ the b -operation on D . Then D_* of Th. 2 is denoted by D_b and called the Kronecker function ring of D .

PROPOSITION 6. *Let D be an integrally closed domain, D_b the Kronecker function ring of D , $*$ an e.a.b. $*$ -operation on D . Then we have $D_b \subset D_*$.*

PROOF. Let $\{V_\lambda; A\}$ be the family of all valuation overrings of D , V_λ^* the trivial extension of V_λ to $K_G(X_i; I)$. By Th. 5, $D_b = \bigcap_A V_\lambda^*$. $*$ is equivalent to some w -operation $*'$. Assume $*'$ is defined by a family $\{V_{\lambda\sigma}; \Sigma\}$ of valuation overrings of D . By Th. 5, $D_{*'} = \bigcap_\Sigma V_{\lambda\sigma}^*$. By Prop. 3, $D_* = D_{*}$. Hence $D_* \supset D_b$.

PROPOSITION 7. *Let D be an integrally closed domain, J an integrally closed overdomain of D , $*$ an e.a.b. $*$ -operation on J . Then, $J_* \supset D_b$, $J_* \cap K = J$.*

PROOF. By Th. 5, $D_b \subset J_b$. By Prop. 6, $J_b \subset J_*$. Hence $D_b \subset J_*$. By Th. 2, (1), $J_* \cap K = J$.

PROPOSITION 8. *Let D be an integrally closed domain, R an overring of D_b . Then, $R = (R \cap K)_b$.*

PROOF. D_b is a bezout domain by Th. 2, (2); hence R is integrally closed. Let $\{W_\lambda; A\}$ be the family of valuation overrings of R . Then, $R = \bigcap_A W_\lambda$. W_λ is the trivial extension of $V_\lambda = W_\lambda \cap K$ to $K_G(X_i; I)$ by Prop. 4. We have $R \cap K = \bigcap_A V_\lambda$. By Th. 5,

$$(R \cap K)_b = \bigcap_A W_\lambda = R.$$

PROPOSITION 9. *Let D be an integrally closed domain. Then, $\dim_v D = \dim D_b$.*

PROOF. Since D_b is prüfer by Th. 2, (2), it follows $\dim_v D_b = \dim D_b$. Let J be a valuation overring of D . Then, J_b is the trivially extended valuation overring of J to $K_G(X_i; I)$ by Th. 5. By Prop. 4, the mapping $J \mapsto J_b$ is a bijection from the set of valuation overrings of D onto that of D_b ; the corresponding two value groups are isomorphic, hence have the same rank; hence $\dim_v D = \dim_v D_b$; hence $\dim_v D = \dim D_b$.

THEOREM 10. *Let D be a prüfer domain, $\{D_\lambda; A\}$ the set of overrings of D . Then the mapping $D_\lambda \mapsto (D_\lambda)_b$ is a bijection onto the set of overrings of D_b .*

PROOF. By Prop. 7, this is a well-defined mapping and an injection. By Prop. 8, this is a surjection.

PROPOSITION 11. *Let $*$ be an e.a.b. $*$ -operation on D . Then, $*$ is of finite type if and only if $aD_* \cap K = a^*$ for every nonzero fractional ideal a of D .*

PROOF. Suppose that $*$ is of finite type. Let a be a nonzero fractional ideal of D and $x \in aD_* \cap K$. x is of the form $a_1 F_1 + \dots + a_n F_n$, $a_i \in a$, $F_i \in D_*$. By Th. 2, (3), it follows $(a_1, \dots, a_n)D_* \cap K = (a_1, \dots, a_n)^*$; hence $aD_* \cap K \subset a^*$. Conversely, let $x \in a^*$, then there are $a_1, \dots, a_n \in a$ such that $x \in (a_1, \dots, a_n)^*$. By Th. 2, (3), $(a_1, \dots, a_n)^* = (a_1, \dots, a_n)D_* \cap K$; hence $a^* \subset aD_* \cap K$. Next, the sufficiency: Let $x \in a^*$. Since $x \in aD_* \cap K$, it follows

$$\begin{aligned} x &= (a_1 f_1 + \dots + a_n f_n)/g, \quad a_i \in a, \quad f_i/g \in D_*; \\ xA_g &= a_1 A_{f_1} + \dots + a_n A_{f_n}; \quad (xA_g^*)^* = (a_1 A_{f_1}^* + \dots + a_n A_{f_n}^*)^* \\ &\subset (a_1 A_g^* + \dots + a_n A_g^*)^* = ((a_1, \dots, a_n)^* A_g^*)^*. \end{aligned}$$

By the e.a.b., it follows $(x) \subset (a_1, \dots, a_n)^*$. Therefore $*$ is of finite type.

PROPOSITION 12. *Assume that D admits an S -representation $D = \bigcap_A V_\lambda$, $*$ the w -operation induced by this representation, V_λ^* the trivial extension of V_λ to $K_G(X_i; I)$. Then $\bigcap_A V_\lambda^*$ is an S -representation of D_* .*

PROOF. By Th. 5, it follows $D_* = \bigcap_{\lambda} V_{\lambda}^*$. If $V_{\mu}^* \supset \bigcap_{\lambda \neq \mu} V_{\lambda}^*$ for some μ , it follows $V_{\mu} \supset \bigcap_{\lambda \neq \mu} V_{\lambda}$. Therefore we get the assertion.

PROPOSITION 13. Let $*_1, *_2$ be two e.a.b. $*$ -operations on D . Then $\alpha^{*1} \subset \alpha^{*2}$ for each nonzero finitely generated ideal α of D if and only if $D_{*1} \subset D_{*2}$.

PROOF. The necessity: Let $f/g \in D_{*1}$. Since $A_f \subset A_f^{*1} \subset A_f^{*2}$, we have $A_f^{*2} \subset A_f^{*1*2} \subset A_f^{*2}$; hence $A_f^{*1*2} = A_f^{*2}$; similarly, $A_g^{*1*2} = A_g^{*2}$. Since $A_f^{*1} \subset A_g^{*1}$, we have $A_f^{*2} = A_f^{*1*2} \subset A_g^{*1*2} = A_g^{*2}$; hence $f/g \in D_{*2}$; hence $D_{*1} \subset D_{*2}$. The sufficiency: Fix $G \in \alpha > 0, i \in I$. Let $X_i^* = X$. Let E^{*1}, E^{*2} be the usual Kronecker function rings of D with respect to X and $*_1, *_2$ respectively. Then we have $E^{*1} \subset E^{*2}$. Hence by [8] p. 409, 13, we have $\alpha^{*1} \subset \alpha^{*2}$ for each nonzero finitely generated ideal α of D .

THEOREM 14. Let D be an integrally closed domain. The followings are equivalent:

- (1) D is a Prüfer domain.
- (2) $D_G(X_i; I)_{\text{loc}} = D_b$.
- (3) $D_G(X_i; I)_{\text{loc}}$ is a prüfer domain.
- (4) D_b is a quotient ring of $D_G(X_i; I)_{\text{loc}}$.

PROOF. (2) $\rightarrow$ (3): Let $f, g \in D_G[X_i; I] - 0$. We will show that $(f, g)D_b$ is principal. Fix $i \in I$ and take a large $\alpha \in G$ and set $h = f + gX_i^{\alpha}$. Then, $(f, g)D_b = hD_b$. (2) $\rightarrow$ (4): Apparent. (1) $\rightarrow$ (2): Let $D_b \ni \xi = f/g \neq 0; A_f \subset A_g$. Since A_g is finitely generated and invertible, A_g^{-1} is also finitely generated; $A_g^{-1} = (u_0, u_1, \dots, u_n)D$. Set $\eta = u_0 + u_1X_i^{\alpha} + \dots + u_nX_i^{n\alpha}$ for $\alpha > 0$. By § 2, Th. 11, it follows $A_{g\eta} = A_gA_{\eta}, A_{f\eta} = A_fA_{\eta}$; hence $A_{g\eta} = D, A_{f\eta} \subset A_gA_{\eta} = D$; hence $\xi = (f\eta)/(g\eta) \in D_G(X_i; I)_{\text{loc}}$. (4) $\rightarrow$ (2): We have $D_b = D_G[X_i; I]_N, N = \{f \in D_G[X_i; I]; 1/f \in D_b\}$. If $1/f \in D_b$, it follows $A_f = D$; hence $D_b \subset D_G(X_i; I)_{\text{loc}}$. (3) $\rightarrow$ (1): Let $\mathfrak{m}$ be a maximal ideal of D . $\mathfrak{m}D_G(X_i; I)_{\text{loc}}$ is a maximal ideal of $D_G(X_i; I)_{\text{loc}}$ by § 7, Prop. 6. Therefore, $D_G[X_i; I]_{\mathfrak{m}D_G(X_i; I)} = D_G(X_i; I)_{\text{loc} \cap \mathfrak{m}D_G(X_i; I)_{\text{loc}}}$ is a valuation ring; hence $D_{\mathfrak{m}} = K \cap D_G[X_i; I]_{\mathfrak{m}D_G(X_i; I)}$ is a valuation ring. Hence D is a prüfer domain.

PROPOSITION 15. Let $*$ be an e.a.b. $*$ -operation on D . Suppose that the v -operation on D is e.a.b.. Then, $D_* \subset D^v$.

The G -powered Kronecker function ring of the v -operation is D^v .

THEOREM 16. (1) Let D be a prüfer v -multiplication ring, H the group of divisor classes of D of finite type. Then H is order isomorphic to the group of divisibility of D^v .

(2) Let $*$ be a $*$ -operation such that, the set H of $*$ -ideals of D of finite type is a group with respect to the $*$ -multiplication. Then H is order isomorphic to the group of divisibility of D_* .

PROOF. D^v is a bezout domain by Th. 2, (2). Therefore the group of divisibility of D^v is order isomorphic to the group S of non-zero finitely generated fractional ideals of D^v . Each element of H is of the form $\text{div } a$ for some finitely generated fractional ideal a of D . We define $\phi: H \rightarrow S$ by $\phi(\text{div } a) = aD^v$. By Th. 2, (3), it follows $a_v D^v = aD^v$; hence ϕ is certainly a mapping. If $\phi(\text{div } a) = D^v$, then

$$a_v = aD^v \cap K = D^v \cap K = D$$

by Th. 2, (1). Therefore ϕ is injective.

PROPOSITION 17. *Let D be a v -domain. The followings are equivalent:*

- (1) D is a prüfer v -multiplication ring.
- (2) D^v is a quotient ring of $D_G[X_i; I]$.

PROOF. (1)→(2): Set $N = \{g \in D_G[X_i; I]; 1/g \in D^v\}$. Let $D^v \ni \xi = f/g; (A_f)_v \subset (A_g)_v$. We have $(A_g a)_v = D$ for some $a = u_0 D + \dots + u_n D$. Fix $i \in I$, $0 < \alpha \in G$ and set $\eta = u_0 + u_1 X_i^\alpha + \dots + u_n X_i^{n\alpha}$. By § 6, Prop. 1,

$$(A_{g\eta})_v = (A_g A_\eta)_v = (A_g a)_v = D;$$

hence $g\eta \in N$. Since

$$(A_{f\eta})_v = (A_f A_\eta)_v \subset (A_g A_\eta)_v = D,$$

it follows $f\eta \in D_G[X_i; I]$; hence $\xi = (f\eta)/(g\eta) \in D_G[X_i; I]_N$. (2)→(1): We have $D^v = D_G[X_i; I]_N$. Let a be a nonzero finitely generated fractional ideal of D ; $a = x_0 D + \dots + x_n D$, $0 \neq x_j \in K$; $x_j = a_j/a$, $a_j, a \in D$; set $\eta = a_0 + a_1 X_i^\alpha + \dots + a_n X_i^{n\alpha}$. We have $a_0/\eta = f/g$, $g \in N$.

$$\text{div } a + \text{div} \left(\frac{a}{a_0} A_f \right) = \text{div } \eta + \text{div } A_f - \text{div } a_0 = \text{div } A_g = D.$$

§9. The other properties.

In this section we state some Theorems, Propositions and Remarks of properties that were not treated in previous sections; namely the flatness, the Galois, the homogeneous, the zerodivisor, the primary, the content.

PROPOSITION 1. $A_G[X_i; I]$ and $A_G(X_i; I)_{\text{loc}}$ are faithfully flat A -modules.

The proof follows from [3] chap. 1, § 3, Example 2) and [11] (0, 6.4.1), (0, 6.2.1).

THEOREM 2. *Let K, L be fields. If L is a Galois extension of K of degree n , then $L_G(X_i; I)$ is a Galois extension of $K_G(X_i; I)$ of degree n . Conversely, if $L_G(X_i; I)$ is a Galois extension of $K_G(X_i; I)$ of degree n , then L is a Galois ex-*

tension of K of degree n . In this case, the Galois groups of L/K and $L_G(X_i; I)/K_G(X_i; I)$ are canonically isomorphic.

PROOF. By [16] 6.2, 6.3.

PROPOSITION 3. Let $\mathfrak{a}$ be a fractional ideal of a domain D . Then, $(\mathfrak{a} \cdot D_G[X_i; I])_v \cap K = \mathfrak{a}_v$.

We take the " v " in respective rings.

PROOF. Suppose that $\mathfrak{a}$ is a nonzero ideal of D and let $\mathfrak{a}D_G[X_i; I] = \mathfrak{A}$, $\mathfrak{a} \in \mathfrak{a}_v$. Let $\mathfrak{A} \subset \xi \cdot D_G[X_i; I]$, $\xi \in D_G(X_i; I)$. Since $\mathfrak{a}/\xi \subset D_G[X_i; I]$, $\xi = b/f(X)$, $f(X) \in D_G[X_i; I]$, $b \in D$. Set $f = \sum a(X_{i_1}^{e_1} \cdots X_{i_n}^{e_n}) X_{i_1}^{e_1} \cdots X_{i_n}^{e_n}$; $\frac{a}{b} a(X_{i_1}^{e_1} \cdots X_{i_n}^{e_n}) \subset D$;

$$\mathfrak{a} \subset \frac{b}{a(X_{i_1}^{e_1} \cdots X_{i_n}^{e_n})} D; \text{ hence } \mathfrak{a} \in \frac{b}{a(X_{i_1}^{e_1} \cdots X_{i_n}^{e_n})} D;$$

$$\mathfrak{a}/\xi = \sum a \frac{a(X_{i_1}^{e_1} \cdots X_{i_n}^{e_n})}{b} X_{i_1}^{e_1} \cdots X_{i_n}^{e_n} \in D_G[X_i; I]; \mathfrak{a} \in \xi D_G[X_i; I];$$

hence $\mathfrak{a} \in \mathfrak{A}_v$; hence $\mathfrak{a}_v \subset \mathfrak{A}_v \cap D$. Conversely, let $\mathfrak{a} \in \mathfrak{A}_v \cap D$, $\mathfrak{a} \subset xD$, $x \in K$. Since $\mathfrak{a}D_G[X_i; I] \subset xD_G[X_i; I]$, $\mathfrak{a} \in xD_G[X_i; I]$; hence $\mathfrak{a} \in xD$; hence $\mathfrak{a} \in \mathfrak{a}_v$; hence $\mathfrak{A}_v \cap D \subset \mathfrak{a}_v$; hence $\mathfrak{A}_v \cap D = \mathfrak{a}_v$.

We may define the homogeneous elements of $A_G[X_i; I]$ in natural way.

PROPOSITION 4. Let D be a domain, f a nonzero homogeneous element of $D_G[X_i; I]$. Then, any divisor of f is also homogeneous.

PROPOSITION 5. Let f be a zerodivisor of $A_G[X_i; I]$. Then there is a nonzero $a \in A$ such that $af=0$.

PROOF. Reduce to [1] chap. 4, §1, exerc. 11).

A ring whose zerodivisors are all nilpotents is said a primary ring ([20] chap. 12, §90). By Prop. 5, $A_G[X_i; I]$ is primary if and only if A is primary.

PROPOSITION 6. Let F be an extension field of a field K such that K is algebraically closed in F . Then, $K_G(X)$ is algebraically closed in $F_G(X)$.

PROOF. Let $\xi \in F_G(X)$ be algebraic over $K_G(X)$. There is $0 \neq f \in K_G[X]$ such that $f\xi$ is integral over $K_G[X]$. Since $F_G[X]$ is integrally closed (§6, Th. 3), it follows $f\xi = g \in F_G[X]$. By §6, Th. 6, $g \in K_G[X]$; hence $\xi = g/f \in K_G(X)$.

PROPOSITION 7. Let $\mathfrak{q}$ be a $\mathfrak{p}$ -primary ideal of A . Then, $\mathfrak{q}A_G[X_i; I]$ is $\mathfrak{p}A_G[X_i; I]$ -primary and $\mathfrak{q}A_G[X_i; I] \cap A = \mathfrak{q}$.

PROOF. We will suppose that $I = \text{one point}$. Suppose that $f = \sum a_\alpha X^\alpha \in \mathfrak{q}A_G[X]$, $g = \sum b_\beta X^\beta \in \mathfrak{p}A_G[X]$ and $fg \in \mathfrak{q}A_G[X]$. Let $\min\{\alpha'; a_{\alpha'} \in \mathfrak{q}\} = \alpha$, $\min\{\beta'; b_{\beta'} \in \mathfrak{p}\} = \beta$. Seeing the coefficient of $X^{\alpha+\beta}$ in fg , we have $p + a_\alpha b_\beta \in \mathfrak{q}$ for $p \in \mathfrak{p}$. We get the contradiction of $b_\beta \in \mathfrak{p}$ from Prop. 5.

THEOREM 8. *Let B be an extension ring of A ; $f, g \in B_G[X_i; I]$; A_f, A_g the A -modules generated by the coefficients of f, g respectively. Then we have $A_f^{k+1}A_g = A_f^k A_{fg}$ for some k .*

PROOF. By [8] (28.3).

We say a ring A a chained ring if the set of ideals of A is linearly ordered by the inclusion.

REMARK 9. $A_G[X]$ is never a chained ring.

PROOF. If $A_G[X]$ is a chained ring, A becomes a domain. Hence $A_G[X]$ is a valuation ring; this is impossible.

PROPOSITION 10. *Let $\mathfrak{m}$ be a maximal ideal of $A_G[X_i; I]$. Then $\mathfrak{m}$ cannot consists entirely of zerodivisors.*

PROOF. Suppose $\mathfrak{m}$ consists entirely of zerodivisors. Fix $i \in I$, $0 < \alpha \in G$. Since $1 - X_i^\alpha$ is not a zerodivisor, $1 - X_i^\alpha \notin \mathfrak{m}$; hence $1 = (1 - X_i^\alpha)f + g$ for $g \in \mathfrak{m}$. By Prop. 5, there is $0 \neq a \in A$ such that $a = (1 - X_i^\alpha)h$. But this is obviously impossible.

A ring A is said subdirectly irreducible if the intersection of all nonzero ideals of A is not zero ([14] chap. 10, § 2).

REMARK 11. $A_G[X]$ is never subdirectly irreducible.

PROOF. Suppose that $A_G[X]$ is subdirectly irreducible. Take nonzero f which is of order > 0 and is contained in every nonzero ideal. Set $f = a_1 X^{\alpha_1} + \dots + a_n X^{\alpha_n}$, $0 < \alpha_1 < \dots < \alpha_n$, $a_i \neq 0$. Take $G \in \alpha > \alpha_n$. We have $f \in (1 + X^\alpha)A_G[X]$; $f = (1 + X^\alpha)(b_1 X^{\beta_1} + \dots + b_m X^{\beta_m})$, $\beta_1 < \dots < \beta_m$, $b_j \neq 0$. Then $\beta_1 > 0$; $\alpha_n = \alpha + \beta_m > \alpha$; a contradiction.

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