Torsion-Free Abelian Group Rings III

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In this paper a ring is a commutative ring with identity. Let X be an indeterminate, G a non-zero torsion-free abelian group (additively written). We consider the group ring $A_G[X]$ of G over a ring A . $A_G[X]$ is the ring of elements $\sum a_\alpha X^\alpha$, $a_\alpha \in A$, $\alpha \in G$ almost all a_α are zero. This paper is a continuation of [26]. Let I be any nonempty set, and $\{X_i; I\}$ a set of indeterminates with the index set of I . We may define the group ring $A_G[X_i; I]$ of G over a ring A with the indeterminates $\{X_i; I\}$. This paper consists of 14 sections. §1 is on the integral (elements), §2 is on the Kronecker functions rings, §3 is on the noetherian, §4 is on the pseudo-prüfer, §5 is on the extension $K_G[X]/k_G[X]$ for fields K/k , §6 is on the Hilbert, §7 is on $A_G[X]_S$ where S is the monics for a totally ordered G , §8 is on the factorial, §9 is on the chain conditions, §10 is on the finite characters, §11 is on the regularly noetherian, §12 is on the Dedekind, §13 is on the rational functions rings, §14 is on the others. Throughout the paper, A is a ring, X an indeterminate, G a non-zero torsion-free abelian group (additively written), $\{X_i; I\}$ a set of indeterminates, D an integral domain, K the quotient field of D , $A_G(X_i; I)$ the total quotient ring of $A_G[X_i; I]$. Definitions of terms are those of [26]. We note that G becomes a totally ordered abelian group.

§1. The integral

Let $\mathfrak{p}^*(D)$ be the ordered group of nonzero principal fractional ideals of an integral domain D (cf. [6]).

PROPOSITION 1. *Let D be a regularly integrally closed domain, $G_f(D)$ the ordered group generated by divisors of D of finite type. Then, $\mathfrak{p}^*(D_G[X])$ is canonically order-isomorphic onto a subgroup of the product ordered group $\mathfrak{p}^*(K_G[X]) \times G_f(D)$.*

PROOF. K is the quotient field of D . We consider a mapping: $(f/g)D_G[X] \mapsto ((f/g)K_G[X], \text{div } A_f - \text{div } A_g)$. That this is a well-defined mapping and a homomorphism follows from [26] §6, Prop. 1.

A ring A is said integrally closed if A is integrally closed in the total quotient

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ring.

PROPOSITION 2. *Assume every non-zero-divisor of A is a unit. If $A_G[X]$ is integrally closed, A is von Neumann regular.*

PROOF. Suppose A has a nonzero nilpotent a . $a^n=0$ for some $n>0$. Take $0 \neq \alpha \in G$. $\alpha/(1+X)^\alpha \in A_G[X]$. $(\alpha/(1+X)^\alpha)^n=0$. $A_G[X]$ is not integrally closed.

PROPOSITION 3. *If $A_G[X]$ is integrally closed, then A is integrally closed and the nilradical of A is zero.*

PROPOSITION 4. *Let B be the total quotient ring of a ring A . $A_G[X]$ is integrally closed if and only if A is integrally closed and $B_G[X]$ is integrally closed.*

PROOF. We use [26] § 6, Th. 6.

PROPOSITION 5. *If A is von Neumann regular, then $A_Z[X_1, \dots, X_n]$ is integrally closed.*

PROOF. By induction on n . Suppose $n=1$ and put $X_1=X$. Let $f/g \in A_Z(X)$ be integral over $A_Z[X]$. We may take $f, g \in A[X]$. By [16] (3.2), $A[X]$ is bezout. There exist $h, f', g', f_1, g_1 \in A[X]$ such that $f=hf'$, $g=hg'$, $h=ff_1+gg_1$. It follows $1=f'f_1+g'g_1$, $1/g'=(f/g)f_1+g_1$. $1/g'$ is integral over $A_Z[X]$. It follows $1/g' \in A_Z[X]$, $f/g \in A_Z[X]$. Suppose the proposition is OK till $n-1$. If $\xi \in A_Z(X_1, \dots, X_n)$ is integral over $A_Z[X_1, \dots, X_n]$, it follows $\xi \in (A_Z(X_1, \dots, X_{n-1}))_Z[X_n]$. By [26] § 6, Th. 6, we have $\xi \in (A_Z[X_1, \dots, X_{n-1}])_Z[X_n]$.

PROPOSITION 6. *If A is von Neumann regular, $A_G[X]$ is integrally closed.*

PROOF. Let $f/g \in A_G(X)$ be integral over $A_G[X]$.

$$(f/g)^n + (f/g)^{n-1}h_{n-1} + \dots + h_0 = 0$$

for $h_i \in A_G[X]$. Let H be the subgroup of G generated by the powers of $f, g, h_{n-1}, \dots, h_0$. $f/g \in A_H(X)$ is integral over $A_H[X]$. By Prop. 5, we have $f/g \in A_H[X]$.

THEOREM 7. *$A_G[X]$ is integrally closed if and only if A is integrally closed and of zero nilradical.*

PROOF. The necessity follows from Prop. 3. The sufficiency. The total quotient ring B of A is von Neumann regular. By Prop. 6, $B_G[X]$ is integrally closed. By Prop. 4, $A_G[X]$ is integrally closed.

LEMMA 8. *Let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals $\neq 0$ of a domain D . If $\text{div } \mathfrak{a} = \text{div } \mathfrak{b}$ on D , then $\text{Div } \mathfrak{a}D_G[X] = \text{Div } \mathfrak{b}D_G[X]$ on $D_G[X]$.*

PROOF. Assume $\xi \in D_G(X)$, $\mathfrak{a}D_G[X] \subset \xi D_G[X]$. $\xi = a/g$ for $a \in D, g \in D_G[X]$.

We set $g = \sum b_\alpha X^\alpha$. $a \in (a/b_a)D$. $b \in (a/b_a)D$. $b \in (a/g)D_G[X]$. $bD_G[X] \subset \xi D_G[X]$. $\text{Div } aD_G[X] \leq \text{Div } bD_G[X]$ on $D_G[X]$. Similarly, $\text{Div } aD_G[X] \geq \text{Div } bD_G[X]$. Hence, $\text{Div } aD_G[X] = \text{Div } bD_G[X]$.

THEOREM 9. $D_G[X]$ is regularly integrally closed if and only if D is regularly integrally closed.

PROOF. The sufficiency is over in [26] § 6, Th. 4. The necessity. Assume $\text{div } a + \text{div}(x_1, \dots, x_n) = \text{div } b + \text{div}(x_1, \dots, x_n)$ on D for $x_i \in K$. $\text{div } a(x_1, \dots, x_n) = \text{div } b(x_1, \dots, x_n)$. By Lem. 8, $\text{Div } a(x_1, \dots, x_n)D_G[X] = \text{Div } b(x_1, \dots, x_n)D_G[X]$ on $D_G[X]$. $\text{Div } aD_G[X] + \text{Div}(x_1, \dots, x_n) = \text{Div } bD_G[X] + \text{Div}(x_1, \dots, x_n)$. Hence, $\text{Div } aD_G[X] = \text{Div } bD_G[X]$. Assume $a \in xD$, $x \in K$. $aD_G[X] \subset xD_G[X]$. $bD_G[X] \subset xD_G[X]$. $b \in xD$. Hence, $\text{div } a \leq \text{div } b$. Similarly, $\text{div } a \geq \text{div } b$. Hence, $\text{div } a = \text{div } b$.

Let B be the total quotient ring of A . If every $x \in B$ which is almost integral over A belongs to A , A is said completely integrally closed ([13] p. 132).

PROPOSITION 10. If $A_G[X]$ is completely integrally closed, then A is completely integrally closed.

LEMMA 11. Let A be a von Neumann regular ring, g a non-zerodivisor of $A[X]$. If $1/g$ is almost integral over $A[X]$, then $1/g \in A[X]$.

PROOF. By the induction on $\deg g = n$. Suppose the proposition is OK till $n-1$. Let b be the coefficient of the highest degree of g . $bA = eA$, $e^2 = e$ for $e \in A$. $A = eA \oplus (1-e)A$. $g = eg + (1-e)g$. Since $(1-e)/((1-e)g)$ is almost integral over $(1-e)A[X]$ and $\deg(1-e)g$ is less than n , we have $(1-e)/((1-e)g) \in (1-e)A[X]$. Since $e/(eg)$ is almost integral over $eA[X]$ and eg is a monic of $eA[X]$, it follows $e/(eg) \in eA[X]$. Hence $1/g \in A[X]$.

LEMMA 12. Assume A is the total quotient ring of itself. Then $A[X]$ is completely integrally closed if and only if A is von Neumann regular.

PROOF. The necessity. The nilradical of A is zero. The sufficiency. Similar with the proof of Prop. 5, using Lem. 11.

LEMMA 13. Assume A is von Neumann regular. If $f/g \in A_Z(X)$ is almost integral over $A_Z[X]$, then $f/g \in A_Z[X]$.

PROOF. We may take $f, g \in A[X]$. Set $g = a_0 + a_1X + \dots + a_nX^n$. We prove the proposition by the induction on $\deg g = n$. Suppose the proposition is OK till $n-1$. If $a_0 = 0$, $f/(a_1 + \dots + a_nX^{n-1})$ is almost integral over $A_Z[X]$; hence $\in A_Z[X]$; hence $f/g \in A_Z[X]$. Hence, we may suppose $a_0 \neq 0$. $a_0A = eA$, $e^2 = e$ for $e \in A$. $((1-e)f)/((1-e)g)$ is almost integral over $(1-e)A_Z[X]$. $((1-e)f)/((1-e)(a_1 + \dots + a_nX^{n-1}))$ is almost integral over $(1-e)A_Z[X]$;

hence $\in (1-e)A_Z[X]$; hence $((1-e)f)/((1-e)g) \in (1-e)A_Z[X]$. $(ef)/(eg)$ is almost integral over $eA_Z[X]$. There exists non-zero-divisor g_1 of $eA[X]$ such that $g_1((ef)/(eg))^m = H_m \in eA_Z[X]$. Then, $H_m \in eA[X]$. By Lem. 12, it follows $(ef)/(eg) \in eA[X]$. Hence, $f/g \in A_Z[X]$.

LEMMA 14. *Assume A is its own total quotient ring. Then $A_Z[X_1, \dots, X_n]$ is completely integrally closed if and only if A is von Neumann regular.*

PROOF. The necessity. By Prop. 3, the nilradical of A is zero. The sufficiency. By the induction on n . By Lem. 13, the proposition is OK for $n=1$. Suppose we are over till $n-1$. Assume $\xi \in A_Z(X_1, \dots, X_n)$ is almost integral over $A_Z[X_1, \dots, X_n]$. By Lem. 13, it follows $\xi \in A_Z(X_1, \dots, X_{n-1})_Z[X_n]$. By the hypothesis and [26] § 6, Th. 8, it follows $\xi \in A_Z[X_1, \dots, X_n]$.

PROPOSITION 15. *Assume A is its own total quotient ring. Then $A_G[X]$ is completely integrally closed if and only if A is von Neumann regular.*

PROOF. The sufficiency. Assume $f/g \in A_G(X)$ be almost integral over $A_G[X]$. There exists a non-zero-divisor h of $A_G[X]$ such that $h(f/g)^n = h_n \in A_G[X]$. Let H be the subgroup of G generated by the powers of h, f, g and let K be the subgroup of G generated by the powers of h, f, g, h_n . $K = Z\alpha_1 \oplus \dots \oplus Z\alpha_m$, $H = Ze_1\alpha_1 \oplus \dots \oplus Ze_k\alpha_k$, $0 < e_i, e_i | e_{i+1}$. Then, $k=m$. h_n is integral over $A_H[X]$. By Prop. 6, it follows $h_n \in A_H[X]$. $f/g \in A_H(X)$ is almost integral over $A_H[X]$. By Lem. 14, it follows $f/g \in A_H[X]$. Hence, $A_G[X]$ is completely integrally closed.

THEOREM 16. *$A_G[X]$ is completely integrally closed if and only if A is completely integrally closed and the nilradical of A is zero.*

PROOF. The sufficiency. By Prop. 15, $B_G[X]$ is completely integrally closed, where B is the total quotient ring of A . Assume $\xi \in A_G(X)$ be almost integral over $A_G[X]$. Then, $\xi \in B_G[X]$. By [26] § 6, Th. 8, it follows $\xi \in A_G[X]$.

The next is on the japanese rings [20] (23.1.1).

PROPOSITION 17. *If $D_G[X]$ is japanese, then D is japanese.*

PROOF. Assume L be a finite field extension of K . By [24] (4.4), the integral closure of $D_G[X]$ in $L_G(X)$ is the integral closure of $D_G[X]$ in $L_G[X]$. Let I be the integral closure of D in L . By [26] § 6, Th. 6, the integral closure of $D_G[X]$ in $L_G[X]$ is $I_G[X]$. $I_G[X] = \sum_1^n D_G[X]F_i$ for $F_i \in I_G[X]$. We set $F_i = \sum b_{i\alpha}X^\alpha$. Then, $I = \sum_{i,\alpha} Db_{i\alpha}$.

REMARK 18. If D is a japanese ring, then $D_Z[X]$ is a japanese ring.

PROOF. By [30] (9.0.2), $D[X]$ is japanese. Hence, $D_Z[X]$ is japanese.

A noetherian ring A is said D -ring if all of its overrings are integrally closed [10]. A subring of the total quotient ring of A containing A is an overring of A .

PROPOSITION 19. *If $A_G[X]$ is a D -ring, then A is a D -ring.*

PROOF. Let B be an overring of A . Then, $B_G[X]$ is an overring of $A_G[X]$. Hence, $B_G[X]$ is integrally closed. By Prop. 3, B is integrally closed. Hence, A is a D -ring.

THEOREM 20. *$D_G[X]$ is a D -domain if and only if D is a field and $G \cong \mathbb{Z}$.*

PROOF. The necessity. $D_G[X]$ is integrally closed. By [10] Th. 1, $D_G[X]$ is a C -domain. By §3, Th. 7, D is a field and $G \cong \mathbb{Z}$. The sufficiency. $D_{\mathbb{Z}}[X]$ has the QR-property. Hence, overrings of $D_{\mathbb{Z}}[X]$ are integrally closed. Hence, $D_{\mathbb{Z}}[X]$ is a D -domain.

THEOREM 21. *$A_G[X]$ is a D -ring if and only if $A_G[X]$ is a C -ring.*

PROOF. [10] Th. 1 and §3, Th. 7.

The next is on the universally japanese rings [20] (0, 23.1.1).

PROPOSITION 22. *If $A_G[X]$ is universally japanese, then A is universally japanese.*

PROOF. An A -algebra of finite type is an $A_G[X]$ -algebra of finite type.

LEMMA 23. *Let S be a ring, R a subring of S such that R is integrally closed in S , $G = \mathbb{Z} \oplus \mathbb{Z}$ with the lexicographic order. Let H be a monic element of $S_G[X]$ and $G' \in S_G[X]$ such that $G'H \in R_G[X]$. Then, $G' \in R_G[X]$.*

PROOF. We set anew $X^{(1,0)} = X$, $X^{(0,1)} = Y$. We may suppose $o_X(G') = o_Y(G') = o_X(H) = o_Y(H) = 0$, where " $o_X(\)$ " is the order as to X . Set $G' = g_0 + g_1 Y + \cdots + g_n Y^n$, $H = h_0 + h_1 Y + \cdots + h_m Y^m$ for $g_i, h_j \in S[X]$. Set $k = \max \{\deg_X g_i, \deg_X h_j; 1 \leq i \leq n, 1 \leq j \leq m\}$. Take $l > k$. $G'(X, X^l)$ and $H(X, X^l)$ are elements of $S[X]$ and $H(X, X^l)$ is monic and $G'(X, X^l)H(X, X^l) \in R[X]$. Hence, by [13] (10.4), it follows $G'(X, X^l) \in R[X]$. The coefficients of g_i are those of $G'(X, X^l)$; hence belong to R . $g_i \in R[X]$. Hence, $G' \in R[X, Y]$.

PROPOSITION 24. *Let S be a ring, R a subring such that R is integrally closed in S , $G = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_n$ with the lexicographic order. Let H be a monic element of $S_G[X]$ and $G' \in S_G[X]$ such that $G'H \in R_G[X]$. Then, $G' \in R_G[X]$.*

PROOF. The induction on n . Suppose the proposition is OK till $n-1$. Set $X^{(0, \dots, 0, 1, \dots, 0)} = X_i$. We may suppose $o_{X_i}(G') = o_{X_i}(H) = 0$. Take a sufficiently large m and substitute X_1^m to X_2 . Then, the coefficients of $G'(X_1, X_2, \dots, X_n)$ are those of $G'(X_1, X_1^m, X_3, \dots, X_n)$ and the coefficients of $H(X_1, X_2, \dots, X_n)$ are those of $H(X_1, X_1^m, X_3, \dots, X_n)$ and $H(X_1, X_1^m, X_3, \dots, X_n)$ is monic. By the

hypothesis, it follows $G'(X_1, X_1^m, X_3, \dots, X_n) \in R[X_1, X_3, \dots, X_n]$. Hence, $G'(X_1, X_2, X_3, \dots, X_n) \in R[X_1, X_2, X_3, \dots, X_n]$.

LEMMA 25. *Let S be a ring, R a subring such that R is integrally closed in S , G a totally ordered abelian group ($\neq 0$), h a monic element of $S_G[X]$ and $g \in S_G[X]$ such that $gh \in R_G[X]$. Assume G be of the torsion-free rank 2. Then, $g \in R_G[X]$.*

PROOF. (i) Let $\{x, y\}$ be real numbers which is rationally independent. Assume $k_1x + l_1y < \dots < k_nx + l_ny$ for $k_i, l_j \in \mathbb{Z}$. Then, there exist natural numbers p, q such that $k_1p + l_1q < \dots < k_np + l_nq$. (ii) Let $\{x, y\}$ be rationally independent real numbers, $G = \mathbb{Z}x + \mathbb{Z}y$ with the usual order of real numbers. Then, our Lemma is OK. Set $X^x = X, X^y = Y$. We may suppose $o_x(g) = o_y(g) = o_x(h) = o_y(h) = 0$. We arrange the powers of g, h as follows: $k_1x + l_1y < \dots < k_nx + l_ny$. Take p, q of (i). Substitute $X \rightarrow Z^p, Y \rightarrow Z^q$ in $g(X, Y), h(X, Y)$ (Z is an indeterminate). The coefficients of g are those of $g(Z^p, Z^q)$. Similar as to h . $h(Z^p, Z^q)$ is a monic of $S[Z]$ and $g(Z^p, Z^q)h(Z^p, Z^q) \in R[Z]$. Hence, it follows $g(Z^p, Z^q) \in R[Z]$. Hence, $g(X, Y) \in R[X, Y]$. Hence, $g \in R_G[X]$. (iii) We may suppose G is finitely generated. When G has proper isolated subgroup, G is order-isomorphic with $\mathbb{Z} \oplus \mathbb{Z}$ with the lexicographic order. Our assertion follows from Prop. 24. When G does not have proper isolated subgroups, G is order-isomorphic with one in (ii). Hence, our assertion follows.

THEOREM 26. *Let S be a ring, R a subring such that R is integrally closed in S , G a totally ordered abelian group ($\neq 0$), h a monic of $S_G[X]$ and $g \in S_G[X]$ such that $gh \in R_G[X]$. Then, $g \in R_G[X]$.*

PROOF. We may suppose G is finitely generated. When G does not have proper isolated subgroups, the assertion follows as Lem. 25. (ii). Generally, G is order isomorphic onto $H_1 \oplus H_2 \oplus \dots \oplus H_k$ with the lexicographic order, where H_i is a subgroup of the reals with the usual order. $H_i = \mathbb{Z}\pi_{i1} + \dots + \mathbb{Z}\pi_{im(i)}$ where $\{\pi_{i1}, \dots, \pi_{im(i)}\}$ is independent reals > 0 . Set $X^{\pi_{ij}} = X_{ij}$. We may suppose $o_{X_{ij}}(g) = o_{X_{ij}}(h) = 0$. When all $m(i) = 1$, our assertion is over in Prop. 24. When $m(l) > 1$ for some l . The powers of g, h are of the form $h_1 + \dots + h_k$ ($h_i \in H_i$). Consider the set of l -components h_l of $h_1 + \dots + h_k$ of powers of g, h . We arrange them as follows: $h^1 < h^2 < \dots < h^a$. $h^1 = \sum b_{1i}\pi_{i1}, \dots, h^a = \sum b_{ai}\pi_{ii}$ for $b_{ji} \in \mathbb{Z}$. Take natural numbers $p_1, \dots, p_{m(l)}$ such that $\sum b_{1i}p_i < \dots < \sum b_{ai}p_i$. Substitute $X_{ii} \rightarrow X_{ii}^{p_i}$ in g, h . By the induction hypothesis on the torsion-free rank of G , it follows $g(X_{11}, \dots, X_{1m(1)}; \dots; X_{l1}, \dots, X_{l1}; \dots; X_{k1}, \dots, X_{km(k)}) \in R[X_{11}, \dots, X_{1m(1)}; \dots; X_{l1}, \dots, X_{lm(l)}; \dots; X_{k1}, \dots, X_{km(k)}]$. Hence, $g(X_{11}, \dots, X_{1m(1)}; \dots; X_{l1}, \dots, X_{lm(l)}; \dots; X_{k1}, \dots, X_{km(k)}) \in R[X_{11}, \dots, X_{1m(1)}; \dots; X_{l1}, \dots, X_{lm(l)}; \dots; X_{k1}, \dots, X_{km(k)}]$. Hence, $g \in R_G[X]$.

§2. The Kronecker functions rings

PROPOSITION 1. *Let $\mathfrak{a}$ be a nonzero ideal of a domain D . Then, $(\mathfrak{a}D_G[X])_v = \mathfrak{a}_v D_G[X]$.*

We take " v " in respective rings.

PROOF. By [26] §9, Prop. 3, it follows $(\mathfrak{a}D_G[X])_v \supset \mathfrak{a}_v$. Hence, $(\mathfrak{a}D_G[X])_v \supset \mathfrak{a}_v D_G[X]$. Conversely, let $\{(a_i/b_i)D; I\}$ be the set of principal fractional ideals containing $\mathfrak{a}$. $\mathfrak{a}_v = \cap (a_i/b_i)D$. Then, $(\cap (a_i/b_i)D) D_G[X] = \cap (a_i/b_i) D_G[X]$. Hence, $\mathfrak{a}_v D_G[X]$ is a divisorial ideal containing $\mathfrak{a}D_G[X]$; hence contains the minimal divisorial ideal $(\mathfrak{a}D_G[X])_v$ containing $\mathfrak{a}D_G[X]$.

PROPOSITION 2. *If $D_G(X)_{\text{loc}}$ is a Dedekind domain, then D_b is a Dedekind domain.*

The notations are of [26].

PROOF. $D_G(X)_{\text{loc}}$ is a prüfer domain. By the proof of [26] §8, Th. 14, (3)→(1), D is a prüfer domain. Hence, D is integrally closed. By [26] §8, Th. 14, it follows $D_G(X)_{\text{loc}} = D_b$. Hence, D_b is a Dedekind domain.

PROPOSITION 3. *If D_b is noetherian, then D_b is a principal ideal domain.*

PROOF. [26] §8, Th. 2.

PROPOSITION 4. *If D_b is a Dedekind domain, $D_G(X)_{\text{loc}}$ is also a Dedekind domain.*

PROOF. By [26] §4, Th. 10, D is an almost Dedekind domain; hence, prüfer. By [26] §8, Th. 14, it follows $D_G(X)_{\text{loc}} = D_b$. Hence, $D_G(X)_{\text{loc}}$ is a Dedekind domain.

LEMMA 5. *Let $*$ be a e.a.b. $*$ -operation on a domain D . Let $F, G' \in K_G[X] - \{0\}$. Then, $(A_{FG'})^* = (A_F A_{G'})^*$.*

PROOF. [26] §8, Lem. 1.

PROPOSITION 6. *Let D be a v -domain. Then the followings are equivalent.*

- (1) D is a prüfer v -multiplication ring.
- (2) D^v is a quotient ring of $D_G[X]$.

PROOF. (1)→(2). Set $N = \{g \in D_G[X]; 1/g \in D^v\}$. Let $f/h \in D^v$, $(A_f)_v \subset (A_h)_v$. $\text{div } A_h + \text{div}(u_0, \dots, u_n) = \text{div } D$ for $u_i \in K$. Set $U = u_0 + u_1 X^\alpha + \dots + u_n X^{n\alpha}$ for $0 \neq \alpha \in G$. By Lem. 5, it follows $(A_{hU})_v = (A_h A_U)_v = D$. Hence, $hU \in N$. $(A_{fU})_v = (A_f A_U)_v \subset (A_h)_v (A_U)_v = D$. Hence, $fU \in D_G[X]$. Hence, $f/h = (fU)/(hU) \in D_G[X]_N$. Hence, $D^v \subset D_G[X]_N$. (2)→(1). Then, $D^v = D_G[X]_N$. Consider

$\text{div}(x_0, x_1, \dots, x_n)$ for $x_i \in K - \{0\}$. $x_i = a_i/b$ for $a_i, b \in D$. Take $0 \neq \alpha \in G$. Set $f = a_0 + a_1 X^\alpha + \dots + a_n X^{n\alpha}$. $a_0/f \in D^v$. Hence, $a_0/f = h/g$ for $g \in N$. $(a_0) = (A_{a_0g})_v = (A_{fh})_v = (A_f A_h)_v$. $\text{div}(x_0, \dots, x_n) + \text{div}(b/a_0)A_h = \text{div } D$.

PROPOSITION 7. *If there exists a G -powered Kronecker functions ring D on a domain D such that D is a Krull domain, then D is a Krull domain.*

PROOF. $D_* \cap K = D$.

Let $\{V_\lambda; A\}$ be a family of valuation overrings of D such that $D = \bigcap V_\lambda$. Let L be an algebraic field extension of K and let $\{W_\sigma; \Sigma\}$ be the family of all valuation rings W on L such that $W \cap K \in \{V_\lambda; A\}$. Set $J = \bigcap W_\sigma$. Let D_* be the G -powered Kronecker functions ring with respect to $\{X_i; I\}$ of the w -operation on D induced by $\{V_\lambda; A\}$. Let J_* be the G -powered Kronecker functions ring with respect to $\{X_i; I\}$ of the w -operation on J induced by $\{W_\sigma; \Sigma\}$.

THEOREM 8. J_* is the integral closure of D_* in $L_G(X_i; I)$.

PROOF. By [26] § 8, Th. 5, it follows $J_* = \bigcap W_\sigma^*$ and $D_* = \bigcap V_\lambda^*$, where W_σ^* , V_λ^* are the trivial extensions of W_σ , V_λ to $L_G(X_i; I)$, $K_G(X_i; I)$ respectively. Let v_λ be the valuation for V_λ and w_σ be the valuation for W_σ . $\bigcap W_\sigma^*$ is integrally closed. Let $f/g \in \bigcap W_\sigma^*$ for $f, g \in J_G[X_i; I]$. Let H be the subgroup of G generated by the powers of f, g . $H = Z\alpha_1 \oplus \dots \oplus Z\alpha_n$. Set $X_i^{\alpha_j} = X_{ij}$. Let w'_α be the trivial extension of w_σ to $L(X_{ij}; i, j)$. $w'_\alpha(f/g) \geq 0$; hence f/g is integral over $\bigcap_\lambda V_\lambda(X_{ij}; i, j)_{\text{loc}}$ ([13] p. 409, 15). Since $V_\lambda(X_{ij}; i, j)_{\text{loc}} \subset V_\lambda^*$, f/g is integral over $\bigcap V_\lambda^*$. Hence, J_* is the integral closure of D_* in $L_G(X_i; I)$.

Next, we state the direct proof of Th. 8.

Let $f/g \in \bigcap W_\sigma^*$; $f, g \in J_G[X_i; I]$. It suffices to show that f/g is integral over $\bigcap V_\sigma^*$. We may suppose $A=1$ point; $\{V_\lambda; A\} = \{V\}$. (i) When L/K is purely inseparable. Let w^* be any extension of v^* to $L_G(X_i; I)$. $(f/g)^{p^e} \in K_G(X_i; I)$ for some e . $p^e w^*(f/g) = w^*((f/g)^{p^e}) = v^*((f/g)^{p^e}) = w_\sigma^*((f/g)^{p^e}) = p^e w_\sigma^*(f/g) \geq 0$ for any σ . Hence, f/g is integral over V^* . (ii) When L/K is finite normal separable. Let w^* be any extension of v^* to $L_G(X_i; I)$. Let $[L:K] = n$ and $\{\tau_1, \dots, \tau_n\}$ be the Galois group of L/K . By [26] § 9, Th. 2, $L_G(X_i; I)/K_G(X_i; I)$ is a Galois extension of degree n and its Galois group is $\{\tau_1^*, \dots, \tau_n^*\}$ where τ_j^* is the natural extension of τ_j to $L_G(X_i; I)$. Fix $\sigma \in \Sigma$. $w^* = w_\sigma^* \tau_j^*$ for some j . Hence, $w^*(f/g) \geq 0$; hence f/g is integral over V^* . (iii) When L/K is finite separable. Let M be the minimal normal extension of K containing L . Let u be any extension of v to M . $u|_L = w_\sigma$ for some σ . $u^*(f/g) = w_\sigma^*(f/g) \geq 0$. By (ii), f/g is integral over V^* . (iv) When L/K is infinite. Let M be the separable closure of K in L . Let $\{U_\alpha; \alpha\}$, $\{u_\alpha; \alpha\}$ be the extensions of V, v to M respectively. By (i), f/g is integral over U_α^* . Hence, f/g is integral over $\bigcap U_\alpha^*$. By (iii), $\bigcap U_\alpha^*$ is integral over V^* .

Hence, f/g is integral over V^* . (v) When L/K is algebraic. Let M be the extension $K(\text{coefficients of } f, g)$. Let $\{U_\alpha; \alpha\}, \{u_\alpha; \alpha\}$ be the extensions of V, v to M respectively. $u_\alpha = w_\sigma | M$ for some σ . $u_\alpha^*(f/g) = w_\sigma^*(f/g) \geq 0$. Hence, by (iv), f/g is integral over V^* .

§3. The noetherian

PROPOSITION 1. *Let $(A, \mathfrak{m})$ be a local ring such that $A_G(X)_{\text{loc}}$ is regular local. Then, A is regular local.*

The terminology "local ring" is a ring with only one maximal ideal.

PROOF. Let $\mathfrak{a}, \mathfrak{b}$ be ideals of A such that $\mathfrak{a} \subseteq \mathfrak{b}$. If $\mathfrak{a}A_G(X)_{\text{loc}} = \mathfrak{b}A_G(X)_{\text{loc}}$, it follows $\mathfrak{a} = \mathfrak{b}$ by [26] § 7, Prop. 2. Hence, A is noetherian. By [26] § 7, Prop. 2, it follows $d = \dim A \leq \dim A_G(X)_{\text{loc}}$. $(a_1, \dots, a_d)A$ is $\mathfrak{m}$ -primary for $a_i \in A$. By [26] § 7, Prop. 2, $(a_1, \dots, a_d)A_G(X)_{\text{loc}}$ is $\mathfrak{m}A_G(X)_{\text{loc}}$ -primary. Hence, $d \geq \dim A_G(X)_{\text{loc}}$. Hence, $\dim A = \dim A_G(X)_{\text{loc}}$. We confer [45] chap. 8, § 11, Th. 25. The vector space $(\mathfrak{m}A_G(X)_{\text{loc}})/(\mathfrak{m}^2A_G(X)_{\text{loc}})$ over $A_G(X)_{\text{loc}}/(\mathfrak{m}A_G(X)_{\text{loc}})$ is of dimension d and is generated by $\{x + \mathfrak{m}^2A_G(X)_{\text{loc}}; x \in \mathfrak{m}\}$. Hence, there exists a basis $\{x_j + \mathfrak{m}^2A_G(X)_{\text{loc}}; x_j \in \mathfrak{m}, 1 \leq j \leq d\}$. Hence, $\mathfrak{m}A_G(X)_{\text{loc}} = (x_1, \dots, x_d)A_G(X)_{\text{loc}}$. Let $x \in \mathfrak{m}$. $x = \sum (x_i f_i)/g$ for $f_i \in A_G[X]$ and $A_G[X] \ni g \notin \mathfrak{m}A_G[X]$. It follows $x \in (x_1, \dots, x_d)A$. Hence, $\mathfrak{m} = (x_1, \dots, x_d)A$. Hence, A is regular.

COROLLARY 2. *Assume $A_G[X]_{\mathfrak{P}}$ is regular for any prime $\mathfrak{P}$ of $A_G[X]$. Then, $A_{\mathfrak{p}}$ is regular for any prime $\mathfrak{p}$ of A .*

PROOF. Let $\mathfrak{p}$ be a prime ideal of A . $\mathfrak{p}A_G[X]$ is a prime ideal of $A_G[X]$. Hence, $A_G[X]_{\mathfrak{p}A_G[X]}$ is regular. $A_G[X]_{\mathfrak{p}A_G[X]} = (A_{\mathfrak{p}})_G[X]_{\mathfrak{p}(A_{\mathfrak{p}})_G[X]}$. By Prop. 1, A is regular.

PROPOSITION 3. *If $A_G[X]_{\mathfrak{P}}$ is regular for every prime ideal $\mathfrak{P}$ of $A_G[X]$, then the torsion-free rank of G is finite.*

PROOF. Prop. 16.

THEOREM 4. *$A_G[X]_{\mathfrak{P}}$ is regular for every prime ideal $\mathfrak{P}$ of $A_G[X]$ if and only if the torsion-free rank of G is finite, $A_{\mathfrak{p}}$ is regular for every prime ideal $\mathfrak{p}$ of A , $(G/F)_{\mathfrak{p}}$ is finite for each $\mathfrak{p} \in \Omega$.*

The notations are of [23].

PROOF. Prop. 3 and [23] Th., (4).

If all of overrings of A are noetherian, A is said a C-ring [10].

PROPOSITION 5. *If $A_G[X]$ is a C-ring, then A is also a C-ring.*

PROOF. Let B be an overring of A . $B_G[X]$ is an overring of $A_G[X]$. Hence, $B_G[X]$ is noetherian. Hence, B is noetherian [9]. Hence, A is a C -ring.

PROPOSITION 6. *If $A_G[X]$ is a C -ring, then $\dim A = 0$ and $G \cong Z$.*

PROOF. A is noetherian. $G = Z\alpha_1 \oplus \cdots \oplus Z\alpha_n$. Set $X^{\alpha_i} = X_i$, $A_Z[X_1, \dots, X_{n-1}] = B$. $B_Z[X_n]$ is a C -ring. Suppose $\dim B > 0$. There exist prime ideals $\mathfrak{p} \supseteq \mathfrak{q}$ of B . $(\mathfrak{p}, 1 - X_n) \supseteq \mathfrak{p}B_Z[X_n] \supseteq \mathfrak{q}B[X_n]$. We confer [10]. $B_Z[X_n]$ is of the restricted dimension ≥ 2 ; a contradiction. Hence, the dimension of B is 0. Hence, it follows $n = 1$. Hence, $G \cong Z$ and $\dim A = 0$.

THEOREM 7. *$A_G[X]$ is a C -ring if and only if $G \cong Z$ and A is a direct sum of a finite number of fields.*

PROOF. (1) If $A_Z[X]$ is a C -ring and A is a primary ring, then A is a field. By Prop. 6, it follows $\dim A = 0$. By Prop. 5, A is a C -ring. The nilradical $\mathfrak{p}$ of A is a prime ideal. $(\mathfrak{p}, 1 - X)$ is a maximal ideal of $A_Z[X]$. $A_Z[X]_{(\mathfrak{p}, 1 - X)}$ is not a Z -ring; hence without nilpotents. Let $a \in \mathfrak{p}$. $ag = 0$ for $A_Z[X] \ni g \notin (\mathfrak{p}, 1 - X)$. $as = 0$ for $A \ni s \notin \mathfrak{p}$. Hence, it follows $a = 0$. Hence, $\mathfrak{p} = 0$. Hence, A is a field. (2) $A_Z[X]$ is a C -ring if and only if A is a direct sum of a finite number of fields. A is of dimension 0 and a C -ring. $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$, $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$, $\mathfrak{p}_i \neq \mathfrak{p}_j$ ($i \neq j$) for primary ideals $\mathfrak{q}_i$ of A . $A \cong A/\mathfrak{q}_1 \oplus \cdots \oplus A/\mathfrak{q}_n$. $A_Z[X] \cong (A/\mathfrak{q}_1)_Z[X] \oplus \cdots \oplus (A/\mathfrak{q}_n)_Z[X]$. $(A/\mathfrak{q}_i)_Z[X]$ is a C -ring. By (1), $A/\mathfrak{q}_i$ is a field. Conversely, let $A = k_1 \oplus \cdots \oplus k_n$ for fields k_i . By [10] Th. 1, $A_Z[X]$ is a C -ring. (3) The necessity follows from Prop. 6.

PROPOSITION 8. *If $A_G[X]$ is a Macaulay ring of [29], then so is A .*

PROOF. We may suppose $G = Z$. By [26] § 5, Prop. 3, A is a locally Macaulay ring of [29]. Suppose $\text{ht } \mathfrak{m} < \dim A$ for some maximal ideal $\mathfrak{m}$ of A . There exist maximal chains containing $(1 - X)$ of prime ideals of $A_Z[X]$ such that $\mathfrak{P}_0 \supseteq \mathfrak{P}_1 \supseteq \cdots \supseteq \mathfrak{P}_n$, $\mathfrak{Q}_0 \supseteq \mathfrak{Q}_1 \supseteq \cdots \supseteq \mathfrak{Q}_m$, $n \neq m$. There arise maximal chains in $A_Z[X]$ of lengths $n + 1$ and $m + 1$. This is a contradiction.

REMARK 9. Even if A is a Macaulay ring of [29], $A_Z[X]$ need not be so.

A COUNTER EXAMPLE. Let L, M be fields, $A = L \oplus M$. We confer [29] (25.12).

LEMMA 10. *If A is noetherian, then $\dim A_Z[X_1, \dots, X_n] = n + \dim A$.*

PROOF. $\dim A[X] = 1 + \dim A$.

PROPOSITION 11. *Let A be a Macaulay ring of [29]. Suppose (*) "for every maximal ideal $\mathfrak{M}$ of $A_Z[X_1, \dots, X_n]$, $\mathfrak{M} \cap A$ is a maximal ideal of A ". Then, $A_Z[X_1, \dots, X_n]$ is a Macaulay ring of [29].*

PROOF. Let $\mathfrak{M}$ be a maximal ideal of $A_Z[X_1, \dots, X_n]$ and let $\mathfrak{M} \cap A = \mathfrak{n}$. By Lem. 10, it follows $\text{ht } \mathfrak{M} \geq n + \text{ht } \mathfrak{n} = n + \dim A = \dim A_Z[X_1, \dots, X_n] \geq \text{ht } \mathfrak{M}$. Hence, $\text{ht } \mathfrak{M} = \dim A_Z[X_1, \dots, X_n]$.

PROPOSITION 12. *If $A, A_Z[X_1, \dots, X_n]$ are Macaulay rings of [29] then (*) holds.*

PROOF. Suppose (*) does not hold. There exists a maximal ideal $\mathfrak{M}$ of $A_Z[X_1, \dots, X_n]$ such that $\mathfrak{M} \cap A = \mathfrak{n}$ is not a maximal ideal of A . $n + \text{ht } \mathfrak{n} = \dim (A_{\mathfrak{n}})_Z[X_1, \dots, X_n] \geq \text{ht } \mathfrak{M}_{\mathfrak{n}} \geq n + \text{ht } \mathfrak{n} A_{\mathfrak{n}} = n + \text{ht } \mathfrak{n}$. $\text{ht } \mathfrak{M} = \text{ht } \mathfrak{M}_{\mathfrak{n}} = n + \text{ht } \mathfrak{n} < n + \dim A = \dim A_Z[X_1, \dots, X_n]$. Hence, $A_Z[X_1, \dots, X_n]$ is not a Macaulay ring of [29].

PROPOSITION 13. *$A_Z[X_1, \dots, X_n]$ is a Macaulay ring of [29] if and only if A is a Macaulay ring of [29] and (*) holds.*

PROPOSITION 14. *There exists a maximal ideal $\mathfrak{M}$ of $D_Z[X_1, \dots, X_n]$ such that $\mathfrak{M} \cap D = 0$ if and only if $D[a^{-1}]$ is a field for some $0 \neq a \in D$.*

The proof is similar with that of [29] (14.10).

THEOREM 15. *$A_G[X]$ is a Macaulay ring of [29] if and only if A is a Macaulay ring of [29], G is finitely generated, A has no prime ideals $\mathfrak{p}$ such that (1) $\mathfrak{p}$ is not a maximal ideal, (2) $(A/\mathfrak{p})[\bar{a}^{-1}]$ is a field for some $\bar{a} \in \mathfrak{p}$.*

PROOF. The necessity. By Prop. 13, A is a Macaulay ring of [29] and (*) holds. Suppose there exist $\mathfrak{p}$ satisfying (1), (2). Apply Prop. 14 to $A/\mathfrak{p}$. There exists a maximal ideal $\mathfrak{M}$ of $A_Z[X_1, \dots, X_n]$ such that $\mathfrak{M} \cap A = \mathfrak{p}$, a contradiction. The sufficiency. Suppose (*) does not hold. There exists a maximal ideal $\mathfrak{M}'$ of $A_Z[X_1, \dots, X_n]$ such that $\mathfrak{M}' \cap A = \mathfrak{p}$ is not a maximal ideal of A . Prop. 14 leads to a contradiction.

If $A_{\mathfrak{p}}$ is noetherian for every prime ideal $\mathfrak{p}$ of A , A is said to be locally noetherian.

PROPOSITION 16. *If $A_G[X]$ is locally noetherian, the torsion-free rank of G is finite.*

PROOF. Suppose the torsion-free rank is infinite. Let $\{\alpha_\lambda; A\}$ be a maximal independent subset of G . Set $H = \sum_{\lambda} Z\alpha_\lambda$. Take $\{\alpha_1, \alpha_2, \alpha_3, \dots\} \subset \{\alpha_\lambda; A\}$. $(1 - X^{\alpha_1}, \dots, 1 - X^{\alpha_n})A_H[X] = \mathfrak{p}_n$ is a prime ideal of $A_H[X]$. Take a prime ideal $\mathfrak{P}_1$ of $A_G[X]$ over $\mathfrak{p}_1$. Take a prime ideal $\mathfrak{P}_2 \supset \mathfrak{P}_1$ of $A_G[X]$ over $\mathfrak{p}_2$. Take $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \dots$ like this. $\bigcup_{i=1}^{\infty} \mathfrak{P}_i = \mathfrak{P}$ is a prime ideal of $A_G[X]$. $A_G[X]_{\mathfrak{P}}$ is not noetherian, a contradiction.

THEOREM 17. *$A_G[X]$ is locally noetherian if and only if G is of finite torsion-*

free rank, A is locally noetherian, $(G/F)_p$ is finite for each $p \in \Omega$.

The notations are of [23].

PROOF. Prop. 16 and [23] Th., (1).

The next is on the global Gorenstein rings [21] p. 164.

PROPOSITION 18. *If $A_G[X]$ is a global Gorenstein ring, then A is a global Gorenstein ring.*

PROOF. A is noetherian. Let $\mathfrak{m}$ be a maximal ideal of A . $(\mathfrak{m}, 1-X)$ is a maximal ideal of $A_Z[X]$. Hence, $A_Z[X]_{(\mathfrak{m}, 1-X)}$ is a Gorenstein ring. $(1-X)/1$ is a non-zerodivisor of $A_Z[X]_{(\mathfrak{m}, 1-X)}$. Hence, $A_Z[X]_{(\mathfrak{m}, 1-X)}/(1-X)A_Z[X]_{(\mathfrak{m}, 1-X)}$ is Gorenstein (cf. [21] p. 164, 13). This last ring is isomorphic with $A_{\mathfrak{m}}$. Hence, A is a global Gorenstein ring. Next, $G \cong \underbrace{Z \oplus \cdots \oplus Z}_n$. $A_G[X] \cong A_Z[X_1, \dots, X_n]$. From the above, $A_Z[X_1, \dots, X_{n-1}]$ is a global Gorenstein ring. Like this, A is a global Gorenstein ring.

PROPOSITION 19. *$A_G[X]$ is a global Gorenstein ring if and only if A is a global Gorenstein ring and G is finitely generated.*

PROOF. [23] Th., (3).

The next is on the $J-0$ [27] (32.B).

PROPOSITION 20. *If A is $J-0$, then $A_Z[X]$ is $J-0$.*

PROOF. There exists $a \in A$ such that $\phi \neq D(a) \subset \text{Spec}(A)$ and $A_{\mathfrak{p}}$ is regular for every $\mathfrak{p} \in D(a)$. Set $D'(a) = \{\mathfrak{P} \in \text{Spec}(A_Z[X]); \mathfrak{P} \bar{\ni} a\}$. $D'(a) \neq \emptyset$. Let $\mathfrak{P} \in D'(a)$, $\mathfrak{P} \cap A = \mathfrak{p}$. Since $\mathfrak{p} \bar{\ni} a$, $A_{\mathfrak{p}}$ is regular. $A_Z[X]_{\mathfrak{P}} \cong (A_{\mathfrak{p}})_Z[X]_{\mathfrak{P}}$ for a prime ideal $\mathfrak{P}'$ of $(A_{\mathfrak{p}})_Z[X]$. Since $(A_{\mathfrak{p}})_Z[X]$ is regular, $(A_{\mathfrak{p}})_Z[X]_{\mathfrak{P}'}$ is regular. Hence, $A_Z[X]_{\mathfrak{P}}$ is regular. Hence, $A_Z[X]$ is $J-0$.

The next is on the $J-2$ [27] (32.B).

PROPOSITION 21. *If $A_G[X]$ is $J-2$, A is also $J-2$.*

PROOF. Let $\mathfrak{p}$ be a prime ideal of A . Then, there exists a prime ideal $\mathfrak{P}$ of $A_G[X]$ such that $A_G[X]/\mathfrak{P} = A/\mathfrak{p}$.

LEMMA 22. *If A is $J-2$, $A_Z[X]$ is also $J-2$.*

PROOF. A finitely generated $A_Z[X]$ -algebra is a finitely generated A -algebra.

PROPOSITION 23. *$A_G[X]$ is $J-2$ if and only if A is $J-2$ and G is finitely generated.*

PROOF. Prop. 21, Lem. 22 and the results of [9].

§4. The pseudo-prüfer

LEMMA 1. Let $\pi \in G$ be of $\text{ht}(0, 0, 0, \dots)$, K a field, $a, b \in K - \{0\}$. Then, $a + bX^\pi$ is irreducible in $K_G[X]$.

PROOF. Similar with that of [24] Lem. 2.2.

LEMMA 2. Let $\pi \in G$ be of $\text{ht}(0, 0, 0, \dots)$, $f \in D_{Z\pi}[X] - D$. If f is irreducible in $D_{Z\pi}[X]$, then f is irreducible in $D_G[X]$.

PROOF. The proof appears in the proof of Prop. 5.

PROPOSITION 3. Let D be a prüfer domain, G contain a $\text{ht}(0, 0, 0, \dots)$ element π . If every irreducible element of $D_G[X]$ is also irreducible in $K_G[X]$, then D is a bezout domain.

PROOF. Suppose D is not bezout. There exist $a, b \in D$ such that (a, b) is not principal. $(a, b)(c, d) = D$ for $c, d \in K$. Let $f = (a + bX^\pi)(c + dX^\pi)$. By [26] § 2, Th. 11, $A_f = (a, b)(c, d) = D$. Hence, $f \in D_G[X]$. f is not irreducible in $K_G[X]$. Hence, f is not irreducible in $D_G[X]$. By Lem. 2, f is not irreducible in $D_{Z\pi}[X]$. Let $f = f_1 \cdots f_n$ be the irreducible decomposition of f in $D_{Z\pi}[X]$; $n \geq 2$. f_i is irreducible in $K_G[X]$; hence irreducible in $K_{Z\pi}[X]$. $f = f_1 \cdots f_n$ is the irreducible decomposition in $K_{Z\pi}[X]$. By Lem. 1, $a + bX^\pi$ is associated with, say f_1 in $K_{Z\pi}[X]$. $f_1 = \alpha(a + bX^\pi)$ for $\alpha \in K$. $D = A_f = A_{\alpha(a + bX^\pi)\alpha^{-1}(c + dX^\pi)} = A_{f_1 f_2} = A_{f_1} A_{f_2}$. Hence, $A_{f_1} = D$. $(\alpha a, \alpha b) = D$. $(a, b) = (\alpha^{-1})$; a contradiction.

PROPOSITION 4. Let D be an integrally closed domain, π a $\text{ht}(0, 0, 0, \dots)$ element of G . Assume that every prime ideal $\mathfrak{P}$ of $D_G[X]$ such that $\mathfrak{P} \cap D = 0$ is principal. Then D is a pseudo-bezout domain.

PROOF. Let $a, b \in D - \{0\}$. Take a maximal $\mathfrak{P}$ among ideals of $D_G[X]$ containing $a + bX^\pi$ and not intersecting with $D - \{0\}$. $\mathfrak{P}$ is prime. $\mathfrak{P} = (f)$ for $f \in D_G[X]$. $a + bX^\pi = fg$ for $g \in D_G[X]$. By Lem. 1, g is a unit of $K_G[X]$. $g = d \in D$. $D_G[X] \ni f = a/d + (b/d)X^\pi = a' + b'X^\pi$. Since (f) is prime, $\text{GCD}(a', b')$ is 1. By [26] § 6, Prop. 1 (3), it follows $\text{div}(a, b) = \text{div } A_{a + bX^\pi} = \text{div } A_{d(a' + b'X^\pi)} = \text{div}(d) + \text{div}(a', b') = \text{div}(d)$. Hence, $d = \text{GCD}(a, b)$.

PROPOSITION 5. Let D be a prüfer v -multiplication ring, π a $\text{ht}(0, 0, 0, \dots)$ element of G . Assume that every irreducible $f \in D_G[X] - D$ in $D_G[X]$ such that $(A_f)_v = D$ is irreducible in $K_G[X]$. Then D is pseudo-bezout.

K is the quotient field of D .

PROOF. Let $a, b \in D - \{0\}$. $\text{div}(a, b) + \text{div}(x_0, \dots, x_n) = \text{div } D$ for $x_i \in K$.

Set $f = (a + bX^n)(x_0 + x_1X^n + \cdots + x_nX^{nn})$. By [26] § 6, Prop. 1, it follows $(A_f)_v = ((a, b)(x_0, \dots, x_n))_v = D$. Hence, $f \in D_G[X]$ and f is reducible in $D_G[X]$. f has not nonunit constant factors in $D_G[X]$. Let $f = gh$ be a proper decomposition in $D_G[X]$ and let H be the subgroup generated by π and the powers of g, h . By [24] (2.1), $Z\pi$ is a direct summand of H . $H = Z\pi \oplus Z\alpha_1 \oplus \cdots \oplus Z\alpha_m$. We may suppose $g, h \in D_{Z\pi}[X]$. Let $g = g_1 \cdots g_k, h = h_1 \cdots h_l$ be irreducible decompositions in $D_{Z\pi}[X]$; $k, l \geq 1$. g_i, h_j are irreducible in $D_G[X]$. Since $(A_{g_i})_v = (A_{h_j})_v = D$, g_i and h_j are irreducible in $K_G[X]$. Since $f = g_1 \cdots g_k h_1 \cdots h_l$ is the irreducible decomposition in $K_{Z\pi}[X]$, $a + bX^n$ is associated with, say g_i in $K_{Z\pi}[X]$. $(a + bX^n)(t/s) = g_i$ for $s, t \in D$. Set $(t/s)a = c \in D, (t/s)b = d \in D$. $\text{div}(a, b) = \text{div}(s/t) + \text{div}(c, d) = \text{div}(s/t)$. Hence, $s/t = \text{GCD}(a, b)$.

The next is on the $(n)'$ ([13] p. 305).

PROPOSITION 6. If $A_G[X]$ is $(n)'$, A is $(n)'$.

The next is on the (n) ([13] p. 305).

PROPOSITION 7. If $A_G[X]$ is (n) , then A is (n) .

REMARK 8. If $A_G[X]$ is (2) , then A is von Neumann regular.

PROOF. If A has nonzero nilpotents, there exists $b \neq 0$ such that $b^2 = 0$. Let $0 \neq \alpha \in G$. $b(1 + X^\alpha) = (1 + X^\alpha)^2 g$ for some g . $b = (1 + X^\alpha)g$; this is obviously impossible. Suppose A have nonunit non-zero-divisor a . $a(1 + X^\alpha) = a^2 f + (1 + X^\alpha)^2 g$ for some f, g . $1 + X^\alpha = af + (1 + X^\alpha)^2 g'$ for some g' . This is also impossible.

PROPOSITION 9. $D_Z[X]$ is (2) if and only if D is a field.

PROPOSITION 10. $D_G[X]$ is (2) if and only if D is a field and $G \subset Q$ in the sense of isomorphism.

Q is the rational numbers.

PROOF. The necessity. By Remark 8, $D = K$ is a field. Let $f, g \in K_G[X] - \{0\}$. There exists the GCD h of f, g in $K_G[X]$ ([24] Prop. 1.4). $f = hf_1, g = hg_1, f_1 g_1 = f_1^2 f' + g_1^2 g'$. Let H be the subgroup generated by the powers of f_1, g_1, f', g' . $K_H[X]$ is factorial. $1 = f_1 f'' + g_1 g''$ for some f'', g'' . Then, $(f, g) = (h)$. Similarly, every finitely generated ideal of $K_G[X]$ is principal; especially $K_G[X]$ is prüfer. Hence, G is isomorphic into Q ([16]). The sufficiency. Let $f, g \in K_G[X] - \{0\}$. There exists the GCD h of f, g . $f = hf_1, g = hg_1$. There exists $n > 0$ such that $f_1, g_1 \in K_Z[Y]$, where $Y = X^{1/n}$. $1 = f_1 f' + g_1 g'$. It follows $fg = f^2(g_1 f') + g^2(f_1 g')$. Hence, $K_G[X]$ is (2) .

PROPOSITION 11. If $A_G[X]$ is (n) , then A is von Neumann regular.

PROOF. Suppose A has nonzero nilpotent. Then there exists $0 \neq b \in A$ such that $b^2 = 0$. Take $0 \neq \alpha \in G$. $b(1 + X^\alpha)^{n-1} = (1 + X^\alpha)^n g$ for some g . $b = (1 + X^\alpha)g$; this is impossible. Hence, A has zero nilradical. Next, suppose A has nonunit non-zerodivisor a . $a(1 + X^\alpha)^{n-1} = a^n f' + (1 + X^\alpha)g'$ for some f', g' . $(1 + X^\alpha)^{n-1} = a^{n-1} f' + (1 + X^\alpha)^n g''$ for some g'' . This is impossible.

PROPOSITION 12. *Let A be a noetherian ring, $n > 1$. Then, $A_G[X]$ is (n) if and only if A is von Neumann regular and G is isomorphic into Q .*

PROOF. The necessity. By Prop. 11, A is von Neumann regular. $(A/\mathfrak{p})_G[X]$ is (n) for a prime ideal $\mathfrak{p}$ of A ; hence G is isomorphic into Q (Th. 14). The sufficiency. $A \cong k_1 \oplus \cdots \oplus k_n$ for fields k_i . By Th. 14, $(k_i)_G[X]$ is (n) . Hence, $A_G[X]$ is (n) .

PROPOSITION 13. *If $A_G[X]$ is (n) , then G is isomorphic into Q .*

PROOF. Th. 14.

THEOREM 14. *Let $n > 1$. $D_G[X]$ is (n) if and only if D is a field and G is isomorphic into Q .*

PROOF. The case $n=2$ is over in Prop. 10. We may suppose $n > 2$. The necessity. By Prop. 11, $D=K$ is a field. Let $f, g \in K_G[X] - \{0\}$. There exists $\text{GCD}(f, g) = h$ ([24] (1.4)). $f = hf_1, g = hg_1, f_1 g_1^{n-1} = f_1^n f' + g_1^n g'$. Let H be the subgroup generated by the powers of f_1, g_1, f', g' . $K_H[X]$ is factorial. $g_1^{n-1} = f_1^{n-1} f' + g_1^n g'', 1 = f_1^{n-1} f'' + g_1 g''$. Hence, $(f, g) = (h)$. Hence, every finitely generated ideal of $K_G[X]$ is principal. By [16], G is isomorphic into Q . The sufficiency. $D=K$. Let $f, g \in K_G[X] - \{0\}$. There exists $\text{GCD}(f, g) = h$. $f = hf_1, g = hg_1$. There exists n such that $f_1, g_1 \in K_Z[Y]$, where $Y = X^{1/n}$. $1 = \text{GCD}(f_1^n, g_1^n)$. $1 = f_1^n f' + g_1^n g'$. For $0 \leq k \leq n$, it follows $f^k g^{n-k} = f^n f'' + g^n g''$. Hence, $(f, g)^n = (f^n, g^n)$.

The next is on the n -generator property ([13] p. 320).

PROPOSITION 15. *If $A_G[X]$ has n -generator property, then A has n -generator property.*

THEOREM 16. *Let $n > 0$. $D_G[X]$ has n -generator property if and only if D is a field and G is isomorphic into Q .*

PROOF. The necessity. Let $\bar{D}$ be the integral closure of D . $\bar{D}_G[X]$ is the integral closure of $D_G[X]$ [24] (4.4). By [14] Cor. 3, $\bar{D}$ is a field and G is isomorphic into Q . Hence D is a field.

PROPOSITION 17. *Let $n > 0$. If $A_G[X]$ has n -generator property, then $\dim A = 0$ and G is isomorphic into Q .*

PROOF. Th. 16.

The next is on the GCD-ring [13] p. 76.

PROPOSITION 18. *If $A_G[X]$ is a GCD-ring, then A is also a GCD-ring.*

PROOF. Similar with that of Prop. 20.

REMARK 19. Even if A is a GCD-ring, $A_2[X]$ need not be a GCD-ring.

A COUNTER EXAMPLE. $A = \mathbb{Z}/(4)$. $f = (1+X)(3+X)$, $g = (1+X)^2 \in A_2[X]$. Suppose there exist GCD h of f, g . $h \in (1+X) \cap (3+X)$. As seen in Remark 21, h is of the form either $(1+X)(3+X)f'$ or $(1+X)\{(3+X)f' + 2\}$. Suppose $h = (1+X)(3+X)f'$. $(1+X)^2 = (1+X)(3+X)f'f_1$. $1+X = (3+X)f'f_1$; a contradiction. The rest is similar.

The next is on the LCM-ring [13] p. 76.

PROPOSITION 20. *If $A_G[X]$ is a LCM-ring, then A is a LCM-ring.*

PROOF. Let φ be the canonical homomorphism of $A_G[X]$ to A . Let $a, b \in A$ and f the LCM of a, b in $A_G[X]$. Then, $\varphi(f)$ is the LCM of a, b in A .

REMARK 21. Even if A is a LCM-ring, $A_2[X]$ need not be a LCM-ring.

A COUNTER EXAMPLE. $A = \mathbb{Z}/(4)$. $f \in (1+X) \cap (3+X)$ is associated with either $(1+X)(3+X)g$ or $(3+X)\{(1+X)g + 2\}$. Suppose there exist h such that $(1+X) \cap (3+X) = (h)$. Then, we see a contradiction.

§5. The extension $K_G[X]/k_G[X]$

Let K/k be a field extension. We concern in this section with the properties in the extension $K_G[X]/k_G[X]$.

PROPOSITION 1. *Let K/k be fields, α an ideal of $k_G[X]$. Then, $\alpha K_G[X] \cap k_G[X] = \alpha$.*

PROOF. Let $\{u_\lambda; \Lambda\}$ be a linear basis of K/k ; let $u_{\lambda_0} = 1$. $\{u_\lambda; \Lambda\}$ is linearly independent over $k_G[X]$. Every element of $K_G[X]$ is uniquely expressed as $\sum f_\lambda u_\lambda$, $f_\lambda \in k_G[X]$, where almost all f_λ are zero. $\alpha K_G[X] = \sum \alpha u_\lambda$. Let $f \in \alpha K_G[X] \cap k_G[X]$. $f = \sum f_\lambda u_\lambda$ for $f_\lambda \in \alpha$. $(f_{\lambda_0} - f) + \sum_{\lambda \neq \lambda_0} f_\lambda u_\lambda = 0$. Hence, $f = f_{\lambda_0} \in \alpha$.

PROPOSITION 2. *Let K/k be fields, α, β ideals of $k_G[X]$. Then, $(\alpha \cap \beta) K_G[X] = \alpha K_G[X] \cap \beta K_G[X]$.*

PROOF. Let $F \in \alpha K_G[X] \cap \beta K_G[X]$. Under the notations of the proof of Prop. 1, we set $F = \sum f_\lambda u_\lambda$, $f_\lambda \in \alpha$, $F = \sum g_\lambda u_\lambda$, $g_\lambda \in \beta$. Since $\{u_\lambda; \Lambda\}$ is linearly independent over $k_G[X]$, it follows $f_\lambda = g_\lambda \in \alpha \cap \beta$. Hence, $F \in (\alpha \cap \beta) K_G[X]$.

PROPOSITION 3. *Let $\mathfrak{a}, \mathfrak{b}$ ideals of $k_G[X]$. Then,*

$$(\mathfrak{a} : \mathfrak{b})_{k_G[X]} K_G[X] = (\mathfrak{a} K_G[X] : \mathfrak{b} K_G[X])_{K_G[X]}.$$

PROOF. Let $F \in (\mathfrak{a} K_G[X] : \mathfrak{b} K_G[X])_{K_G[X]}$. With the notations of Prop. 1, $F = \sum f_\lambda u_\lambda$, $f_\lambda \in k_G[X]$. Let $g \in \mathfrak{b}$. $Fg = \sum g_\lambda u_\lambda$ for $g_\lambda \in \mathfrak{a}$. Since $\sum f_\lambda g u_\lambda = \sum g_\lambda u_\lambda$, it follows $f_\lambda g = g_\lambda$. Hence, $f_\lambda \in (\mathfrak{a} : \mathfrak{b})_{k_G[X]}$. $F = \sum f_\lambda u_\lambda \in (\mathfrak{a} : \mathfrak{b})_{k_G[X]} K_G[X]$.

PROPOSITION 4. *Assume $\{t_\lambda; \Lambda\}$ be algebraically independent subset of K over k and $K = k(t_\lambda; \Lambda)$, $\mathfrak{p}$ a prime ideal of $k_G[X]$. Then, (1) $\mathfrak{p} K_G[X]$ is a prime ideal of $K_G[X]$. (2) $\{t_\lambda; \Lambda\}$ is algebraically independent over the quotient field of $k_G[X]/\mathfrak{p}$. (3) The quotient field of $K_G[X]/\mathfrak{p} K_G[X]$ is the field obtained by adjoining $\{t_\lambda; \Lambda\}$ to the quotient field of $k_G[X]/\mathfrak{p}$.*

PROOF. We set $k_G[X] = R$, $K_G[X] = S$, $\mathfrak{p} S = \mathfrak{p}^e$. (1) Let $F' G' \in \mathfrak{p}^e$ for $F', G' \in S$. There exists a finite subset $\{t_1, \dots, t_n\} \subset \{t_\lambda; \Lambda\}$ such that $F', G' \in k(t_1, \dots, t_n)_G[X]$. By Prop. 1, we may suppose Λ is one point; $\{t_\lambda; \Lambda\} = \{t\}$. We may suppose $F', G' \in k[t]_G[X]$. $F' = \sum f_j t^j$, $G' = \sum g_j t^j$ for $f_j, g_j \in R$. Suppose $F' \notin \mathfrak{p}^e$, $G' \in \mathfrak{p}^e$. Set $\min\{j; f_j \notin \mathfrak{p}\} = \alpha$, $\min\{j; g_j \in \mathfrak{p}\} = \beta$. Take a linear base $\{u_\lambda; \Lambda\}$ containing $\{1, t, t^2, \dots\}$ of K/k . $F' G'$ is uniquely written as the form $\sum h_\lambda u_\lambda$, $h_\lambda \in \mathfrak{p}$. Then, $f_\alpha g_\beta \in \mathfrak{p}$. Hence, $f_\alpha \in \mathfrak{p}$ or $g_\beta \in \mathfrak{p}$; a contradiction. (2) Let $F(\mathfrak{p})$, $F(\mathfrak{p}^e)$ the quotient fields of $R/\mathfrak{p}$, $S/\mathfrak{p}^e$ respectively. Suppose $\{t_\lambda; \Lambda\}$ is algebraically dependent over $F(\mathfrak{p})$. There is a finite subset $\{t_1, \dots, t_n\} \subset \{t_\lambda; \Lambda\}$ such that $\{t_1, \dots, t_n\}$ is algebraically dependent over $F(\mathfrak{p})$. Set $K' = k(t_1, \dots, t_n)$, $S' = K'_G[X]$. Then, $\mathfrak{p} S'$ is a prime ideal of S' . Hence, we may suppose Λ is one point; $\{t_\lambda; \Lambda\} = \{t\}$. Let $\bar{f}_m t^m + \dots + \bar{f}_0 = 0$ ($f_j \in R$, $\bar{f}_m \neq 0$) be the minimal equation of t over $F(\mathfrak{p})$. $f_m t^m + f_{m-1} t^{m-1} + \dots + f_0 \in \mathfrak{p}^e$. If we set like $\{u_\lambda; \Lambda\} \supset \{1, t, t^2, \dots\}$ in the proof of Prop. 1, it follows $f_j \in \mathfrak{p}$. A contradiction. (3) is immediate.

PROPOSITION 5. *In Prop. 4, we have $\text{transc. deg}_k F(\mathfrak{p}) = \text{transc. deg}_k F(\mathfrak{p}^e)$.*

The notations are of in the proof of Prop. 4.

PROOF. By Prop. 4, $\{K, F(\mathfrak{p})\}$ is free over k . There is a transcendence basis of $F(\mathfrak{p})/k$ of the form $\{X^\beta + \mathfrak{p}; \beta \in \mathfrak{B}\}$. Then $\{X^\beta + \mathfrak{p}^e; \beta \in \mathfrak{B}\}$ is transcendental over K . Then, $\{X^\beta + \mathfrak{p}^e; \beta \in \mathfrak{B}\}$ is a transcendence basis of $F(\mathfrak{p}^e)/K$.

PROPOSITION 6. *Assume K/k be purely-transcendental extension fields. Let $\mathfrak{q}$ be a $\mathfrak{p}$ -primary ideal of $k_G[X]$. Then, $\mathfrak{q} K_G[X]$ is a $\mathfrak{p} K_G[X]$ -primary ideal of $K_G[X]$.*

PROOF. By Prop. 4, it follows $\sqrt{\mathfrak{q} K_G[X]} = \mathfrak{p} K_G[X]$. We may suppose $K = k(t)$. Let $F' G' \in \mathfrak{q}^e$ for $F', G' \in S$ and $G' \notin \mathfrak{q}^e$. We may suppose $F', G' \in k[t]_G[X]$. By [29] (6.15), $\mathfrak{q} R[t]$ is primary in $R[t]$. Hence, $F'^n \in \mathfrak{q}^e$ for some n .

PROPOSITION 7. *Let K be an extension field of a field k , $\mathfrak{p}$ a prime ideal of $k_G[X]$. Then, (1) $\mathfrak{p}K_G[X]$ is primary in $K_G[X]$ if and only if every zerodivisor of $(k_G[X]/\mathfrak{p}) \otimes_k K$ is nilpotent; (2) $\mathfrak{p}K_G[X]$ is prime in $K_G[X]$ if and only if $(k_G[X]/\mathfrak{p}) \otimes_k K$ is a domain; (3) $\sqrt{\mathfrak{p}K_G[X]} = \mathfrak{p}K_G[X]$ if and only if the nilradical of $(k_G[X]/\mathfrak{p}) \otimes_k K$ is zero.*

PROOF. (1) Adjoin $\otimes_k K$ to the exact sequence $0 \rightarrow \mathfrak{p} \rightarrow k_G[X] \rightarrow k_G[X]/\mathfrak{p} \rightarrow 0$.
 (2) $K_G[X]/\mathfrak{p}K_G[X] \cong (k_G[X]/\mathfrak{p}) \otimes_k K$.

PROPOSITION 8. *Let K/k be fields, $\mathfrak{p}$ a prime ideal of $k_G[X]$. (1) If K/k is a separable extension ([44] p. 113, Def. 2), then $\sqrt{\mathfrak{p}K_G[X]} = \mathfrak{p}K_G[X]$. (2) If the quotient field of $k_G[X]/\mathfrak{p}$ is separable over k , then $\sqrt{\mathfrak{p}K_G[X]} = \mathfrak{p}K_G[X]$. (3) If k is perfect, then $\sqrt{\mathfrak{p}K_G[X]} = \mathfrak{p}K_G[X]$.*

PROOF. (1) By [44] p. 195, Th. 39, the nilradical of $F(\mathfrak{p}) \otimes_k K$ is zero. Since $(R/\mathfrak{p}) \otimes K \subset F(\mathfrak{p}) \otimes K$, the nilradical of $(R/\mathfrak{p}) \otimes K$ is zero. Hence, the nilradical of $S/\mathfrak{p}^e$ is zero.

PROPOSITION 9. *Let k be a field, $\mathfrak{p}$ a prime ideal of $k_G[X]$. Assume $\sqrt{\mathfrak{p}K_G[X]} = \mathfrak{p}K_G[X]$ for every extension field K of k . Then, $F(\mathfrak{p})$ is a separable extension of k .*

PROOF. We may suppose the characteristic of k be $p > 0$. Set $K = k^{1/p}$. $\mathfrak{p}^e = \mathfrak{p}K_G[X]$ is a primary ideal of $K_G[X]$ (Prop. 13). Hence, $\mathfrak{p}^e$ is a prime ideal of $K_G[X]$. K and $k_G[X]/\mathfrak{p}$ are linearly disjoint over k in $K_G[X]/\mathfrak{p}^e$. $F(\mathfrak{p}^e)$ is a common extension of K , $F(\mathfrak{p})$; K and $F(\mathfrak{p})$ are linearly disjoint over k in $F(\mathfrak{p}^e)$. Hence, $F(\mathfrak{p})$ is separable over k .

LEMMA 10. *Let $\mathfrak{q}$ be a primary ideal of $k_G[X]$ and $\sqrt{\mathfrak{q}} = \mathfrak{p}$. Assume G be finitely generated and $\mathfrak{P}$ a prime ideal of $\mathfrak{q}K_G[X] = \mathfrak{q}^e$ in $K_G[X]$. Then, $\mathfrak{P} \cap k_G[X] = \mathfrak{p}$.*

PROOF. $\mathfrak{q}^e : \mathfrak{P} \supseteq \mathfrak{q}^e$. Hence, $\mathfrak{q}^e : \mathfrak{P}^{ee} \supseteq \mathfrak{q}^e$ where $\mathfrak{P}^e = \mathfrak{P} \cap k_G[X]$. By Prop. 3, $\mathfrak{q}^e : \mathfrak{P}^{ee} = (\mathfrak{q} : \mathfrak{P}^e)^e$. Hence, $(\mathfrak{q} : \mathfrak{P}^e)^e \supseteq \mathfrak{q}^e$. By Prop. 1, $\mathfrak{q} : \mathfrak{P}^e \supseteq \mathfrak{q}$. Hence, $\mathfrak{P}^e = \mathfrak{p}$.

PROPOSITION 11. *Let K be a purely-inseparable field extension of k , $\mathfrak{q}$ a primary ideal of $k_G[X]$, G finitely generated. Then, $\mathfrak{q}K_G[X]$ is a primary ideal of $K_G[X]$.*

PROOF. $K_G[X]$ is noetherian. Let $\mathfrak{P}, \mathfrak{P}'$ be prime ideals of $\mathfrak{q}K_G[X]$. Let $F \in \mathfrak{P}$. $F^{p^e} \in k_G[X] \cap \mathfrak{P} = \mathfrak{P}^e$. By Lem. 10, it follows $F^{p^e} \in \mathfrak{p}$. Hence, $F^{p^e} \in \mathfrak{P}'$. Hence, $F \in \mathfrak{P}'$. Hence, $\mathfrak{P} \subset \mathfrak{P}'$. Similarly, $\mathfrak{P} \supset \mathfrak{P}'$. Hence, $\mathfrak{q}K_G[X]$ has only one prime ideal.

LEMMA 12. Assume H be a subgroup of G , $\mathfrak{q}$ a primary ideal of $k_G[X]$. Then, $\mathfrak{q} \cap k_H[X]$ is a primary ideal of $k_H[X]$.

PROPOSITION 13. Let K/k be a purely-inseparable field extension, $\mathfrak{q}$ a primary ideal of $k_G[X]$. Then, $\mathfrak{q}K_G[X] = \mathfrak{q}^e$ is a primary ideal of $K_G[X]$.

PROOF. Let $F'G' \in \mathfrak{q}^e$ for $F', G' \in K_G[X]$, $G' \notin \mathfrak{q}^e$. $F'G' = \sum f_i F_i$ for $f_i \in \mathfrak{q}$, $F_i \in K_G[X]$. Let H be the subgroup generated by the powers of $F', G', f_1, \dots, f_m, F_1, \dots, F_m$. By Lem. 12, $\mathfrak{q} \cap k_H[X]$ is a primary ideal of $k_H[X]$. By Prop. 11, $(\mathfrak{q} \cap k_H[X])K_H[X]$ is a primary ideal of $K_H[X]$. Since $F', G' \in K_H[X]$, $F'G' \in (\mathfrak{q} \cap k_H[X])K_H[X]$, $G' \in (\mathfrak{q} \cap k_H[X])K_H[X]$, it follows $F'^n \in (\mathfrak{q} \cap k_H[X])K_H[X]$ for some n . $F'^n \in \mathfrak{q}^e$. Hence, $\mathfrak{q}^e$ is primary.

PROPOSITION 14. Let $\mathfrak{p}$ be a prime ideal of $k_G[X]$. Assume k is quasi-maximally algebraic in $F(\mathfrak{p})$. Then, $\mathfrak{p}K_G[X]$ is primary for every extension K/k .

The terminology "quasi-maximally algebraic" means that every separably algebraic element of $F(\mathfrak{p})$ over k is contained in k ([45] p. 227).

PROOF. Reduce to the case where G is finitely generated. $k_G[X] = k_Z[X_1, \dots, X_n]$. k is q.m.a. in the quotient field of $k[X_1, \dots, X_n]/(\mathfrak{p} \cap k[X_1, \dots, X_n])$. By [45] V 11, Th. 38, $(\mathfrak{p} \cap k[X_1, \dots, X_n])K[X_1, \dots, X_n]$ is a primary ideal of $K[X_1, \dots, X_n]$. Hence, $\mathfrak{p}K_Z[X_1, \dots, X_n]$ is a primary ideal of $K_Z[X_1, \dots, X_n]$.

PROPOSITION 15. Let $\mathfrak{p}$ be a prime ideal of $k_G[X]$. Assume $\mathfrak{p}^e = \mathfrak{p}K_G[X]$ is primary in $K_G[X]$ for every extension K/k . Then k is quasi-maximally algebraic in $F(\mathfrak{p})$.

PROOF. Let $\alpha \in F(\mathfrak{p})$ be separably algebraic over k . Let $G(T) = T^q + a_{q-1}T^{q-1} + \dots + a_0$ be the minimal polynomial of α over k . Let $c'_1, \dots, c'_q$ be the roots of $G(T)$. Set $K = k(c'_1, \dots, c'_q)$. $\alpha = \bar{A}/\bar{B}$ where $\bar{A}, \bar{B} \in k_G[X]/\mathfrak{p}$ for $A, B \in k_G[X]$. $A^q + a_{q-1}A^{q-1}B + \dots + a_0B^q \in \mathfrak{p} \subset \mathfrak{p}^e$. $\sqrt{\mathfrak{p}^e} = \mathfrak{P}$ is a prime ideal of $K_G[X]$. $(A - c'_1B) \dots (A - c'_qB) \in \mathfrak{p}^e \subset \mathfrak{P}$. Hence, $A - c'_{i_0}B \in \mathfrak{P}$ for some i_0 . For every automorphism σ of K/k , we have $\mathfrak{P}^\sigma = \mathfrak{P}$. Hence, $A - c'_iB \in \mathfrak{P}$ for every i . If $q \geq 2$, it follows $(c'_2 - c'_1)B \in \mathfrak{P}$. Since $\mathfrak{P} \ni c'_2 - c'_1$ (Prop. 1), it follows $B \in \mathfrak{P}$. $B^m \in \mathfrak{P}^e$ for some m . $B^m \in \mathfrak{p}^e \cap k_G[X] = \mathfrak{p}$ (Prop. 1). Hence, $B \in \mathfrak{p}$; a contradiction. Hence, $q = 1$. Hence, $\alpha \in k$.

PROPOSITION 16. Let $\mathfrak{p}$ be a prime ideal of $k_G[X]$. Assume $F(\mathfrak{p})$ be regular over k . Then $\mathfrak{p}K_G[X]$ is prime in $K_G[X]$ for every extension K/k .

PROOF. Reduce to the case where G is finitely generated. $k_G[X] = k_Z[X_1, \dots, X_n]$. Next, reduce to the fact that $(\mathfrak{p} \cap k[X_1, \dots, X_n])K[X_1, \dots, X_n]$ is prime in $K[X_1, \dots, X_n]$ (cf. [45] V 11, Th. 39).

PROPOSITION 17. *Let $\mathfrak{p}$ be a prime ideal of $k_G[X]$. Assume $\mathfrak{p}^e = \mathfrak{p}K_G[X]$ be prime in $K_G[X]$ for every extension K/k . Then, $F(\mathfrak{p})$ is a regular extension of k .*

PROOF. By Prop. 9, $F(\mathfrak{p})$ is a separable extension over k . By Prop. 15, k is quasi-maximally algebraic in $F(\mathfrak{p})$.

§6. The Hilbert

PROPOSITION 1. *If $A_G[X]$ is Hilbert, then A is also Hilbert.*

PROOF. A is a homomorphic image of $A_G[X]$.

PROPOSITION 2. *Assume $\mathfrak{M} \cap A$ is a maximal ideal of A for every maximal ideal $\mathfrak{M}$ of $A_G[X]$. Then, A is a Hilbert ring.*

PROOF. Let $\mathfrak{p}$ be a prime ideal of A and $\mathfrak{p} \ni a \in A$. Take $0 \neq \alpha \in G$. There exists a maximal ideal $\mathfrak{M}$ of $A_G[X]$ containing $(\mathfrak{p}, 1 + aX^\alpha)A_G[X]$. $\mathfrak{M} \ni a$. The maximal ideal $\mathfrak{M} \cap A$ of A does not contain a and contains $\mathfrak{p}$. Hence, $\mathfrak{p}$ is an intersection of maximal ideals.

PROPOSITION 3. *Let A be a Hilbert ring. Assume $M \cap A$ is maximal in A for every free G and for every maximal ideal M of $A_G[X]$. Then $M \cap A$ is maximal in A for any G and for any maximal ideal M of $A_G[X]$.*

PROOF. Consider any G . Let $\{\alpha_\lambda; A\}$ be a maximal independent subset of G . Set $H = \sum_\lambda Z\alpha_\lambda$. Then $A_G[X]$ is integral over $A_H[X]$.

PROPOSITION 4. *Let A be a Hilbert ring. Let M be a maximal ideal of $A_Z[X]$. Then, $A \cap M$ is a maximal ideal of A .*

PROOF. Let D be a Hilbert domain. Let M be a maximal ideal of $D_Z[X]$ such that $D \cap M = 0$. Set $x = \bar{X} \in D_Z[X]/M$, $y = \bar{X}^{-1} \in D_Z[X]/M$. $D[x, y]$ is a field which is algebraic over D . If D has zero pseudoradical, then $D[x, y]$ has also zero pseudoradical. This is impossible. Hence, D has nonzero pseudoradical; that is to say, D is a G -domain. Since D is Hilbert, D is a field.

REMARK 5. Assume D be a Hilbert domain, G be free, M a maximal ideal of $D_G[X]$. Then, $D \cap M$ need not be maximal in D .

A COUNTER EXAMPLE. $X_1, X_2, \dots; T$ indeterminates. Let $\{\phi_1, \phi_2, \phi_3, \dots\}$ be the set of units of $Z[T]_{(T)}$. Define a homomorphism $Z[X_1, X_2, \dots] \rightarrow Z[T]_{(T)}$ by $X_i \mapsto \phi_i$. Extend the homomorphism to $\theta: Z_Z[X_1, X_2, \dots] \rightarrow Z[T]_{(T)}$. Similar with Example 7; θ is a surjection. Set $M = \theta^{-1}(T Z[T]_{(T)})$. $M \cap Z = 0$. We set $D = Z$, $G = \underbrace{Z \oplus Z \oplus Z \oplus \dots}_{\aleph_0}$.

EXAMPLE 6. Let k be a field. Then $k_Q[X]$ is Hilbert.

PROOF. Set $Y = X^1$. $k_Z[Y]$ is 1-dimensional. $k_Q[X]$ is integral over $k_Z[Y]$. Hence, $\dim k_Q[X] = 1$. The Jacobson radical of $k_Q[X]$ is zero ([26] § 3, Th. 3).

EXAMPLE 7. Assume A be a countable ring, $G = \underbrace{Z \oplus Z \oplus \cdots}_{\aleph_0}$. Then, $A_G[X]$ is not Hilbert.

PROOF. At first, let k be a countable field, $X_1, X_2, \dots$; T a set of indeterminates. Let $\{\varphi_1, \varphi_2, \varphi_3, \dots\}$ be the set of units of $k[T]_{(T)}$. We define a homomorphism $\theta: k_Z[X_1, X_2, \dots] \rightarrow k[T]_{(T)}$ by $X_i \mapsto \varphi_i$. $1 - T = \varphi_i$ for some i . $\theta(1 - X_i) = T$. Hence, θ is a surjection. Set $\ker(\theta) = \mathfrak{P}$. $k_Z[X_1, X_2, \dots]/\mathfrak{P} \cong k[T]_{(T)}$. J -rad. $(k_Z[X_1, X_2, \dots]/\mathfrak{P}) \neq 0$. $k_Z[X_1, X_2, \dots]$ is not Hilbert. Suppose $A_G[X]$ be Hilbert. Let $\mathfrak{p}$ be a maximal ideal of A . Then, $(A/\mathfrak{p})_Z[X_1, X_2, \dots] \cong A_G[X]/\mathfrak{p}A_G[X]$ is also a Hilbert ring; a contradiction.

THEOREM 8. Assume G be finitely generated, $\mathfrak{a}$ an ideal of A . Then, $\mathfrak{a}$ is a G -ideal of A if and only if $\mathfrak{M} \cap A = \mathfrak{a}$ for some maximal ideal $\mathfrak{M}$ of $A_G[X]$.

PROOF. (1) For every G -ideal $\mathfrak{P}$ of $A_Z[X]$, $A \cap \mathfrak{P}$ is a G -ideal of A . Set $A[X] \cap \mathfrak{P} = \mathfrak{P}_0$. Denote the total quotient ring by $q(\quad)$. $q(A_Z[X]/\mathfrak{P}) = (A_Z[X]/\mathfrak{P})[\xi_1, \dots, \xi_m]$ for $\xi_i \in q(A_Z[X]/\mathfrak{P})$. $A_Z[X]/\mathfrak{P} = (A[X]/\mathfrak{P}_0)[\bar{X}^{-1}]$, where $\bar{X}^{-1} = X^{-1} + \mathfrak{P}$. Hence, $\mathfrak{P}_0$ is a G -ideal of $A[X]$. By [13] p. 389, 15, $A \cap \mathfrak{P} = A \cap \mathfrak{P}_0$ is a G -ideal of A . (2) Let D be a G -domain. Then, there exists a maximal ideal $\mathfrak{M}$ of $D_Z[X]$ such that $\mathfrak{M} \cap D = 0$. $q(D) = K = D[u^{-1}]$ for $u \in D$. There exists a homomorphism $\theta: D_Z[X] \rightarrow K$, $X \mapsto u^{-1}$. Set $\mathfrak{M} = \ker(\theta)$. (3) Let $\mathfrak{p}$ be a G -ideal of A . Then, there exists a maximal ideal $\mathfrak{M}$ of $A_Z[X]$ such that $\mathfrak{M} \cap A = \mathfrak{p}$. (4) $A_G[X] \cong A_Z[X_1, \dots, X_n]$. The assertion follows from (3), (1).

PROPOSITION 9. Let R be a ring. Assume $M \cap R$ be a G -ideal of R for every free G and for every maximal M of $R_G[X]$. Then, $M \cap R$ is a G -ideal of R for any G and for every maximal M of $R_G[X]$.

PROOF. Consider any G . Let $\{\alpha_\lambda; \lambda\}$ be a maximal independent subset of G . Set $H = \sum_{\lambda} Z\alpha_\lambda$. Let M be a maximal ideal of $R_G[X]$. $R_H[X] \cap M$ is a maximal ideal of $\hat{R}_H[X]$. Hence, $(R_H[X] \cap M) \cap R$ is a G -ideal of R .

REMARK 10. Assume G be free and M a maximal ideal of $A_G[X]$. Then, $M \cap A$ need not be a G -ideal of A .

A COUNTER EXAMPLE. Remark 5.

PROPOSITION 11. Let A be a ring. Assume every G -ideal of A is a contraction of a maximal ideal of $A_G[X]$ for every free G . Then, every G -ideal of A is a contraction of a maximal ideal of $A_G[X]$ for any G .

PROOF. Let $\{\alpha_\lambda; A\}$ be a maximal independent subset of G and let $H = \sum_\lambda Z\alpha_\lambda$. Let I be a G -ideal of A . There exists a maximal ideal M' of $A_H[X]$ such that $M' \cap A = I$. Since $A_G[X]$ is integral over $A_H[X]$, the assertion is OK.

PROPOSITION 12. Assume G be free. Then every G -ideal $\mathfrak{p}$ of A is a contraction of a maximal ideal of $A_G[X]$.

PROOF. $G = \bigoplus_\lambda Z\alpha_\lambda$. Fix $\lambda_0 \in A$. Set $X^{\alpha_{\lambda_0}} = X_{\lambda_0}$. By Th. 8, there exists a maximal ideal M_0 of $A_Z[X_{\lambda_0}]$ such that $M_0 \cap A = \mathfrak{p}$. Set $B = A_Z[X_{\lambda_0}]$. $A_G[X] = B_Z[X_\lambda; \lambda \in A - \{\lambda_0\}]$. Take a maximal ideal $M \supset M_0$ of $A_G[X]$. Then, $M \cap A = \mathfrak{p}$.

PROPOSITION 13. Assume G be finitely generated. Let M be a maximal ideal of $K_G[X]$. Then, $K_G[X]/M$ is algebraic over K .

REMARK 14. Assume G be free, M a maximal ideal of $k_G[X]$. Then, $k_G[X]/M$ need not be algebraic over k .

A COUNTER EXAMPLE. $Q(T) - \{0\} = \{\varphi_1, \varphi_2, \varphi_3, \dots\}$. Define a homomorphism $\theta: Q_Z[X_1, X_2, \dots] \rightarrow Q(T)$ by $X_i \mapsto \varphi_i$. Set $M = \ker(\theta)$. We set $G = \underbrace{Z \oplus Z \oplus \dots}_{\aleph_0}$, $k = Q$.

PROPOSITION 15. Let A be a ring. Assume $\mathfrak{P} \cap A$ be a G -ideal of A for every free G and for every G -ideal $\mathfrak{P}$ of $A_G[X]$. Then, $\mathfrak{P} \cap A$ is a G -ideal of A for any G and for every G -ideal $\mathfrak{P}$ of $A_G[X]$.

PROOF. (1) Let E be a G -domain, D a subdomain such that E is integral over D . Then, D is also a G -domain. There exists $0 \neq b \in E$ such that b is contained in every nonzero prime ideal $\mathfrak{P}$ of E . $b^n + b^{n-1}a_1 + \dots + a_n = 0$ for $a_i \in D$. We may suppose $a_n \neq 0$. Let $\mathfrak{p}$ be a nonzero prime ideal of D . There exists a prime ideal $\mathfrak{P}$ of E over $\mathfrak{p}$. Then, $a_n \in \mathfrak{P} \cap D = \mathfrak{p}$. Hence, D is a G -domain. (2) Take a maximal independent subset $\{\alpha_\lambda; A\}$ of G . Set $H = \sum_\lambda Z\alpha_\lambda$ and let $\mathfrak{P}$ be a G -ideal of $A_G[X]$. $\mathfrak{P} \cap A_H[X] = \mathfrak{P}'$ is a G -ideal of $A_H[X]$ by (1). Hence, $\mathfrak{P}' \cap A = \mathfrak{P} \cap A$ is a G -ideal of A .

PROPOSITION 16. Let R be a ring. Assume $J\text{-rad.}(R/(\mathfrak{P} \cap R)) = 0$ imply $J\text{-rad.}(R_G[X]/\mathfrak{P}) = 0$ for every free G and for every prime $\mathfrak{P}$ of $R_G[X]$. Then, $J\text{-rad.}(R/\mathfrak{P} \cap R) = 0$ implies $J\text{-rad.}(R_G[X]/\mathfrak{P}) = 0$ for any G and for every prime $\mathfrak{P}$ of $R_G[X]$.

PROOF. (1) If a domain E is integral over a subdomain D and if $J\text{-rad.} D$ is zero, then $J\text{-rad.} E$ is zero. (2) Consider a maximal independent subset $\{\alpha_\lambda; A\}$ of G and set $H = \sum_\lambda Z\alpha_\lambda$ and consider $R_H[X]$. The assertion follows from (1).

PROPOSITION 17. Assume G be finitely generated. Let $\mathfrak{P}$ be a prime ideal

of $R_G[X]$. Then, if $J\text{-rad.}(R/(\mathfrak{P} \cap R))$ is zero, then $J\text{-rad.}(R_G[X]/\mathfrak{P})$ is zero.

PROOF. We may suppose $G=Z$. Let D be a domain, $\mathfrak{P}$ a prime ideal of $D[X]$ such that $\mathfrak{P} \cap D=0$. Assume $J\text{-ad.} D$ be zero. We will show $J\text{-rad.}(D_Z[X]/\mathfrak{P})$ is zero. Suppose $J\text{-rad.}(D_Z[X]/\mathfrak{P}) \neq 0$. There exists $g \in \mathfrak{P}$ such that every maximal ideal of $D_Z[X]$ containing $\mathfrak{P}$ contains g . We may suppose $o(g)=0$. By [26] § 3, Th. 3, it follows $\mathfrak{P} \neq 0$. Set $n = \min \{\deg f; 0 \neq f \in \mathfrak{P}, o(f)=0\}$ and take $0 \neq f \in \mathfrak{P}, o(f)=0$ such that $\deg(f)=n$. We introduce the order to G . $f = a_0 + a_1X + \dots + a_nX^n$. f is irreducible in $K[X]$. There exists $0 \neq b \in D$ such that $b \in (f, g)D[X]$. Let $\mathfrak{m}$ be a maximal ideal of D . Suppose $\mathfrak{m} \ni a_0a_n$. Suppose $\mathfrak{m}D_Z[X] + \mathfrak{P} = D_Z[X]$. $1 = p(X) + m(X)$ for $p(X) \in \mathfrak{P}, m(X) \in \mathfrak{m}D_Z[X]$. $a_n^i p(X) \in f(X)D_Z[X]$ for some i . $a_n^i = f(X)h(X) + a_n^i m(X)$ for some $h(X) \in D_Z[X]$. This is a contradiction. Hence, there exists a maximal ideal $\mathfrak{M}$ of $D_Z[X]$ containing $\mathfrak{m}D_Z[X] + \mathfrak{P}$. $b \in \mathfrak{M} \cap D = \mathfrak{m}$. Hence, every maximal ideal of D contains a_0a_nb .

REMARK 18. Assume G is free, $\mathfrak{P}$ a prime ideal of $A_G[X]$ and $J\text{-rad.}(A/(\mathfrak{P} \cap A))$ is zero. Then, $J\text{-rad.}(A_G[X]/\mathfrak{P})$ need not be zero.

A COUNTER EXAMPLE. $A = k = \text{countable fields}$, $G = \underbrace{Z \oplus Z \oplus \dots}_{\aleph_0}$. We confer the proof of Example 7.

§7. $A_G[X]_{\text{monics}}$

Let G be a totally ordered abelian group ($\neq 0$). In this section we concern with the quotient ring $A_G[X]_{\text{monics}}$.

PROPOSITION 1. Let D be a Dedekind domain, S the set of monics of $D_G[X]$, where G is a totally ordered abelian group $\neq 0$. If $D_G[X]_S$ is factorial, then D is also factorial.

PROOF. (1) Every element of D is decomposed into a product of irreducible elements. (2) Every irreducible element p of D is an irreducible element of $D_G[X]_S$. (3) If $a, b \in D - \{0\}$ are associated in $D_G[X]_S$, then they are associated in D . (4) D is factorial.

PROPOSITION 2. Assume G be totally ordered. Then $D_G[X]_{\text{monics}}$ is factorial if and only if D is factorial.

PROOF. The necessity. D satisfies the maximal condition for principal ideals. The rest is similar with Prop. 1. The sufficiency. Similar with the proof of Prop. 12.

PROPOSITION 3. Assume D be a Krull domain, G totally ordered of type

$(0, 0, 0, \dots)$. Then, canonically $C(D) \cong C(D_G[X]_{\text{monics}})$.

PROOF. By [24] Prop. 3.3, $D_G[X]$ is a Krull domain. By [24] Prop. 5.3, canonically $C(D) \cong C(D_G[X])$. By Nagata theorem [28], we have the canonical onto homomorphism: $C(D) \rightarrow C(D_G[X]_{\text{monics}})$. Let $\text{cl}(\mathfrak{p}) \mapsto 0$. $\mathfrak{p}D_G[X]_S = (f/g)D_G[X]_S$, where S is the monics. Let a be the coefficient of f of the highest degree. Then, $\mathfrak{p} = aD$. Hence, $\text{cl}(\mathfrak{p}) = 0$.

PROPOSITION 4. Assume G be totally ordered abelian group $\neq 0$, S the monics of $D_G[X]$ and assume $D_G[X]_S$ be a Krull domain. Then, D is a Krull domain and canonically $C(D) \subset C(D_G[X]_S)$.

PROOF. $D_G[X]$ is a flat D -module ([26] §9, Prop. 1). $D_G[X]_S$ is a faithfully flat D -module. Hence, D is a Krull domain ([12] Prop. 6.10). There exists the canonical homomorphism: $C(D) \rightarrow C(D_G[X]_S)$. Let $\mathfrak{a}$ be a divisorial ideal of D such that $\mathfrak{a}D_G[X]_S = fD_G[X]_S$ is principal. We may suppose $f \in D_G[X]$. Let c be the coefficient of f of the highest degree. Then, $\mathfrak{a} = cD$.

LEMMA 5. Let $f, g \in D_G[X]$. Assume either the leading coefficient or the last coefficient of g is a unit of D and g divide f in $K_G[X]$. Then, g divides f in $D_G[X]$.

PROPOSITION 6. Let D be a Dedekind domain which is not a field. Assume G be totally ordered of type $(0, 0, 0, \dots)$. Let S be the monics of $D_G[X]$. Then, every element of $C(D_G[X]_S)$ is of the form $\text{cl}(\mathfrak{P})$, where $\mathfrak{P}$ is a proper prime ideal of $D_G[X]_S$.

PROOF. Let $0 \neq a$ be a nonunit of D , $\pi \in G$ be of $\text{ht}(0, 0, 0, \dots)$. Suppose $fg \in (1 + aX^\pi)D_G[X]$ for $f, g \in D_G[X]$. By §4, Lem. 1, $1 + aX^\pi$ divides f or g in $K_G[X]$. By Lem. 5, $1 + aX^\pi$ divides f or g in $D_G[X]$. Hence, $(1 + aX^\pi)$ is a prime ideal of $D_G[X]$. $\mathfrak{P} = (1 + aX^\pi)D_G[X]_S$ is a proper prime ideal of $D_G[X]_S$ and $\text{cl}(\mathfrak{P}) = 0$. Next, let $\text{cl}(\mathfrak{Q})$ be a nonzero element of $C(D_G[X]_S)$. By Prop. 3, there exists an ideal $\mathfrak{a}$ of D such that $\text{cl}(\mathfrak{a}D_G[X]_S) = \text{cl}(\mathfrak{Q}^{-1})$. There are a, b such that $\mathfrak{a} = (a, b)$. $\mathfrak{p} = (a + bX^\pi)K_G[X] \cap D_G[X]$ is a prime ideal of $D_G[X]$. By [26] §6, Prop. 10, $\mathfrak{p} = (a + bX^\pi)\mathfrak{a}^{-1}D_G[X]$. $\text{cl}(\mathfrak{p}D_G[X]_S) = \text{cl}(\mathfrak{a}^{-1}D_G[X]_S) = -\text{cl}(\mathfrak{Q}^{-1}) = \text{cl}(\mathfrak{Q})$.

Next, let D be a local principal ideal domain, $G = \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$ with the lexicographic order, S the monics of $D_G[X]$. Set $X^{(\overset{0}{\dots} \overset{0}{\dots} \overset{0}{\dots} \overset{k}{\dots} \overset{1}{\dots} \overset{0}{\dots})} = X_i$ ($i \leq k-1$), $X^{(0, \dots, 0, 1)} = Y$ and set $D_G[X]_S = D_1$. Generally, if a, b of a domain do not have common divisors without units, we say a, b are relatively prime.

LEMMA 7. $F, G \in D[X_1, \dots, X_{k-1}, Y] - \{0\}$. If F, G are relatively prime in $D[X_1, \dots, X_{k-1}, Y]$, then they are relatively prime in D_1 .

LEMMA 8. Suppose D_1 is a PID for $k=1, 2, 3, \dots, k-1$. Let $F, G \in D[X_1, \dots, X_{k-1}, Y]$. Then, $(F, G)D_1$ is principal.

PROOF. We express $X_1, \dots, X_{k-1}$ by X .

$$F = f_0 + f_1 Y + \dots + f_n Y^n,$$

$$G = g_0 + g_1 Y + \dots + g_m Y^m;$$

$f_i, g_j \in D[X]$; $f_n \neq 0 \neq g_m$. We prove by the induction on $n+m$. Suppose the proposition is OK till $\sigma-1$. Let $n+m=\sigma$. Suppose $(F, G)D_1$ is not principal. At first, $F \neq 0 \neq G$. Next, $f_0 \neq 0$. Similarly, $g_0 \neq 0$. Let H be the GCD of F, G in $D[X, Y]$. $F = HF_1, G = HG_1$. By Lem. 7, F_1, G_1 are relatively prime in D_1 . $(F_1, G_1)D_1$ is not principal. Y -degree of F_1 is n , Y -degree of G_1 is m .

$$F_1 = f_0^1 + f_1^1 Y + \dots + f_n^1 Y^n,$$

$$G_1 = g_0^1 + g_1^1 Y + \dots + g_m^1 Y^m;$$

$f_i^1, g_j^1 \in D[X]$. $f_0^1 \neq 0 \neq g_0^1$. There exists a maximal ideal $\mathfrak{M} \supset (F_1, G_1)D_1$. Take an irreducible factor in $D[X, Y]$ of F_1 belonging to $\mathfrak{M}$. Say that F_2 . Take an irreducible factor G_2 in $D[X, Y]$ of G_1 belonging to $\mathfrak{M}$. Let n' be the Y -degree of F_2 , m' the Y -degree of G_2 ; $n' \leq n$. $m' \leq m$. If $n' < n$, $(F_2, G_2)D_1$ is principal. $D_1 = (F_2, G_2)D_1 \subset \mathfrak{M}$; a contradiction. Hence, $n' = n$. Similarly, $m' = m$.

$$F_2 = f_0^2 + f_1^2 Y + \dots + f_n^2 Y^n,$$

$$G_2 = g_0^2 + g_1^2 Y + \dots + g_m^2 Y^m;$$

$f_i^2, g_j^2 \in D[X]$. We may suppose $n \leq m$. Suppose $G_2 f_n^2 - F_2 g_m^2 Y^{m-n} = 0$. Then, $n=m$, $G_2 f_n^2 = F_2 g_m^2$. F_2 is associated with $f' \in D[X]$ in $D[X, Y]$, G_2 is associated with $g' \in D[X]$ in $D[X, Y]$. Hence, $(F_2, G_2)D_1$ is principal, a contradiction. Hence, $G_2 f_n^2 - F_2 g_m^2 Y^{m-n} \neq 0$. Take an irreducible factor H in $D[X, Y]$ of $G_2 f_n^2 - F_2 g_m^2 Y^{m-n}$ belonging to $\mathfrak{M}$. Y -degree of H is $\leq m-1$. Hence, $(F_2, H)D_1$ is principal. $(F_2, H)D_1 = F_2 D_1$. Hence, F_2 divides f_n^2 in D_1 . Hence, F_2 is associated with $f' \in D[X]$ in $D[X, Y]$. Suppose $g_m^2/f' \in D[X]$. $g_m^2 = f' f''$ for $f'' \in D[X]$. Y -degree of $G_2 - f' f'' Y^m$ is $\leq m-1$. Hence, $(F_2, G_2 - f' f'' Y^m)D_1$ is principal. $(F_2, G_2 - f' f'' Y^m)D_1 = F_2 D_1$; a contradiction. Hence, $g_m^2/f' \notin D[X]$. By Lem. 7, $1 = f'(f'''/h) + g_m^2(g'/h)$ for $f''', g', h \in D[X]$, $h \in S$. $G_2 g' + f' f''' Y^m \in S$. A contradiction.

PROPOSITION 9. In the above, $D_G[X]_S$ is a PID for every $k < \infty$.

PROOF. Lem. 8.

THEOREM 10. Let D be a semilocal PID, $G = \underbrace{Z \oplus \dots \oplus Z}_k$ with the lexicographic order, S the monics of $D_G[X]$. Then, $D_G[X]_S$ is a PID.

PROOF. Similar with that of Prop. 9.

PROPOSITION 11. *Let D be a domain. If $D_G[X]_{\text{monics}}$ is a PID for every finitely generated totally ordered G , then, $D_G[X]_{\text{monics}}$ is a PID for any totally ordered G .*

PROOF. Set $D_1 = D_G[X]_{\text{monics}}$. We show first that D_1 is bezout. Let $f, g \in D_G[X]$. Let H be the ordered subgroup of G generated by the powers of f, g . $(f, g)D_H[X]_{\text{monics}} = hD_H[X]_{\text{monics}}$ for some $h \in D_H[X]$. Then, $(f, g)D_1 = hD_1$. Next, we show that D_1 is noetherian. Suppose D_1 is not noetherian. There exists a prime ideal $\mathfrak{P}$ of D_1 which is not finitely generated. Since $D_Z[X]_{\text{monics}}$ is PID, D satisfies the maximal condition on principal ideals. Hence, D satisfies the maximal condition on principal ideals. Take an irreducible element $f \in \mathfrak{P}$. We may suppose $f \in D_G[X]$. $fD_1 \not\subseteq \mathfrak{P}$. There is an irreducible element g of D_1 such that $fD_1 \ni g \in \mathfrak{P} \cap D_G[X]$. $(f, g)D_1 = hD_1$ for some $h \in D_G[X]$. $D_1 = hD_1 = (f, g)D_1 \subset \mathfrak{P}$. A contradiction.

PROPOSITION 12. *Let G be totally ordered. Assume $D_G[X]_{\text{monics}}$ is a PID for every local PID D . Then, $D_G[X]_{\text{monics}}$ is a PID for any semi-local PID D .*

PROOF. Let D be a semi-local PID. Set $D_G[X]_{\text{monics}} = D_1$. (1) At first, D_1 satisfies the maximal condition on principal ideals. (2) Next, D_1 is a factorial domain. Proof. Let f be an irreducible element of D_1 . We may suppose $f \in D_G[X]$. Let $gh \in fD_1$ for $g, h \in D_1$. May suppose $g, h \in D_G[X]$. Then, g or h is divided by f . (3) D_1 is a bezout domain. Proof. Let $f, g \in D_G[X]$. May suppose $f, g \neq 0$. May suppose f, g are relatively prime in D_1 . Let $p_1D, \dots, p_nD$ be the set of maximal ideals of D . $(D_{p_i})_G[X]_{\text{monics}}$ is a PID. Set $S_i = \{f \in D_G[X]; p_i \text{ does not divide the terminal coefficient of } f\}$. $(D_{p_i})_G[X]_{\text{monics}} = D_G[X]_{S_i}$. Since f, g are relatively prime in $D_G[X]_{S_i}$, $(f, g)D_G[X]_{S_i} = D_G[X]_{S_i}$. $ff_i + gg_i = h_i$ for $f_i, g_i \in D_G[X]$, $h_i \in S_i$. May suppose the terminal coefficient of h_i is divided by $p_1, p_2, \dots, p_{i-1}, p_{i+1}, \dots, p_n$. May suppose $\deg h_1 = \deg h_2 = \dots = \deg h_n$. The terminal coefficient of $\sum h_i$ is a unit of D . $(f, g)D_1 = D_1$. (4) D_1 is noetherian. Suppose D_1 is not noetherian. There exists a prime ideal $\mathfrak{P}$ of D_1 which is not finitely generated. Let $f \in D_G[X] \cap \mathfrak{P}$ be an irreducible element of D_1 . $fD_1 \not\subseteq \mathfrak{P}$. There exists an irreducible element g of D_1 such that $fD_1 \ni g \in \mathfrak{P} \cap D_G[X]$. $(f, g)D_1 = hD_1$. $D_1 = (f, g)D_1 \subset \mathfrak{P}$; a contradiction. (5) D_1 is a PID.

EXAMPLE 13. $G = \mathbb{Z} \oplus \mathbb{Z}$ with the lexicographic order. Then, $Z_G[X]_{\text{monics}}$ is 1-dimensional.

PROOF. $(1, 0) < (0, 1)$. $X^{(1,0)} = X$, $X^{(0,1)} = Y$. Let S' be the monics of $Z[X]$. By [13] (40.6), $D = Z[X]_{S'}$ is a Dedekind domain. Let S'' be the monics

of $D[Y]$. $D[Y]_{S''}$ is also Dedekind. $D[Y]_{S''} \cong Z_G[X]_{\text{monics}}$.

PROPOSITION 14. *Let D be a Dedekind domain with infinitely many maximal ideals. $G = \underbrace{Z \oplus \cdots \oplus Z}_n$ with the lexicographic order. Then, $D_G[X]_{\text{monics}}$ is 1-dimensional.*

PROOF. $G(m) = \underbrace{Z \oplus \cdots \oplus Z}_m$. Let $S(m)$ be the monics of $D_{G(m)}[X]$ and set $D_{G(m)}[X]_{S(m)} = D(m)$. By [13] (40.6), $D(1)$ is 1-dimensional. Suppose $D(n')$ is 1-dimensional for $n' < n$. $D(n) = D_{G(n)}[X]_{S(n)} = D(n-1)[X]_{\text{monics}}$.

EXAMPLE 15. $G = \underbrace{Z \oplus Z \oplus \cdots}_{\aleph_0}$ with the lexicographic order. Then, $Z_G[X]_{\text{monics}}$ is 1-dimensional.

PROOF. Let $\mathfrak{P}$ be a ht 1 prime of $Z_G[X]_{\text{monics}}$. $\mathfrak{P} = fZ_G[X]_{\text{monics}}$ for $f \in Z_G[X]$. There exists n such that $f \in Z_H[X]$ for $\underbrace{Z \oplus Z \oplus \cdots \oplus Z \oplus 0 \oplus 0 \oplus \cdots}_n = H$. Set $\underbrace{0 \oplus \cdots \oplus 0 \oplus Z \oplus Z \oplus \cdots}_n = K$ and set $Z_H[X]_{\text{monics}} = D$. By Prop. 14, D is 1-dimensional. $fD = \mathfrak{p}$ is a maximal ideal of D . Set $D/\mathfrak{p} = k$. $Z_G[X]_{\text{monics}}/\mathfrak{P} = D_K[X]_{\text{monics}}/\mathfrak{p}D_K[X]_{\text{monics}} = k_K[X]_{\text{monics}} = \text{field}$. Hence, $\mathfrak{P}$ is maximal.

EXAMPLE 16. Let D be a Dedekind domain with infinitely many maximal ideals, Λ a totally ordered set $\neq \emptyset$, $G_\lambda = Z$, $G = \bigoplus_\lambda G_\lambda$ with the lexicographic order. Then $D_G[X]_{\text{monics}}$ is 1-dimensional.

PROOF. Let $\mathfrak{P}$ be a proper prime of $D_G[X]_{\text{monics}}$. Take $D_G[X] \ni f \in \mathfrak{P}$. Take $\mathfrak{P} \cap D_G[X] \ni g \neq 0$. $f, g \in D_{G'}[X]$ for a finite $\Lambda' \subset \Lambda$ and $G' = \bigoplus_{\Lambda'} G_\lambda$. By Prop. 14, $\mathfrak{p} = \mathfrak{P} \cap D_{G'}[X]_{\text{monics}}$ is a maximal ideal of $D_{G'}[X]_{\text{monics}}$. $\sum f_1 - 1 \in \mathfrak{p}$ for $f_1 \in D_{G'}[X]_{\text{monics}}$. Hence, $\mathfrak{P}$ is maximal.

PROPOSITION 17. *Let D be a Dedekind domain with infinitely many maximal ideals. Assume $D_G[X]_{\text{monics}}$ is 1-dimensional for every finitely generated totally ordered G . Then, $D_G[X]_{\text{monics}}$ is 1-dimensional for any totally ordered G .*

PROOF. Let $\mathfrak{P}$ be a proper prime of $D_G[X]_{\text{monics}}$. Take $\mathfrak{P} \ni f \in D_G[X]$. Take $D_G[X] \cap \mathfrak{P} \ni g \neq 0$. $f, g \in D_H[X]$ for a finitely generated subgroup H of G . $\mathfrak{P} \cap D_H[X]_{\text{monics}} = \mathfrak{p}$ is a maximal ideal of $D_H[X]_{\text{monics}}$. $\sum f_1 - 1 \in \mathfrak{p}$ for $f_1 \in D_H[X]_{\text{monics}}$. Hence, $\mathfrak{P}$ is maximal.

REMARK 18. Let D be a Dedekind domain with class group cyclic of prime order. Assume every proper overring of $D_G[X]_{\text{monics}}$ is bezout for every finitely generated totally ordered G . Then every proper overring of $D_G[X]_{\text{monics}}$ is bezout for any totally ordered G .

§8. The factorial

At first, on the HFD [43].

PROPOSITION 1. *If $D_G[X]$ is a HFD, then D is a HFD.*

Next, on the euclidean rings [41] Def. 1.

PROPOSITION 2. *$A_G[X]$ is a euclidean ring if and only if $G \cong \mathbb{Z}$ and A is a finite direct sum of fields.*

PROOF. The sufficiency. $(K_i)_{\mathbb{Z}}[X]$ is euclidean for fields K_i . Then, $(K_1)_{\mathbb{Z}}[X] \oplus \cdots \oplus (K_n)_{\mathbb{Z}}[X]$ is also euclidean ([41] Prop. 6).

PROPOSITION 3. *Let L/K be fields such that K is algebraically closed in L . Then, every irreducible element of $K_G[X]$ is also irreducible in $L_G[X]$.*

PROOF. (1) $K(X_1, \dots, X_n)$ is algebraically closed in $L(X_1, \dots, X_n)$. (2) Every irreducible element of $K[X]$ is irreducible in $L[X]$ ([13] p. 410, 18). (3) Let f be an irreducible element of $K[X_1, \dots, X_n]$ such that X_n -degree of f is >0 . Then, f is an irreducible element of $K(X_1, \dots, X_{n-1})[X_n]$. (4) If $f, g \in K[X_1, \dots, X_n]$ are associated in $L[X_1, \dots, X_n]$, then they are associated in $K[X_1, \dots, X_n]$. (5) Every irreducible element $f = f(X_1, \dots, X_n)$ of $K[X_1, \dots, X_n]$ is irreducible in $L[X_1, \dots, X_n]$. Suppose f is properly decomposed in $L[X_1, \dots, X_n]$ as $f = FG$. Suppose there exist i such that $\deg_{X_i} F > 0 < \deg_{X_i} G$. May suppose $i = n$. f is an irreducible element of $K(X_1, \dots, X_{n-1})[X_n]$ by (3). Since $K(X_1, \dots, X_{n-1})$ is algebraically closed in $L(X_1, \dots, X_{n-1})$ ((1)), f is an irreducible element of $L(X_1, \dots, X_{n-1})[X_n]$ ((2)). A contradiction. Hence, may suppose $F \in L[X_1, \dots, X_k]$, $G \in L[X_{k+1}, X_{k+2}, \dots, X_n]$. $f = \sum h(e_1, \dots, e_{n-k}) X_{k+1}^{e_1} \cdots X_n^{e_{n-k}}$, $h(\) \in K[X_1, \dots, X_k]$. $b(e_1, \dots, e_{n-k})F = h(e_1, \dots, e_{n-k})$ for $b(\) \in L$. Hence, any two nonzero $h(e_1, \dots, e_{n-k})$, $h(e'_1, \dots, e'_{n-k})$ are associated in $L[X_1, \dots, X_k]$. By (4), they are associated in $K[X_1, \dots, X_k]$. f is divided in $K[X_1, \dots, X_n]$ by $h(e_1, \dots, e_{n-k})$. This is a contradiction. (6) Every irreducible element f of $K_G[X]$ is irreducible in $L_G[X]$. Suppose $f = FG$ in $L_G[X]$ properly. Let H be the subgroup generated by the powers of f, F, G . $H = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$. $X^{\alpha_i} = X_i$. May suppose $\text{ord}_{X_i} f = \text{ord}_{X_i} F = \text{ord}_{X_i} G = 0$. f is an irreducible element of $K[X_1, \dots, X_n]$. By (5), f is irreducible in $L[X_1, \dots, X_n]$. A contradiction.

PROPOSITION 4. *Let L/K be fields such that K is algebraically closed in L . Then every prime element f of $K_G[X]$ is a prime element of $L_G[X]$.*

PROOF. By Prop. 3, f is an irreducible element of $L_G[X]$. Let $FG = fH$ for $F, G, H \in L_G[X]$. Let G' be the subgroup generated by the powers of f, F, G ,

H. Since $L_G[X]$ is factorial, $F \in fL_G[X]$ or $G \in fL_G[X]$. Hence, f is prime in $L_G[X]$.

The next is on the Gaussian [13] p. 351, 7.

PROPOSITION 5. *If $A_G[X]$ is Gaussian, then A is Gaussian.*

PROPOSITION 6. *$D_G[X]$ is a Gaussian domain if and only if D is a field and G is isomorphic into Q .*

PROOF. [13] p. 351, 7(d) and [16].

REMARK 7. Let $(A, \sqrt[3]{0})$ be a local ring, then $A_Z[X]$ need not be Gaussian.

A COUNTER EXAMPLE. $k = \mathbb{Z}/(2)$, $A = k[u]$, $u^2 = 0$. $B = A_Z[X]$. $B[Y] \ni f = g = 1 + X + uY$. $A_f A_g = (1 + X^2)B$. $A_f A_g = (1 + X^2, u + uX)B$. If B is Gaussian, $u + uX \in (1 + X^2)A_Z[X]$. $u + uX = (1 + X^2)(a_1 X^1 + a_{1+1} X^{1+1} + \dots + a_h X^h)$, $a_i \in A$. This is obviously impossible.

PROPOSITION 8. *If $A_G[X]$ is Gaussian, A is von Neumann regular and G is isomorphic into Q .*

PROOF. Suppose A has nonzero nilpotents. There exists $u \neq 0$ such that $u^2 = 0$. Take $0 \neq \alpha \in G$. $(A_G[X])[Y] \ni f = 1 + X^\alpha + uY$, $g = 1 + X^\alpha - uY$. Then, $A_f = (1 + X^\alpha, u) = A_g$, $A_f A_g = ((1 + X^\alpha)^2, u + uX^\alpha)$, $A_f A_g = (1 + X^\alpha)^2$. $u + uX^\alpha = (1 + 2X^\alpha + X^{2\alpha})h$ for $h \in A_G[X]$. This is obviously impossible. Hence, A is of zero nilradical. Let $\mathfrak{p}$ be a prime ideal of A . By Prop. 6, $\mathfrak{p}$ is maximal and $G \subset Q$, because $(A/\mathfrak{p})_G[X]$ is Gaussian.

THEOREM 9. *$A_G[X]$ is Gaussian if and only if A is von Neumann regular and G is isomorphic into Q .*

PROOF. The sufficiency. At first, $\dim A = 0$. Let $\mathfrak{M}$ be a maximal ideal of $A_G[X]$. Set $\mathfrak{M} \cap A = \mathfrak{p}$. $\mathfrak{p}$ is a maximal ideal of A . By [13] p. 113, 29, $A_{\mathfrak{p}}$ is a field. By Prop. 6, $(A_{\mathfrak{p}})_G[X]$ is Gaussian. $\mathfrak{M}' = \{f/s; f \in \mathfrak{M}, s \in A \setminus \mathfrak{p}\}$ is a maximal ideal of $(A_{\mathfrak{p}})_G[X]$. By [13] p. 351, 7, $(A_{\mathfrak{p}})_G[X]_{\mathfrak{M}'}$ is Gaussian. $(A_{\mathfrak{p}})_G[X]_{\mathfrak{M}'} \cong A_G[X]_{\mathfrak{M}}$. Hence, $A_G[X]$ is Gaussian.

If each element of a domain D is a finite product of irreducible elements, D is said to be atomic.

PROPOSITION 10. *If $D_G[X]$ is atomic, then G is of type $(0, 0, 0, \dots)$.*

PROOF. Let $0 \neq \alpha \in G$ and let p be a prime number such that α is divided by every p^n . We will derive a contradiction. Let $1 - X^\alpha = f_1 \cdots f_n$ be an irreducible decomposition. Let H be the subgroup generated by α and the powers of $f_1, \dots, f_n$. $H = \mathbb{Z}\pi \oplus \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_k$, $m\pi = \alpha$. Hence, may suppose $f_i \in D_{\mathbb{Z}\pi}[X]$. May sup-

pose $\alpha(f_i)=0$. $1-X^\pi$ divides some f_i in $K_{Z^\pi}[X]$. By § 7, Lem. 5, $1-X^\pi$ divides f_i in $D_{Z^\pi}[X]$. $f_i=(1-X^\pi)g$. $1-X^\pi$ is reducible in $D_G[X]$. A contradiction. The rest is similar.

The next is on the Fossum's locally factorial [12].

PROPOSITION 11. *If $D_G[X]$ is locally factorial of Fossum, then D is factorial.*

PROPOSITION 12. *If $D_G[X]$ is Fossum's locally factorial, then G is of type $(0, 0, 0, \dots)$.*

PROOF. When D is not a field. Take nonunit $0 \neq b$ of D . $D_G[X]_b = (D_b)_G[X]$. By [15], G is of type $(0, 0, 0, \dots)$. When $D=K$ is a field. Let $0 \neq \alpha \in G$ and p a prime number. Suppose $p^{n+1}\alpha_{n+1} = \alpha$ is solvable for all n . $g = 1 - X^\alpha + X^{2\alpha}$. $(1 - X^\alpha) \subset (1 - X^{\alpha/p}) \subset (1 - X^{\alpha/p^2}) \subset \dots$. Since $K_G[X]_g$ is factorial, $1 - X^{\alpha/p^{n+1}} = (1 - X^{\alpha/p^n})(f/g^n)$ for some n and $f \in K_G[X]$. $g^n = (1 + X^{\alpha/p^{n+1}} + X^{2\alpha/p^{n+1}} + \dots + X^{(p-1)\alpha/p^{n+1}})f$. $g, 1 - X^\alpha$ have a nonunit common divisor. This is obviously impossible. The rest is similar.

THEOREM 13. *If $D_G[X]$ is Fossum's locally factorial, then $D_G[X]$ is factorial.*

PROOF. Prop. 11, Prop. 12, [15].

PROPOSITION 14. *Assume D_m is factorial for every maximal ideal m of D , G be of type $(0, 0, 0, \dots)$. Then, $D_G[X]_{\mathfrak{M}}$ is factorial for every maximal ideal $\mathfrak{M}$ of $D_G[X]$.*

PROOF. Set $\mathfrak{M} \cap D = \mathfrak{p}$, $D_G[X] \setminus \mathfrak{M} = T$. Since $D_{\mathfrak{p}}$ is factorial, $(D_{\mathfrak{p}})_G[X]$ is factorial. Hence, $(D_{\mathfrak{p}})_G[X]_T$ is factorial. Hence, $D_G[X]_{\mathfrak{M}}$ is factorial.

PROPOSITION 15. *Assume $D_G[X]_{\mathfrak{M}}$ be factorial for every maximal ideal $\mathfrak{M}$ of $D_G[X]$. Then, $D_{\mathfrak{p}}$ is factorial for every prime ideal $\mathfrak{p}$ of D .*

PROOF. Let $\mathfrak{p}$ be a prime ideal of D . $D_G[X]_{\mathfrak{p}D_G[X]}$ is factorial. $D_G[X]_{\mathfrak{p}D_G[X]} = (D_{\mathfrak{p}})_G(X)_{\text{loc}}$. By § 13, Prop. 4, $D_{\mathfrak{p}}$ is factorial.

PROPOSITION 16. (1) *If D is a factorial local domain, then $D_G(X)_{\text{loc}}$ is factorial.* (2) *Let $\mathfrak{p}$ be a prime ideal of D . Then, $D_{\mathfrak{p}}$ is factorial if and only if $D_G[X]_{\mathfrak{p}D_G[X]}$ is factorial.*

PROOF. (1) By § 7, Prop. 2, $D_G[X]_{\text{monics}}$ is factorial (G becomes totally ordered). $D_G(X)_{\text{loc}}$ is a quotient of $D_G[X]_{\text{monics}}$. Hence, $D_G(X)_{\text{loc}}$ is factorial. (2) The necessity. By (1), $(D_{\mathfrak{p}})_G[X]_{\mathfrak{p}(D_{\mathfrak{p}})_G[X]}$ is factorial. The sufficiency. § 13, Prop. 4.

REMARK 17. Assume $Q_G[X]_M$ be factorial for every maximal ideal M of $Q_G[X]$, then G need not be of type $(0, 0, 0, \dots)$.

A COUNTER EXAMPLE. G in § 10, Remark 3.

§ 9. The chain conditions

At first, on the saturated chain conditions ([13] p. 383).

PROPOSITION 1. *If $A_G[X]$ satisfies the saturated chain condition, A satisfies the saturated chain condition.*

PROPOSITION 2. *$A_G[X]$ satisfies the saturated chain condition if and only if $(A/\mathfrak{p})_G[X]$ satisfies the saturated chain condition for every minimal prime ideal $\mathfrak{p}$ of A .*

The next is on the catenary ([20] p. 23).

PROPOSITION 3. *If $A_G[X]$ is catenary, then A is catenary.*

The next is on the chain condition for prime ideals in noetherian domains [45] p. 326.

PROPOSITION 4. *If the chain condition for prime ideals in noetherian $D_G[X]$ holds, then so is in D .*

PROOF. Similar with that of Prop. 7.

The next is on the H-domain ([38]).

REMARK 5. Even if $D_G[X]$ is an H-domain, D need not be an H-domain.

A COUNTER EXAMPLE. Take a non-H-domain D . $G = G_1 \oplus G_2 \oplus G_3 \oplus \dots$, $G_i = \mathbb{Z}$. $\dim D_G[X] = \infty$. Let $\mathfrak{P}$ be a ht 1 prime of $D_G[X]$ and set $\mathfrak{p} = D \cap \mathfrak{P}$. When $\mathfrak{p} \neq 0$. $D_G[X]/\mathfrak{P} \cong (D/\mathfrak{p})_G[X]$. $\dim D_G[X] = \dim D_G[X]/\mathfrak{P} + 1$. Next, the case $D \cap \mathfrak{P} = 0$. Take an irreducible element f of $K_G[X]$ belonging to $\mathfrak{P}$. Let $h \in D_G[X] \cap fK_G[X]$. Then, $D_G[X] \cap fK_G[X] = \mathfrak{P}$. Set $X^{(0, \dots, 0, \frac{1}{i}, 0, \dots)} = X_i$. $D_G[X] = D_{\mathbb{Z}}[X_1, X_2, \dots]$. May suppose $f \in D[X_1, \dots, X_n]$. Set $(\mathfrak{P}, 1 - X_{n+1}, 1 - X_{n+2}, \dots, 1 - X_{n+m}) = \mathfrak{P}_m$. Then, $\mathfrak{P}_m$ is a prime ideal of $D_{\mathbb{Z}}[X_1, X_2, \dots]$. Hence, $\dim D_G[X]/\mathfrak{P} = \infty$. $\dim D_G[X] = \dim D_G[X]/\mathfrak{P} + 1$. Hence, $D_G[X]$ is an H-domain.

The next is on the equidimensional [20] (0, 16.1.4).

REMARK 6. Even if $A_G[X]$ is equidimensional, A need not be equidimensional.

A COUNTER EXAMPLE. $A = \mathbb{Q} \oplus \mathbb{Z}$. $G = \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \dots}_{\aleph_0}$. A is not equidimen-

sional. Let $\mathfrak{P}$ be a minimal prime ideal of $A_G[X]$. $\mathfrak{p} = \mathfrak{P} \cap A$ is a minimal prime of A . $\mathfrak{P} = \mathfrak{p}A_G[X]$. $\dim A_G[X]/\mathfrak{p}A_G[X] = \infty$. Hence, $A_G[X]$ is equidimensional.

The next is on the biequidimensional [20] (0, 16.1.4).

PROPOSITION 7. *If $A_G[X]$ is noetherian and biequidimensional, then so is A .*

PROOF. We may suppose $G = Z$. $\infty > n = \dim A_Z[X]$. Suppose A is not biequidimensional. There exist two maximal chains of prime ideals containing $1 - X$: $\mathfrak{P}_0 \subsetneq \mathfrak{P}_1 \subsetneq \cdots \subsetneq \mathfrak{P}_k$, $\mathfrak{Q}_0 \subsetneq \mathfrak{Q}_1 \subsetneq \cdots \subsetneq \mathfrak{Q}_l$ such that $k \neq l$. Set $\mathfrak{P}_k \cap A = \mathfrak{p}$. If $\mathfrak{P}_k$ is a minimal prime of $A_Z[X]$, $\mathfrak{P}_k = \mathfrak{p}A_Z[X]$. It follows $\mathfrak{p} \ni 1$; a contradiction. Hence, there exists a prime ideal $\mathfrak{P}_{k+1} \subsetneq \mathfrak{P}_k$. By the altitude theorem of Krull ([29] (9.3)), $\text{ht } \mathfrak{P}_k \leq 1$. Hence, $\dim A_Z[X] = k + 1$. Similarly, $\dim A_Z[X] = l + 1$. Hence, $k + 1 = l + 1$; a contradiction.

PROPOSITION 8. *If $A_G[X]$ satisfies the chain condition for primes of [29], then so does A .*

The next is on the dimension formula [45].

PROPOSITION 9. *Consider the proposition, "if $D_G[X_1, \dots, X_n]$ satisfies the chain condition for primes of [45] for every n , then D satisfies the dimension formula". If the proposition holds for $G = Z$, then the proposition holds generally.*

PROOF. Prop. 4.

PROPOSITION 10. *If the dimension formula for noetherian $D_G[X]$ holds, then also holds for D .*

PROOF. (1) For every noetherian local domain $(D, \mathfrak{p})$, $\text{ht}(\mathfrak{p}D_Z[X]) = \text{ht } \mathfrak{p}$.
 (2) Let $\mathfrak{p}$ be a prime ideal in a noetherian domain D . Then $\text{ht}(\mathfrak{p}D_Z[X]) = \text{ht } \mathfrak{p}$.
 (3) Proof of Prop. 10. By (2).

THEOREM 11. *The dimension formula for noetherian $D_G[X]$ holds if and only if it holds for the noetherian D .*

PROOF. The sufficiency. We may suppose $G = Z$.

The next is on the taut [39].

REMARK 12. Even if $A_Z[X]$ is taut, A need not be taut.

A COUNTER EXAMPLE. $A = Q \oplus Z$.

REMARK 13. Assume $\dim k_G[X] = 1$. Then, $k_G[X]$ need not be finitely generated ring extension over k .

A COUNTER EXAMPLE. $G = Q$.

PROPOSITION 14. If $\dim k_G[X] = 1$, G is of type $(0, 0, 0, \dots)$, then $k_G[X]$ is finitely generated ring extension over k .

PROOF. $(1 - X^\alpha; \alpha \neq 0)$ is a prime ideal of $k_G[X]$. Let π be of $\text{ht}(0, 0, 0, \dots)$. $1 - X^\pi$ is a prime element of $k_G[X]$ ([24] (2.2)). Hence, $(1 - X^\alpha; \alpha \neq 0) = (1 - X^\pi)$. Hence, $G = Z\pi$. $k_G[X] = k[X^\pi, X^{-\pi}]$.

PROPOSITION 15. $k_G[X]$ is 1-dimensional if and only if $G \subseteq Q$.

PROPOSITION 16. Let A be noetherian. Then, $\dim A_G[X] = \dim A + \text{torsion-free rank of } G$.

PROOF. Confer § 3, Lem. 10.

§ 10. The finite characters

If D_p is a Krull domain for every prime ideal p of D , D is said to be an almost Krull domain.

PROPOSITION 1. If $D_G[X]$ is almost Krull, then so is D .

PROPOSITION 2. If D is almost Krull, G of the type $(0, 0, 0, \dots)$, then $D_G[X]$ is almost Krull.

PROOF. Let $\mathfrak{P}$ be a prime ideal of $D_G[X]$, $\mathfrak{P} \cap D = p$, $D_G[X] \setminus \mathfrak{P} = M$. $(D_p)_G[X]$ is a Krull domain ([24] (3.3)). Hence, $D_G[X]_{\mathfrak{P}} = (D_p)_G[X]_M$ is a Krull domain.

REMARK 3. Even if $Q_G[X]$ is almost Krull, G need not be of type $(0, 0, 0, \dots)$.

A COUNTER EXAMPLE. $G = \{b/2^a; a, b \in \mathbb{Z}\}$. If $\mathfrak{P}$ is a nonzero prime of $Q_G[X]$, $Q_G[X]_{\mathfrak{P}}$ is a regular local ring.

The next is on the generalized Krull domain ([13] p. 524).

PROPOSITION 4. If D is a generalized Krull domain, G of type $(0, 0, 0, \dots)$, then $D_G[X]$ is a generalized Krull domain.

PROOF. Let $\{v_\lambda; \lambda\}$ be a family of valuations on K defining the generalized Krullness of D . Let v'_λ be the trivial extension of v_λ to $K_G(X)$. Let $\{f_\sigma; \sigma\}$ be a complete representatives of mutually non-associated irreducible elements of $K_G[X]$. Let w'_σ be the f_σ -adic valuation on $K_G(X)$. Then, $\{v'_\lambda, w'_\sigma; \lambda, \sigma\}$ defines the generalized Krullness of $D_G[X]$.

PROPOSITION 5. If $D_G[X]$ is a generalized Krull domain, then so is D .

REMARK 6. Even if $D_G[X]$ is a generalized Krull domain, G need not be of the type $(0, 0, 0, \dots)$.

A COUNTER EXAMPLE. $D=K=Z/(p)$. $G=\{a/p^n; a, n \in Z\}$. Let $\{f_\lambda; A\}$ be the complete representatives of mutually non-associated irreducible elements of $K_Z[X]$. Let v_λ be the f_λ -adic valuation on $K(X)$. We extend v_λ to V_λ of $K_G(X)$ canonically. $\{V_\lambda; A\}$ determines the generalized Krullness of $D_G[X]$.

PROPOSITION 7. Assume the characteristic of D is zero. Then, $D_G[X]$ is a generalized Krull domain if and only if D is a generalized Krull domain and G is of the type $(0, 0, 0, \dots)$.

PROOF. The necessity. $K_G[X]$ is also a generalized Krull domain ([13] (43.6)). Let $p_1 < p_2 < p_3 < \dots$ be a sequence of prime numbers and $0 \neq \alpha \in G$ such that p_i divides α . Let $\{v_\lambda; A\}$ be the defining family of valuations of $K_G[X]$. The number n of the v_λ such that $v_\lambda(1 - X^\alpha) > 0$ is finite. It is impossible that $v_\lambda(1 - X^{\alpha/p_1}) > 0$, $v_\lambda(1 + X^{\alpha/p_1} + X^{2\alpha/p_1} + \dots + X^{(p_1-1)\alpha/p_1}) > 0$. Hence, the number of the v_λ such that $v_\lambda(1 - X^{\alpha/p_1}) > 0$ is at most $n-1$. Like this, we see that $1 - X^{\alpha/(p_1 \cdots p_n)}$ is a unit of $K_G[X]$. This is a contradiction. The rest is similar.

The next is on the Krull type ([13] p. 537).

PROPOSITION 8. If D is Krull type, G of type $(0, 0, 0, \dots)$, then $D_G[X]$ is of Krull type.

PROPOSITION 9. If $D_G[X]$ is Krull type, then D is Krull type.

PROOF. Let $\{v_\lambda; A\}$ be the defining family of valuations on $D_G[X]$. Then, $\{v_\lambda|K; A\}$ defines the Krull type of D .

REMARK 10. Even if D is Krull type, $D_G[X]$ need not be Krull type.

A COUNTER EXAMPLE. $D=Z$ or $D=Q$. $G=Q$. Suppose $D_G[X]$ is Krull type. $\{v_\lambda; A\}$ is the defining family. The number of the v_λ such that $v_\lambda(1 - X) > 0$ is finite n . The number of the v_λ such that $v_\lambda(1 - X^{1/2}) > 0$ is at most $n-1$. Like this, the number of the v_λ such that $v_\lambda(1 - X^{1/2^n})$ is zero. Hence, $1 - X^{1/2^n}$ is a unit. This is impossible.

PROPOSITION 11. Assume the characteristic of D is 0. Then, $D_G[X]$ is Krull type if and only if D is Krull type and G is of type $(0, 0, 0, \dots)$.

PROOF. Prop. 8, Prop. 9, [13] p. 537.

The next is on the finite character of rational type [6] § 1, exerc. 28).

PROPOSITION 12. If $D_G[X]$ is of finite character of rational type, then so is

D.

PROPOSITION 13. *If D is of finite character of rational type, G of type $(0, 0, 0, \dots)$, then $D_G[X]$ is of finite character of rational type.*

PROPOSITION 14. *Let the characteristic of D is zero. Then, $D_G[X]$ is of finite character of rational type if and only if D is of finite character of rational type and G is of type $(0, 0, 0, \dots)$.*

PROOF. The necessity. By Prop. 12, D is of finite character of rational type. $K_G[X]$ is also of finite character of rational type ([13] (43.6)). The rest is similar with that of Prop. 7.

The next is on the finite real character [13] p. 524.

PROPOSITION 15. *If $D_G[X]$ is of finite real character, then so is D .*

PROPOSITION 16. *If D is of finite real character and G of type $(0, 0, 0, \dots)$, then $D_G[X]$ is of finite real character.*

PROPOSITION 17. *Let the characteristic is zero. Then, $D_G[X]$ is of finite real character if and only if so is D and G be of type $(0, 0, 0, \dots)$.*

PROOF. The sufficiency follows from Prop. 16.

The next is on the K domains ([35] DEF. 1.1).

PROPOSITION 18. *If $D_G[X]$ is a K domain, then so is D .*

PROOF. Let $\{v'_\lambda; A\}$ be the defining family of valuations of $D_G[X]$. Set $v'_\lambda|K = v_\lambda$. We will show that the center $P(v)$ of $v = v'_\lambda|K$ on D is divisorial. Set $v'_\lambda = v'$. $P(v') = \bigcap (f_\mu/g_\mu)D_G[X]$. Take $0 \neq a \in P(v)$. $a = (f_\mu/g_\mu)h_\mu$ for some h_μ . $P(v') = \bigcap_M (a/h_\mu)D_G[X]$. $h_\mu = \sum b_{\mu\alpha}X^\alpha$. Then, $P(v) = P(v') \cap D = \bigcap_{0 \neq b_{\mu\alpha}} (a/b_{\mu\alpha})D$.

PROPOSITION 19. *Assume D be a K domain, G of type $(0, 0, 0, \dots)$. Then, $D_G[X]$ is a K domain.*

PROOF. Let $\{v_\lambda; A\}$ be the defining family of valuations of D . Let w_λ be the trivial extension of v_λ to $K_G(X)$. Let $\{f_\sigma; \Sigma\}$ be the complete representatives of mutually non-associated irreducible elements of $K_G[X]$ and let u_σ be the f_σ -adic valuation. Let W_λ, U_σ be the valuation rings of w_λ, u_σ respectively. Then, we see that $\bigcap_{\lambda_0 \neq \lambda} W_\lambda \bigcap_\Sigma U_\sigma \not\subset W_{\lambda_0}, \bigcap_\lambda W_\lambda \bigcap_{\sigma_0 \neq \sigma} U_\sigma \not\subset U_{\sigma_0}$. By [35] Prop. 1.7, $D_G[X]$ is a K domain.

EXAMPLE 20. $Q_Q[X]$ is not a K domain.

PROOF. If $Q_Q[X]$ is a K domain, $Q_Q[X]$ becomes a Dedekind domain ([33], [34] p. 433). This is impossible.

The next is on the finite character ([13] p. 524).

PROPOSITION 21. *If $D_G[X]$ is of finite character, then so is D .*

PROPOSITION 22. *If D is of finite character, G of type $(0, 0, 0, \dots)$, then $D_G[X]$ is of finite character.*

PROPOSITION 23. *Assume the characteristic of D is zero. Then, $D_G[X]$ is of finite character if and only if so is D and G is of type $(0, 0, 0, \dots)$.*

PROOF. Prop. 22, Prop. 21, [13] (43.6).

PROPOSITION 24. *Assume the characteristic of D is $p > 0$, $D_G[X]$ be of finite character, $G \ni \alpha \neq 0$, q a prime number $\neq p$. Then,*

- (1) *An infinite number of prime numbers do not divide α .*
- (2) *q^n does not divide α for some n .*

PROOF. A little modification of the proof of Prop. 7.

The next is on the Krull ring [22].

PROPOSITION 25. *If $A_G[X]$ is a Krull ring, then A is a Krull ring or the total quotient ring of A is A .*

REMARK 26. Even if A is a Krull ring, $A_Z[X]$ need not be a Krull ring.

A COUNTER EXAMPLE. $A = \mathbb{Z} \oplus \mathbb{Z}/(4)$.

§11. The r-noetherian

LEMMA 1. *Assume G be finitely generated and $G \ni \{\alpha_1, \dots, \alpha_{n+1}\}$ is independent. Then, $1 - X^{\alpha_{n+1}} \in \sqrt{(1 - X^{\alpha_1}, \dots, 1 - X^{\alpha_n})A_G[X]}$.*

PROOF. There exist $m_1 > 0$ and a $\text{ht}(0, 0, 0, \dots)$ element β_1 such that $m_1\beta_1 = \alpha_1$. $\{\beta_1, \alpha_2, \dots, \alpha_{n+1}\}$ is independent. $(1 - X^{\alpha_1}, 1 - X^{\alpha_2}, \dots, 1 - X^{\alpha_n}) \subset (1 - X^{\beta_1}, 1 - X^{\alpha_2}, \dots, 1 - X^{\alpha_n})$. By [24] (2.1), $Z\beta_1$ is a direct summand of $G = H_1$; $G = Z\beta_1 \oplus H_2$. Next, $\alpha_2 = m_{21}\beta_1 + m_{22}\beta_2$, β_2 is a $\text{ht}(0, 0, 0, \dots)$ element of H_2 , $m_{ij} \in \mathbb{Z}$. $\{\beta_1, \beta_2, \alpha_3, \dots, \alpha_{n+1}\}$ is independent. $(1 - X^{\beta_1}, 1 - X^{\alpha_2}, \dots, 1 - X^{\alpha_n}) \subset (1 - X^{\beta_1}, 1 - X^{\beta_2}, 1 - X^{\alpha_3}, \dots, 1 - X^{\alpha_n})$. $G = Z\beta_1 \oplus Z\beta_2 \oplus H_3$. Next, $\alpha_3 = m_{31}\beta_1 + m_{32}\beta_2 + m_{33}\beta_3$, β_3 is a $\text{ht}(0, 0, 0, \dots)$ element of H_3 . $\{\beta_1, \beta_2, \beta_3, \alpha_4, \dots, \alpha_{n+1}\}$ is independent. $(1 - X^{\beta_1}, 1 - X^{\beta_2}, 1 - X^{\alpha_3}, \dots, 1 - X^{\alpha_n}) \subset (1 - X^{\beta_1}, 1 - X^{\beta_2}, 1 - X^{\beta_3}, 1 - X^{\alpha_4}, \dots, 1 - X^{\alpha_n})$. $G = Z\beta_1 \oplus Z\beta_2 \oplus Z\beta_3 \oplus H_4$. Continue this process. In n -times, we have $\alpha_n = m_{n1}\beta_1 + m_{n2}\beta_2 + \dots + m_{nn}\beta_n$, β_n is a $\text{ht}(0, 0, 0, \dots)$ element of H_n . $\{\beta_1, \dots, \beta_n, \alpha_{n+1}\}$ is independent.

$$(1 - X^{\beta_1}, \dots, 1 - X^{\beta_{n-1}}, 1 - X^{\alpha_n}) \subset (1 - X^{\beta_1}, \dots, 1 - X^{\beta_{n-1}}, 1 - X^{\beta_n}).$$

$$G = Z\beta_1 \oplus \cdots \oplus Z\beta_n \oplus H_{n+1}.$$

Next, $\alpha_{n+1} = m_{n+1,1}\beta_1 + \cdots + m_{n+1,n+1}\beta_{n+1}$, β_{n+1} is a $\text{ht}(0, 0, 0, \dots)$ element of H_{n+1} .

$$G = Z\beta_1 \oplus \cdots \oplus Z\beta_{n+1} \oplus H_{n+2}$$

$$= Z\beta_1 \oplus \cdots \oplus Z\beta_{n+1} \oplus Z\beta_{n+2} \oplus \cdots \oplus Z\beta_{n+l}.$$

Suppose $1 - X^{\alpha_{n+1}} \in \sqrt{(1 - X^{\alpha_1}, \dots, 1 - X^{\alpha_n})}$. For some $k > 0$,

$$(1 - X^{\alpha_{n+1}})^k \in (1 - X^{\alpha_1}, \dots, 1 - X^{\alpha_n}) \subset (1 - X^{\beta_1}, \dots, 1 - X^{\beta_n}).$$

It follows $(1 - X^{m_{n+1,1} + \dots + m_{n+1,n+1}})^k \in (1 - X^{\beta_1}, \dots, 1 - X^{\beta_n})$. Setting $X^{\beta_i} = X_i$, $(1 - X^{m_{n+1,1} + \dots + m_{n+1,n+1}})^k = (1 - X_1)g_1 + \cdots + (1 - X_n)g_n$ for $g_i \in A_Z[X_1, \dots, X_n]$. This is obviously impossible.

LEMMA 2. Assume $G \supset \{\alpha_1, \dots, \alpha_{n+1}\}$ is independent. Then, in $A_G[X]$, $1 - X^{\alpha_{n+1}} \notin \sqrt{(1 - X^{\alpha_1}, \dots, 1 - X^{\alpha_n})}$.

PROOF. Reduce to the case where G is finitely generated.

The next is on the noetherian spectrum [19] (0, 2.2.1).

PROPOSITION 3. If $\text{Spec}(A_G[X])$ is noetherian, then G is of finite torsion-free rank.

PROOF. Suppose the contrary. There exists $G \supset \{\alpha_1, \alpha_2, \alpha_3, \dots\}$ which is independent. $\mathfrak{A}_n = (1 - X^{\alpha_1}, \dots, 1 - X^{\alpha_n})$. By Lem. 2, it follows $V(\mathfrak{A}_1) \supsetneq V(\mathfrak{A}_2) \supsetneq \cdots$.

PROPOSITION 4. If $\text{Spec}(A_G[X])$ is noetherian, then $\text{Spec}(A)$ is noetherian.

REMARK 5. Even if $\text{Spec}(A)$ is noetherian, $\text{Spec}(A_G[X])$ need not be so.

A COUNTER EXAMPLE. $A = k$, $\text{ch. } k = 0$. $k_Q[X] \supset \sqrt{1 - X^{1/2^i}} = \mathfrak{A}_i$. $\mathfrak{A}_i \subset \mathfrak{A}_{i+1}$. Suppose $\mathfrak{A}_i = \mathfrak{A}_j$, $i < j$. $1 - X^{1/2^{i+1}} \in \sqrt{1 - X^{1/2^i}}$. $(1 - X^{1/2^{i+1}}) = (1 - X^{1/2^i})f$. $(1 - X^{1/2^{i+1}})^{n-1} = (1 + X^{1/2^{i+1}})f$. $Y = X^{1/m}$ for some m . $(1 - Y^a)^{n-1} = (1 + Y^a)g(Y)$, $g(Y) \in k[Y^a, Y^{-a}]$. This is impossible.

An ideal $\mathfrak{a}$ of A is said to be a regular ideal if $\mathfrak{a}$ contains a non-zero-divisor of A . If A satisfies the maximal condition on regular ideals, A is said to be r -noetherian.

PROPOSITION 6. If $A_G[X]$ is r -noetherian, the torsion-free rank of G is finite.

PROOF. If $G \supset \{\alpha_1, \alpha_2, \alpha_3, \dots\}$ is independent, set $(1 - X^{\alpha_1}, \dots, 1 - X^{\alpha_n}) = \mathfrak{a}_n$. By Lem. 2, $\mathfrak{a}_n \subsetneq \mathfrak{a}_{n+1}$.

$$e_1 = e'_{\sigma(1)}, e_2 = e'_{\sigma(1)}e'_{\sigma(2)}, e_3 = e'_{\sigma(1)}e'_{\sigma(2)}e'_{\sigma(3)}, \dots, e_n = e'_{\sigma(1)} \cdots e'_{\sigma(n)}; F = Z\gamma_1 + \cdots + Z\gamma_n.$$

$$\begin{cases} \theta_1 = \beta_1 + a_1\beta_k, \\ \theta_2 = \beta_2 + a_2\beta_k, \\ \dots\dots\dots \\ \theta_{k-1} = \beta_{k-1} + a_{k-1}\beta_k, \\ \theta_k = a_k\beta_k, \\ \theta_{k+1} = \beta_{k+1}, \\ \dots\dots\dots \\ \theta_n = \beta_n. \end{cases}$$

Suppose $\alpha_{H,G} = \alpha_{K,G}$. Then, $\alpha_{H,G} = \alpha_{E,G} = \alpha_{K,G}$; $\alpha_{E,F} = \alpha_{K,F}$. For $f_i \in A_F[X]$, it follows

$$1 - X^{\beta_k} = (1 - X^{\theta_1})f_1 + \cdots + (1 - X^{\theta_n})f_n.$$

Set $X^{\gamma_i} = X_i$.

$$\begin{aligned} 1 - X_k^{e_k} &= (1 - X_1^{e_1} X_k^{a_1 e_k})f_1 + \cdots + (1 - X_{k-1}^{e_{k-1}} X_k^{a_{k-1} e_k})f_{k-1} \\ &\quad + (1 - X_k^{a_k e_k})f_k + (1 - X_{k+1}^{e_{k+1}})f_{k+1} + \cdots + (1 - X_n^{e_n})f_n. \end{aligned}$$

There are a monomial M of $X_1, \dots, X_n$ and $F_i \in A[X_1, \dots, X_n]$ such that

$$\begin{aligned} M(1 - X_k^{e_k}) &= (1 - X_1^{e_1} X_k^{a_1 e_k})F_1 + \cdots + (1 - X_{k-1}^{e_{k-1}} X_k^{a_{k-1} e_k})F_{k-1} \\ &\quad + (1 - X_k^{a_k e_k})F_k + (1 - X_{k+1}^{e_{k+1}})F_{k+1} + \cdots + (1 - X_n^{e_n})F_n. \end{aligned}$$

Substitute as follows $X_1 \rightarrow X_k^{(a_k - a_1)e'_{\sigma(2)} \cdots e'_{\sigma(k)}}, \dots, X_{k-1} \rightarrow X_k^{(a_k - a_{k-1})e'_{\sigma(k)}}, X_{k+1} \rightarrow 1, \dots, X_n \rightarrow 1$. Then, we have

$$X_k^a (1 - X_k^{e_k}) = (1 - X_k^{a_k e_k})F; 0 \leq a, F \in A[X_k].$$

Since $a_k > 1$, this is obviously impossible.

PROPOSITION 10. *If $A_G[X]$ is r -noetherian, then G is finitely generated.*

PROOF. By Prop. 6, the torsion-free rank n of G is finite. Let $\{\alpha_1, \dots, \alpha_n\}$ be independent. $G_0 = Z\alpha_1 + \cdots + Z\alpha_n$. If G is not finitely generated, there are finitely generated subgroups G_i of G such that $G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots$ and $G = \bigcup_{i=0}^{\infty} G_i$. $G_i = Z\alpha_{i1} + \cdots + Z\alpha_{in}$, $\mathfrak{A}_i = (1 - X^{\alpha_{i1}}, \dots, 1 - X^{\alpha_{in}})$. By Lem. 8, $\mathfrak{A}_i \subset \mathfrak{A}_{i+1}$. Suppose $\mathfrak{A}_i = \mathfrak{A}_{i+1}$.

$$\begin{cases} 1 - X^{\alpha_{i+1,1}} = (1 - X^{\alpha_{i1}})f_{11} + \cdots + (1 - X^{\alpha_{in}})f_{1n}, \\ \dots\dots\dots \\ 1 - X^{\alpha_{i+1,n}} = (1 - X^{\alpha_{i1}})f_{n1} + \cdots + (1 - X^{\alpha_{in}})f_{nn}, \end{cases}$$

for $f_{jh} \in A_G[X]$. There exists k such that $G_i, G_{i+1} \subset G_k, f_{jh} \in A_{G_k}[X]$. This is a contradiction by Lem. 9. Hence, $\mathfrak{A}_i \not\subseteq \mathfrak{A}_{i+1}$. This is a contradiction.

THEOREM 11. $A_G[X]$ is r -noetherian if and only if A is noetherian and G is finitely generated.

PROOF. For ideals $\mathfrak{a} \subseteq \mathfrak{b}$ of A , $(\mathfrak{a}, 1 - X^\alpha)A_G[X] \subseteq (\mathfrak{b}, 1 - X^\alpha)A_G[X]$ (for fixed $\alpha \neq 0$). The rest follows from Prop. 10.

§ 12. The Dedekind

The first is on the π -ring ([13] p. 572).

PROPOSITION 1. If $D_G[X]$ is a π -domain, then so is D .

PROOF. We confer [13] (46.7). By [24] (3.3), D becomes a Krull domain. Let $\mathfrak{p}$ be a ht 1 prime of D . By [24] (5.1), $\mathfrak{p}D_G[X]$ is a ht 1 prime of $D_G[X]$. Hence, $\mathfrak{p}(D_G[X]: \mathfrak{p}D_G[X]) = D_G[X]$. $1 = \sum_1^n p_i F_i$, $p_i \in \mathfrak{p}$, $F_i \in (D_G[X]: \mathfrak{p}D_G[X])$. Take $0 \neq p \in \mathfrak{p}$. $F_i = f_i/p$, $f_i \in D_G[X]$. Let a_i be the coefficient of f_i of degree 0. Then, $\mathfrak{p}(a_1/p, \dots, a_n/p)D = D$. That is, $\mathfrak{p}$ is invertible.

LEMMA 2. Every idempotent of $A_G[X]$ belongs to A .

PROOF. Let $f = f^2$. Let H be the subgroup of G generated by the powers of f . $H = \langle X_1, \dots, X_n \rangle$. $X^{\alpha_i} = X_i$. f is an idempotent of $A_Z[X_1, \dots, X_n]$. By [32] (3.1), $f \in A_Z[X_1, \dots, X_{n-1}]$. Like this, $f \in A$.

PROPOSITION 3. If $A_G[X]$ is a π -ring, then so is A .

PROOF. $A_G[X] = B_1 \oplus \dots \oplus B_n \oplus B_{n+1} \oplus \dots \oplus B_{n+m}$, $n \geq 0$, $m \geq 0$; $B_1, \dots, B_n$ are π -domains; $B_{n+1}, \dots, B_{n+m}$ are special primary rings ([13] (46.11)). $1 = e_1 + \dots + e_{n+m}$, $e_i \in B_i$. By Lem. 2, $e_i \in A$. $B_i = (e_i A)_G[X]$. $(e_{n+m} A)_G[X] = B_{n+m}$ is a local ring. This is a contradiction. Hence, $m = 0$. By Prop. 1, $e_1 A, \dots, e_n A$ are π -domains. $A = e_1 A \oplus \dots \oplus e_n A$. Hence, A is a π -ring.

THEOREM 4. $A_G[X]$ is a π -ring if and only if $A = \mathfrak{a}_1 \oplus \dots \oplus \mathfrak{a}_n$, $(\mathfrak{a}_i)_G[X]$ is a π -domain.

PROOF. The necessity follows from the proof of Prop. 3. The sufficiency, [13] (46.11).

REMARK 5. Even if A is a π -ring, $A_Z[X]$ need not a π -ring.

A COUNTER EXAMPLE. $A = \mathbb{Q}[x]$, $x^2 = 0$. If $A_Z[X]$ is a π -ring, in Th. 4, $1 = e_1 + \dots + e_n$, $e_i \in \mathfrak{a}_i$. Since e_i is an idempotent, $e_i = 1$. Hence, $n = 1$. $A_Z[X]$ becomes a π -domain. This is a contradiction.

PROPOSITION 6. *If $A_G[X]$ is a π -ring, then G is of type $(0, 0, 0, \dots)$.*

PROOF. In the proof of Prop. 3, $(e_i A)_G[X]$ is a π -domain. By [13] (46.7), $(e_i A)_G[X]$ is a Krull domain. By [24], G is of type $(0, 0, 0, \dots)$.

The next is on the Dedekind ring [18].

PROPOSITION 7. *If $A_G[X]$ is a Dedekind ring, then so is A .*

PROPOSITION 8. *$A_G[X]$ is a Dedekind ring if and only if $A_G[X]$ is a Krull ring and a prüfer ring.*

LEMMA 9. *Let $(A, \mathfrak{p})$ be a local ring, $\mathfrak{B} = (\mathfrak{p}, 1 - X^\alpha; \alpha \in G)$. If $A_G[X]_{\mathfrak{B}}$ is a Dedekind ring, then so is A .*

PROOF. Let $\mathfrak{a}$ be a regular ideal. Then $\mathfrak{a}A_G[X]_{\mathfrak{B}}$ is a regular ideal of $A_G[X]_{\mathfrak{B}}$. $\mathfrak{a}A_G[X]_{\mathfrak{B}} = (f_1, \dots, f_n)A_G[X]_{\mathfrak{B}}$, $f_i \in A_G[X]$. The image $f_i(1)$ of f_i under the canonical homomorphism $A_G[X] \rightarrow A$ belongs to $\mathfrak{a}$. $\mathfrak{a} = (f_1(1), \dots, f_n(1))A$ is finitely generated. Hence, A satisfies the ascending chain condition on regular ideals. Next, let $\mathfrak{a}$ be a finitely generated regular ideal of A . Set $\mathfrak{B} = (A_G[X]_{\mathfrak{B}} : \mathfrak{a}A_G[X]_{\mathfrak{B}})$. Then, $\mathfrak{a}A_G[X]_{\mathfrak{B}} \cdot \mathfrak{B} = A_G[X]_{\mathfrak{B}}$. Let $a \in \mathfrak{a}$ be a regular element of A . $\mathfrak{a}A_G[X]_{\mathfrak{B}} \cdot (h_1, \dots, h_n)A_G[X]_{\mathfrak{B}} = aA_G[X]_{\mathfrak{B}}$ for $h_i \in A_G[X]$. There exists the canonical homomorphism $\theta: A_G[X]_{\mathfrak{B}} \rightarrow A$. Then, $\mathfrak{b} = \theta((h_1, \dots, h_n)A_G[X]_{\mathfrak{B}})$ is a regular ideal of A . $\mathfrak{a}\mathfrak{b} = aA$. Hence, $\mathfrak{a}$ is invertible.

The next is on the almost Dedekind rings [22] (2.15).

PROPOSITION 10. *If $A_G[X]$ is an almost Dedekind ring, then so is A .*

PROOF. Let $\mathfrak{p}$ be a regular maximal ideal of A . $\mathfrak{B} = (\mathfrak{p}, 1 - X^\alpha; \alpha \in G)A_G[X]$ is a regular maximal ideal of $A_G[X]$. Hence, $A_G[X]_{\mathfrak{B}}$ is a Dedekind ring. Set $\mathfrak{B}' = (\mathfrak{p}_{\mathfrak{p}}, 1 - X^\alpha; \alpha \in G)(A_{\mathfrak{p}})_G[X]$. Then, $A_G[X]_{\mathfrak{B}} \cong (A_{\mathfrak{p}})_G[X]_{\mathfrak{B}'}$. By Lem. 9, $A_{\mathfrak{p}}$ is a Dedekind ring.

§13. The rational functions rings

PROPOSITION 1. *If $D_G(X)_{\text{loc}}$ is a Dedekind domain, then so is D .*

PROOF. By the proof of §2, Prop. 2, D is integrally closed. By [26] §4, Th. 10, D is an almost Dedekind domain. Let $0 \neq d \in D$. Let $\mathfrak{m}$ be a maximal ideal of D containing d . $\mathfrak{m}D_G(X)_{\text{loc}}$ is a maximal ideal of $D_G(X)_{\text{loc}}$ containing d ([26] §7, Prop. 6). And $\mathfrak{m}D_G(X)_{\text{loc}} \cap D = \mathfrak{m}$ ([26] §7, Prop. 2). Hence, d belongs to only finitely many maximal ideals of D . Hence, D is a Dedekind domain.

REMARK 2. Even if D is a Dedekind domain, $C(D_G(X)_{\text{loc}})$ need not be isomorphic with $C(D)$.

A COUNTER EXAMPLE. Let D be a non-factorial Dedekind domain. $D_Z(X)_{\text{loc}}$ is a PID ([21] p. 74, 16 (g)).

PROPOSITION 3. Let D be a Dedekind domain. Then $D_G(X)_{\text{loc}}$ is a PID.

PROOF. Let $\mathfrak{A}$ be an ideal of $D_G(X)_{\text{loc}}$. Suppose $\mathfrak{A}$ is not finitely generated. Then, $(\xi_1) \subsetneq (\xi_1, \xi_2) \subsetneq (\xi_1, \xi_2, \xi_3) \subsetneq \cdots \subset \mathfrak{A}$. By [26] § 7, Prop. 7(2), $D_G(X)_{\text{loc}}$ is bezout. Hence, $(\xi_1, \dots, \xi_i) = f_i D_G(X)_{\text{loc}}$ for $f_i \in D_G[X]$. $f_i = f_{i+1}(h/g)$, $h/g \in D_G(X)_{\text{loc}}$. By [26] § 2, Th. 11, $A_{f_i} = A_{f_i} A_g = A_{f_i g} = A_{f_{i+1} h} = A_{f_{i+1}} A_h$. Since h/g is non-unit of $D_G(X)_{\text{loc}}$, $A_{f_{i+1}} A_h \subsetneq A_{f_{i+1}}$. Hence, $A_{f_1} \subsetneq A_{f_2} \subsetneq \cdots$. A contradiction. Hence, $D_G(X)_{\text{loc}}$ is noetherian. By [26] § 7, Prop. 7(2), $D_G(X)_{\text{loc}}$ is a PID.

PROPOSITION 4. Assume D be a local domain such that $D_G(X)_{\text{loc}}$ is factorial. Then, D is factorial.

PROOF. (1) D is integrally closed. Let E be the integral closure of D . By [26] § 6, Prop. 9, $E_G(X)_{\text{loc}}$ is integral over $D_G(X)_{\text{loc}}$. Hence, $E_G(X)_{\text{loc}} = D_G(X)_{\text{loc}}$. Hence, $E = D$. (2) Every nonunit of D is a nonunit of $D_G(X)_{\text{loc}}$. (3) Every element of D is a finite product of irreducible elements of D . (4) Every irreducible element of D is irreducible in $D_G(X)_{\text{loc}}$. Let a be an irreducible element of D . Let $a = (f/g)(f'/g')$ be a decomposition in $D_G(X)_{\text{loc}}$. Since $A_{ff'} = aD$, $A_f, A_{f'}$ are principal ideals of D ([26] § 6, Prop. 12). Set $A_f = bD$, $A_{f'} = b'D$. By [26] § 6, Prop. 1 (4), $bb'D = A_f A_{f'} = A_{ff'} = aD$. $a = bb'c$ for $c \in D$. Since a is irreducible in D , $A_f = D$ or $A_{f'} = D$. Hence, f/g or f'/g' is a unit of $D_G(X)_{\text{loc}}$. (5) If $a, b \in D$ are associated in $D_G(X)_{\text{loc}}$, then they are associated in D . (6) D is factorial.

PROPOSITION 5. Assume A be a regular ring such that $A_G(X)_{\text{loc}}$ is noetherian local. Then, $A_G(X)_{\text{loc}}$ is a regular local ring.

PROOF. By [26] § 7, Prop. 6, A is local. By [26] § 7, Prop. 2, A is noetherian. Set $\dim A = d$; $\dim A_G(X)_{\text{loc}} \geq d$. The maximal ideal $\mathfrak{m}$ of A is generated by d elements $x_1, \dots, x_d$. $\mathfrak{m} A_G(X)_{\text{loc}} = (x_1, \dots, x_d) A_G(X)_{\text{loc}}$. Hence, $d \geq \dim A_G(X)_{\text{loc}}$. Hence, $\dim A_G(X)_{\text{loc}} = d$.

PROPOSITION 6. Assume $R_G(X)_{\text{loc}}$ is noetherian local for a ring R . Then the multiplicity of $R_G(X)_{\text{loc}}$ is that of R .

PROOF. By [26] § 7, Prop. 2, Prop. 6, R is noetherian local. Let $\mathfrak{m}$ be the maximal ideal of R , $R^* = R_G(X)_{\text{loc}}$. By [26] § 7, Prop. 1, R^* is a faithfully flat R -module. Set $\text{length}_{R/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = l$.

$$\begin{aligned} (\mathfrak{m}^i R^*)/(\mathfrak{m}^{i+1} R^*) &\cong (\mathfrak{m}^i/\mathfrak{m}^{i+1}) \otimes_R R^* \cong \underbrace{(R/\mathfrak{m} \oplus \cdots \oplus R/\mathfrak{m})}_l \\ &\otimes_R R^* \cong \underbrace{R^*/\mathfrak{m} R^* \oplus \cdots \oplus R^*/\mathfrak{m} R^*}_l. \end{aligned}$$

Hence, $\text{length}_{R^*/\mathfrak{m} R^*}(\mathfrak{m}^i R^*)/(\mathfrak{m}^{i+1} R^*) = \text{length}_{R/\mathfrak{m}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. Hence,

$$\text{length}_{R^*} R^*/\mathfrak{m}^{i+1} R^* = \text{length}_R R/\mathfrak{m}^{i+1}.$$

PROPOSITION 7. Let $(A, \mathfrak{m})$ be a noetherian local ring. Then,
 $\bigcap_0^\infty \mathfrak{m}^i A_G(X)_{\text{loc}} = 0.$

PROOF. Let $f/g \in \bigcap \mathfrak{m}_i A_G(X)_{\text{loc}}$. Let H be the subgroup of G generated by the powers of f, g . By [7] Lem. 1, $A_G[X]$ is a faithfully flat $A_H[X]$ -module. Hence, $A_G(X)_{\text{loc}}$ is a flat $A_H(X)_{\text{loc}}$ -module ([19] (0, 6.3.2)). $A_H(X)_{\text{loc}} \hookrightarrow A_G(X)_{\text{loc}}$. By [29] (18.3), $\mathfrak{m}^i A_G(X)_{\text{loc}} \cap A_H(X)_{\text{loc}} = \mathfrak{m}^i A_H(X)_{\text{loc}}$. Hence $f/g \in \mathfrak{m}^i A_H(X)_{\text{loc}}$ for all i . Since $A_H(X)_{\text{loc}}$ is noetherian, $\bigcap \mathfrak{m}^i A_H(X)_{\text{loc}} = 0$. Hence, $f/g = 0$.

REMARK 8. Even if A and $A_G(X)_{\text{loc}}$ are noetherian local, G need not be of finite torsion-free rank.

A COUNTER EXAMPLE. $A = k$ a field, $G = \underbrace{Z \oplus Z \oplus \cdots}_{\aleph_0}$.

PROPOSITION 9. Let A be a noetherian local ring. Assume $\dim A_G(X)_{\text{loc}} = \dim A$ for every free G , then $\dim A_G(X)_{\text{loc}} = \dim A$ for any G .

PROOF. Let $\{\alpha_\lambda; A\}$ be a maximal independent subset of G . $K = \sum Z\alpha_\lambda$. $\dim A_K(X)_{\text{loc}} = \dim A = n$. $\text{ht } \mathfrak{p} A_G[X] \leq \text{ht } \mathfrak{p} A_K[X] = n$. Hence, $\dim A_G(X)_{\text{loc}} = \dim A$.

§14. The others

PROPOSITION 1. Let $\mathfrak{q}$ be a $\mathfrak{p}$ -primary in A . Then, $\mathfrak{q} A_G[X]$ is $\mathfrak{p} A_G[X]$ -primary in $A_G[X]$.

PROOF. Let $f, g \in A_G[X]$, $fg \in \mathfrak{q}^e = \mathfrak{q} A_G[X]$, $g \notin \mathfrak{q}^e$. We may suppose $\mathfrak{q} = 0$. Since f is a zerodivisor of $A_G[X]$, $af = 0$ for $0 \neq a \in A$ ([26] §9, Prop. 5). $f = \sum a_\alpha X^\alpha$. $a_\alpha a = 0$. $a_\alpha \in \mathfrak{p}$. Hence, $f \in \mathfrak{p}^e = \mathfrak{p} A_G[X]$.

The next is on the primitive rings [31] §12.

REMARK 2. $A_G[X]$ is never a primitive ring.

PROOF. [31] (32.3).

THEOREM 3. $A_G[X]$ is arithmetical if and only if A is von Neumann regular, $G \hookrightarrow Q$.

PROOF. The necessity. By [13] p. 321, 20, $A_G[X]$ is a prüfer ring. By [16] (3.1), A is von Neumann regular, $G \hookrightarrow Q$. The sufficiency. Let M be a maximal ideal of $A_G[X]$, $M \cap A = \mathfrak{p}$, $A \setminus \mathfrak{p} = S$. $\mathfrak{P} = M_S$ is a maximal ideal of $(A_\mathfrak{p})_G[X] = A_G[X]_S$. $A_\mathfrak{p} = k$ is a field. Let $0 \neq f, g \in \mathfrak{P}$. Let H be the subgroup generated by the powers of f, g . $H = Z\alpha$. Set $\mathfrak{P} \cap k_H[X] = \mathfrak{P}'$. $k_H[X]_{\mathfrak{P}'}$ is a discrete valua-

tion ring. Hence, f divides g or g divides f in $k_H[X]_{\mathfrak{P}}$. Assume f divides g . Then, $g \in f k_G[X]_{\mathfrak{P}}$. $k_G[X]_{\mathfrak{P}}$ is a chained ring ([13] p. 184, 8). Since $A_G[X]_M = k_G[X]_{\mathfrak{P}}$, $A_G[X]$ is an arithmetical ring ([13] p. 321, 19).

REMARK 4. Even if f is a unit of $A_G[X]$, where G is totally ordered, the leading coefficient need not be a unit of A .

We confer [25] (1.1).

PROPOSITION 5. *If $A_G[X]$ is a multiplication ring, then A is von Neumann regular and $G \cong \mathbb{Z}$.*

PROOF. [16] (3.1).

PROPOSITION 6. *If $A_G[X]$ is a multiplication ring, then so is A .*

PROOF. Let $b \subset a$ be ideals of A . $bA_G[X] = aA_G[X] \cdot \mathfrak{B}$ for an ideal $\mathfrak{B}$ of $A_G[X]$. Let c be the set of coefficients of elements of $\mathfrak{B}$ of degree 0. Then, $b = ac$.

THEOREM 7. *$A_G[X]$ is a multiplication ring if and only if A is a finite direct sum of fields and $G \cong \mathbb{Z}$.*

PROOF. The necessity. By Prop. 5, A is von Neumann regular and $G \cong \mathbb{Z}$. Let $\mathfrak{p}$ be a prime ideal of A . $\mathfrak{P} = (\mathfrak{p}, 1 - X)$ is a ht 1 prime of $A_{\mathbb{Z}}[X]$. Hence, $\mathfrak{P}$ is a finitely generated ideal (cf. the proof of [3] 3 (6.1)). $\mathfrak{P} = (f_1, \dots, f_n)$ for $f_i \in A_{\mathbb{Z}}[X]$. $f_i(1) \in \mathfrak{p}$. Let $a \in \mathfrak{p}$. $a = \sum f_i g_i$, $g_i \in A_{\mathbb{Z}}[X]$. $a = \sum f_i(1)g_i(1)$. $\mathfrak{p} = (f_1(1), \dots, f_n(1))$. Hence, A is noetherian.

The next is on the QQR-property ([13] p. 339, 2).

PROPOSITION 8. *$D_G[X]$ has QQR-property if and only if D is a field and $G \subsetneq \mathbb{Q}$.*

PROOF. The necessity. Let E be the integral closure of D . $E_G[X]$ has also the QQR-property. By [24] (4.4), $E_G[X]$ is integrally closed. By [13] p. 340, 4, $E_G[X]$ is prüfer. By [16], E is a field and $G \subsetneq \mathbb{Q}$. The sufficiency is similar.

PROPOSITION 9. (1) *If $A_G[X]$ is lasker, then so is A .* (2) *If $A_G[X]$ is strongly lasker, then so is A .*

REMARK 10. Even if k is a field, G of type $(0, 0, 0, \dots)$, $k_G[X]$ need not be lasker.

A COUNTER EXAMPLE. $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \dots$. $k_G[X] = k_{\mathbb{Z}}[X_1, X_2, X_3, \dots] = D$. $M_1 = \{1 - X_1\}$, $M_2 = \{1 - X_2, 1 - X_3\}$, $M_3 = \{1 - X_4, \dots, 1 - X_7\}, \dots$. $\mathfrak{A} = (1 - X_1, (1 - X_2)(1 - X_3), (1 - X_4) \cdots (1 - X_7), (1 - X_8) \cdots (1 - X_{15}), \dots)$. Take $g_i \in M_i$, let $\mathfrak{P} = (g_1, g_2, g_3, \dots)$. $\mathfrak{P}$ is a prime ideal containing $\mathfrak{A}$. $\mathfrak{P}$ is a minimal prime ideal

containing $\mathfrak{A}$. Such $\mathfrak{B}$ exist infinitely many. Hence, $\mathfrak{A}$ cannot be a finite intersection of primary ideals.

EXAMPLE 11. $D = k[X_1, X_2, X_3, \dots]$ is not lasker.

PROPOSITION 12. If $D_G[X]$ has the QR-property, so does D .

PROPOSITION 13. $D_G[X]$ has the QR-property if and only if D is a field and $G \subseteq Q$.

PROOF. The necessity. $D_G[X]$ is a prüfer domain. By [16], D is a field and $G \subseteq Q$. The sufficiency. By [16], $D_G[X]$ is bezout. By [13] (27.5), $D_G[X]$ has the QR-property.

If every overring E of D (i.e. $D \subset E \subset K$) is a finite intersection of quotient rings of D , D is said to have FQR-property.

PROPOSITION 14. $D_G[X]$ has the FQR-property if and only if $D_G[X]$ has the QR-property.

PROOF. Prop. 13, Prop. 8.

The next is on the irreducible rings ([31] p. 61).

PROPOSITION 15. If $A_G[X]$ is irreducible, then so is A .

The next is on the root closed ([13] p. 173).

PROPOSITION 16. If $A_G[X]$ is root closed, then so is A .

REMARK 17. Even if A is root closed, $A_Z[X]$ need not be so.

A COUNTER EXAMPLE. $A = \mathbb{Z}/(4)$. $(2/(1+X))^2 \in A_Z[X]$.

REMARK 18. Fix G . Assume D be a domain of transcendental degree 1 over a field. There is not a field such that $D \cong k_G[X]$. Then D need not be G -strongly invariant.

A ring A is said to be G -strongly invariant provided whenever $\sigma: A_G[X_1, \dots, X_n] \cong B_G[Y_1, \dots, Y_n]$ is an isomorphism, then $\sigma(A) = B$.

A COUNTER EXAMPLE. $D = \mathbb{Q}_Z[x]$, x an indeterminate. $G = \mathbb{Z} \oplus \mathbb{Z}$.

REMARK 19. Assume A is a Dedekind domain which is not G -strongly invariant. Then, there need not exist $s \in A$ and a field k such that $A[1/s] = k_G[X]$.

A COUNTER EXAMPLE. $A = \mathbb{Q}_Z[x]$, x an indeterminate. $G = \mathbb{Z} \oplus \mathbb{Z}$. If $A[1/s] = k_Q[X]$, $1 = \text{transc. degree}_Q A[1/s] = \text{transc. degree}_Q k_Q[X] \geq 2$. A contradiction.

The next is on the strongly invariant [11] (0.3).

PROPOSITION 20. *If $A_G[X]$ is strongly invariant, then so is A .*

PROOF. Let B be a ring and $\sigma: A[X_1, \dots, X_n] \cong B[Y_1, \dots, Y_n]$. We get a natural homomorphism $\sigma': (A[X_1, \dots, X_n])_G[X] \cong (B[Y_1, \dots, Y_n])_G[X]$. Hence, $(A_G[X])[X_1, \dots, X_n] \cong (B_G[X])[Y_1, \dots, Y_n]$. Hence, $\sigma'(A_G[X]) = B_G[X]$. Hence, $\sigma(A) = B$.

The next is on the coherence [4] § 2, exerc. 12) α).

PROPOSITION 21. *If $A_G[X]$ is coherent, then so is A .*

PROOF. $A_G[X]$ is a faithfully flat A -module ([26] § 9, Prop. 1). Let α be any cardinal number. The direct product $\prod A_G[X]$ of the number α of $A_G[X]$ is flat as A -module (e.g. [27] (3.B)). Let $(a_n^1)_{n \in \alpha}, (a_n^2)_{n \in \alpha}, \dots, (a_n^m)_{n \in \alpha}$ be a finite number of elements of $\prod A$; $a_1, \dots, a_m \in A$ such that

$$a_1(a_n^1) + \dots + a_m(a_n^m) = (0).$$

By [4] I, § 2, Cor. 1 of Prop. 13,

$$(a_n^i) = a_{1i}(f_n^1) + \dots + a_{mi}(f_n^m), \quad \sum a_{ji}a_i = 0,$$

$(f_n^j) \in \prod A_G[X]$, $a_{ji} \in A$. Let b_n^j be the constant term of f_n^j . Then, $(a_n^i) = \sum a_{ji}(b_n^j)$. Hence, $\prod A$ is a flat A -module.

The next is on the hereditary [8] I, § 5.

PROPOSITION 22. *If $A_G[X]$ is hereditary, then so is A .*

PROOF. Set $\mathfrak{A} = (1 - X^\alpha; \alpha \in G)A_G[X]$. We have a natural onto homomorphism $\gamma: A_G[X] \rightarrow A$. Let $\mathfrak{a}$ be an ideal of A ; Q', Q'' be A -modules; $\alpha: \mathfrak{a} \rightarrow Q''$ an A -homomorphism; $\beta: Q' \rightarrow Q''$ an onto A -homomorphism. Q', Q'' become $A_G[X]$ -modules. β is an $A_G[X]$ -homomorphism. α is also an $A_G[X]$ -homomorphism. There exists an $A_G[X]$ -homomorphism $\gamma|_{\mathfrak{a}A_G[X]}: \mathfrak{a}A_G[X] \rightarrow \mathfrak{a}$. Hence, there exists an $A_G[X]$ -homomorphism $\delta: \mathfrak{a}A_G[X] \rightarrow Q'$ such that $\beta\delta = \alpha(\gamma|_{\mathfrak{a}A_G[X]})$. $\beta(\delta|_{\mathfrak{a}}) = \alpha$. $\delta|_{\mathfrak{a}}$ is an A -homomorphism. Hence, $\mathfrak{a}$ is a projective A -module.

PROPOSITION 23. *If $A_G[X]$ is semi-hereditary, then so is A .*

PROOF. Similar with that of Prop. 22.

PROPOSITION 24. *$A_G[X]$ is semi-hereditary if and only if A is von Neumann regular and $G \subseteq Q$.*

PROOF. The necessity. By [17] Th. 20, $A_G[X]$ is prüfer. By [16], A is von Neumann regular and $G \subseteq Q$. The sufficiency. $A_G[X]$ is a prüfer ring and the nilradical of A is zero. By [26] § 3, Th. 3, the nilradical of $A_G[X]$ is also zero. Hence, the nilradical of $A_G(X)$ is also zero. Hence, $A_G(X)$ is von Neumann regular.

The next is on the condition (M) ([40] p. 15).

PROPOSITION 25. $D_G[X]$ satisfies the (M) if and only if D satisfies (M) and G is of type $(0, 0, 0, \dots)$.

PROOF. The necessity. Let $a_i \in D$, $a_i D \subset a_{i+1} D$, $i = 1, 2, 3, \dots$. There exists n such that $a_i D_G[X] = a_{i+1} D_G[X]$ for all $i \geq n$. It follows $a_i D = a_{i+1} D$ for all $i \geq n$. As the proof of [24] (2.3), G is seen of type $(0, 0, 0, \dots)$. The sufficiency. Let $0 \neq f_i \in D_G[X]$, $f_i D_G[X] \subset f_{i+1} D_G[X]$, $i = 1, 2, 3, \dots$. $K_G[X]$ is a Krull domain ([24] (3.3)). Hence, there exists n such that $f_i K_G[X] = f_{i+1} K_G[X]$ for all $i \geq n$. We may suppose $n = 1$. It follows $f_i = f_{i+1} a_i$, $a_i \in D$. Set $\alpha(f_1) = \alpha$ and let c_i be the coefficient of X^α in f_i . $c_i D \subset c_{i+1} D$. There exists m such that $c_i D = c_{i+1} D$ for all $i \geq m$. Hence, a_i is a unit for any $i \geq m$. Hence, $f_i D_G[X] = f_{i+1} D_G[X]$ for all $i \geq m$.

PROPOSITION 26. If $A_G[X]$ satisfies (M), then so does A and G is of type $(0, 0, 0, \dots)$.

PROOF. Let $a_1, a_2, \dots \in A$, $a_i A \subset a_{i+1} A$ for $i = 1, 2, 3, \dots$. There exists n such that $a_i A_G[X] = a_{i+1} A_G[X]$ for all $i \geq n$. $a_{i+1} = a_i b_i$, $b_i \in A$ for all $i \geq n$. Hence, $a_i A = a_{i+1} A$ for all $i \geq n$. Hence, A satisfies (M). For a prime number p and $0 \neq \alpha \in G$, suppose $p^{n+1} \alpha_{n+1} = \alpha$ is solvable. $(1 - X^{\alpha_{n+1}}) \subseteq (1 - X^\alpha)$. Hence, $A_G[X]$ does not satisfy (M). The rest is similar.

The next is on the M -ring [36].

PROPOSITION 27. $D_G[X]$ is a M -ring if and only if D is a field and $G \cong \mathbb{Z}$.

PROOF. The necessity. By [36] 2, Th. 1, it follows $\dim D_G[X] \leq 1$. If D is not a field, take a maximal ideal $\mathfrak{p}$ of D . Since $D_G[X]/\mathfrak{p}D_G[X] \cong (D/\mathfrak{p})_G[X]$, $\mathfrak{p}D_G[X]$ is not a maximal ideal of $D_G[X]$. Hence, $\dim D_G[X] > 1$. A contradiction. Hence, $D = K$ is a field. By [24] Prop. 4.4, [37] I, Th. 6, $K_G[X]$ is a Dedekind domain. By [16], $G \cong \mathbb{Z}$. The sufficiency. $K_{\mathbb{Z}}[X]$ is an integrally closed Mori-ring. By [36] 2, Cor. of Th. 4, $K_{\mathbb{Z}}[X]$ is a M -ring.

REMARK 28. Even if G is free, every homogeneous ideal $\mathfrak{a}$ of $k_G[X, Y]$ need not be a finite intersection of primary ideals.

The homogeneous ideal of $k_G[X, Y]$ is an ideal generated by homogeneous elements.

A COUNTER EXAMPLE. $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \dots$. $X^{(0, \dots, 0, \frac{1}{i}, 0, \dots)} = X_i$, $Y^{(0, \dots, 0, \frac{1}{i}, 0, \dots)} = Y_i$. $k_G[X, Y] = k_{\mathbb{Z}}[X_1, X_2, \dots; Y_1, Y_2, \dots]$. Set $\mathfrak{A} = (X_1 - Y_1, (X_2 - Y_2)(X_3 - Y_3), (X_4 - Y_4) \cdots (X_7 - Y_7), (X_8 - Y_8) \cdots (X_{15} - Y_{15}), \dots)$. The fact that $\mathfrak{A}$ is not the finite intersection follows as the proof of Remark 10.

A ring A is said to be G -invariant provided that whenever $A_G[X_1, \dots, X_n] \cong B_G[Y_1, \dots, Y_n]$ for a ring B , then $A \cong B$.

REMARK 29. Assume D is a domain of transcendence degree 1 over a field and G be free. Then, D need not be G -invariant.

A COUNTER EXAMPLE. Let k be a field, x an indeterminate, $D = k_Z[x]$, $G = \underbrace{Z \oplus Z \oplus \dots}_{\infty}$, $E = k$.

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