On Certain Compact Submanifolds of Euclidean Space

Kohei HATSUSE

Let M^n be a differentiable manifold. When f is an immersion of M^n into a Euclidean space E^m , we can define a function L_p on M^n by

$$L_p(x) = \langle p - f(x), p - f(x) \rangle \quad x \in M^n$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in E^m . For almost all points $p \in E^m$, L_p is a Morse function on M^n . Nomizu and Rodríguez [2] proved the following

THEOREM. *Let M^n , $n \geq 2$, be a compact connected manifold immersed into E^m . If every Morse function L_p on M^n has exactly two critical points, then M^n is imbedded as a Euclidean n -sphere S^n .*

If a Morse function L_p on M^n has exactly two critical points, then both of the numbers $C_0(L_p)$ of critical points of index zero and $C_n(L_p)$ of critical points of index n are equal to one. The purpose of this note is to investigate M^n such that $C_0(L_p) = 1$ or $C_n(L_p) = 1$ for every Morse function L_p on M^n .

We shall mean C^∞ differentiable by "differentiable" and an $n(\geq 2)$ -dimensional differentiable manifold M^n by a manifold M^n .

1. Lemmas

The notions and notations will follow from Milnor [1]. Let $f: M^n \rightarrow E^m$ be an immersion and let $\perp(M)$ be the normal bundle to M^n . If L_p is a Morse function on M^n , then $p \in E^m$ is not a focal point of M^n . Let $x \in M^n$ and let $e(x) \in \perp(M)$, that is, a vector normal to $f(M^n)$ at $f(x)$. We denote by $A_{e(x)}$ a symmetric linear transformation of a tangent space $T_x M^n$ into itself which corresponds to the second fundamental form of M^n . If x is a critical point of Morse function L_p on M^n and if $p = f(x) + te(x)$, $t > 0$, then the index at x is equal to the number of eigenvalues α of $A_{e(x)}$ such that $0 < 1/\alpha < t$, counting multiplicities.

We shall identify $x \in M^n$ with $f(x)$ if there is no confusion. We begin with the following

LEMMA 1. *Let M^n be a manifold immersed into E^{n+1} . Then M^n is convex if and only if there exists a unit vector $e(x)$ normal to M^n at each $x \in M^n$ and*

Received January 20, 1977.

* Department of Mathematics, Faculty of Science, Ibaraki University, Mito, Japan.

$$L_{p(t,x)}(x) \leq L_{p(t,x)}(y) \quad y \in M^n$$

for all $t > 0$ where $p(t, x) = x + te(x)$.

PROOF. Let $x \in M^n$. If M^n is convex then there exists a unit vector $e(x)$ normal to M^n at x such that $\langle e(x), y - x \rangle \leq 0$ for all $y \in M^n$. Let $t > 0$ then

$$L_{p(t,x)}(y) - L_{p(t,x)}(x) = 2t \langle e(x), x - y \rangle + \langle x - y, x - y \rangle \geq 0.$$

Therefore $L_{p(t,x)}(x) \leq L_{p(t,x)}(y)$. Conversely, we suppose that there exists a unit vector $e(x)$ normal to M^n and $L_{p(t,x)}(x) \leq L_{p(t,x)}(y)$ for all $t > 0$. Then $\langle e(x), y - x \rangle \leq 0$ for all $y \in M^n$. Because, if there exists a point $y_0 \in M^n$ such that $\langle e(x), y_0 - x \rangle > 0$, then

$$L_{p(t,x)}(x) > L_{p(t,x)}(y_0)$$

for $t > \langle x - y_0, x - y_0 \rangle / 2 \langle e(x), y_0 - x \rangle$. Therefore M^n is convex.

LEMMA 2. Let M^n be a convex manifold immersed into E^{n+1} . Let $e(x)$ be a unit vector normal to M^n at x as in Lemma 1. Then every eigenvalue of $A_{e(x)}$ is non-positive.

PROOF. The mapping $e: M^n \ni x \rightarrow e(x) \in \perp(M)$ defines a differentiable vector field along M^n . We suppose that $A_{e(x)}$ has a positive eigenvalue. Let α be the largest positive eigenvalue of $A_{e(x)}$ whose multiplicity is k , and let β be the next largest positive eigenvalue of $A_{e(x)}$. Take $t > 0$ such that $1/\alpha < t < 1/\beta$ (if α is the only positive eigenvalue, just considered $1/\alpha < t$). Then, from Lemma of [2], there exists a point $x' \in M^n$ and it is a critical point of a Morse function $L_{p(t,x')}$ on M^n of index k . On the other hand, Lemma 1 implies the index at x' is zero. Therefore Lemma is proved.

LEMMA 3. Let M^n be a compact connected manifold immersed into E^{n+1} . If M^n is convex then M^n is diffeomorphic to n -sphere S^n .

PROOF. Since M^n is compact, there exists an open ball D of radius r about the origin of E^{n+1} such that $M^n \subset D$. The boundary of D is an n -sphere S^n . Every ray $p(t, x) = x + te(x)$, $t > 0$, starting from $x \in M^n$ meets S^n at only one point $p(t(x), x)$. If we define a mapping $\phi: M^n \rightarrow S^n$ by

$$\phi(x) = p(t(x), x) \quad x \in M^n$$

then ϕ is differentiable since

$$t(x) = -\langle e(x), x \rangle + \{r^2 - \langle x, x \rangle + \langle e(x), x \rangle^2\}^{1/2}.$$

We denote by $\exp$ the exponential mapping of $\perp(M)$ into E^{n+1} and denote by π the projection of $\perp(M)$ into M^n . Then, by the definition of ϕ , we obtain $\phi(x) = \exp t(x)e(x)$. Let $(v^1, \dots, v^{n+1}, U)$ be a cubical coordinate system centered

at $\phi(x)$ in E^{n+1} such that $U \cap S^n$ is an n -dimensional slice defined by $v^{n+1} = v^{n+1}(\phi(x))$. Let $(u^1, \dots, u^n, V)$ be a coordinate system at x in M^n such that $\phi(V) \subset U \cap S^n$. There exists a function u in $\perp(M)$, and $u^1 \circ \pi, \dots, u^n \circ \pi, u$ form a system of coordinates at $t(x)e(x)$. By virtue of Lemma 2, $\exp$ is regular at $t(x)e(x)$. Therefore, ϕ is a univalent mapping of V into $U \cap S^n$, and $u^i \circ \phi^{-1} = u^i \circ \pi \circ \exp^{-1}|_{\phi(V)}$ is a differentiable function of $v^1, \dots, v^n$. This implies ϕ is an immersion. Consequently, ϕ is a diffeomorphism of M^n into S^n since S^n is simply connected when $n \geq 2$.

Lemma of Nomizu and Rodríguez [2] can be stated as in the following

LEMMA ([2], p. 119). *Let M^n be a manifold immersed into E^m . Let $p \in E^m$ and assume that L_p has a non-degenerate critical point $x \in M^n$ of index k . Then there exist sequences $\{x_\lambda\}$ of points in M^n and $\{p_\lambda\}$ of points in E^m such that*

- (1) $\{x_\lambda\}$ and $\{p_\lambda\}$ converge to x and p respectively,
- (2) L_{p_λ} is a Morse function on M^n , and x_λ is a critical point of L_{p_λ} of index k .

2. Theorems

LEMMA 4. *Let M^n be a manifold immersed into E^m , $m > n + 1$. Let $x \in M^n$ and let $e(x)$ be a unit vector normal to M^n at x . We put $p(t, x) = x + te(x)$ when $t > 0$ and $q(s, x) = x + se(x)$ when $s < 0$. Then M^n belongs to a linear variety E^{m-1} if one of the following properties holds for every unit normal vector $e(x)$ at x :*

$$(a) \quad L_{p(t,x)}(x) \leq L_{p(t,x)}(y) \quad y \in M^n$$

for all $t > 0$.

$$(b) \quad L_{q(s,x)}(x) \leq L_{q(s,x)}(y) \quad y \in M^n$$

for all $s < 0$.

PROOF. Let S be a set of all unit vectors $e(x)$ normal to M^n at x . Then S can be considered as a Euclidean $(m-n-1)$ -sphere. We suppose that there exists a unit vector $e'(x)$ normal to M^n at x such that (a) does not hold for it. If we denote by B a set of such unit vectors $e'(x)$, then B is an open subset of S . Because, if B is not an open subset of S , there exists a sequence $\{e_\lambda(x)\}$ of unit vectors in $S - B$ which converges to a suitable $e'(x) \in B$. For each $e_\lambda(x)$, (a) holds. Therefore

$$\begin{aligned} L_{p'(t,x)}(x) &= \lim_{\lambda} L_{p_\lambda(t,x)}(x) \\ &\leq \lim_{\lambda} L_{p_\lambda(t,x)}(y) = L_{p'(t,x)}(y) \end{aligned}$$

for all $t > 0$ where $p'(t, x) = x + te'(x)$ and $p_\lambda(t, x) = x + te_\lambda(x)$. This contradicts $e'(x) \in B$.

When $e'(x) \in B$, (a) holds for $-e'(x)$ and hence $B \ni S$. Let $e(x)$ be a boundary point of B . Then there exists a sequence $\{e'_\mu(x)\}$ of unit vectors in B which converges to $e(x)$. For each $e'_\mu(x)$, (b) holds. Therefore

$$\begin{aligned} L_{q(s,x)}(x) &= \lim_{\mu} L_{q'_\mu(s,x)}(x) \\ &\leq \lim_{\mu} L_{q'_\mu(s,x)}(y) = L_{q(s,x)}(y) \end{aligned}$$

for all $s < 0$ where $q'_\mu(s, x) = x + se'_\mu(x)$. Since $B \ni S$, we obtain $e(x) \notin B$. These imply both of (a) and (b) hold for $e(x)$. Therefore

$$\langle e(x), y - x \rangle = 0$$

for all $y \in M^n$. Thus M^n belongs to a linear variety E^{m-1} perpendicular to $e(x)$.

THEOREM 1. *Let M^n be a compact connected manifold immersed into E^m . Let $e(x) \in \perp(M)$ and let $\alpha_1, \dots, \alpha_n$ be eigenvalues of $A_{e(x)}$. Assume that*

- (1) $\alpha_i \alpha_j \geq 0$ ($i, j = 1, \dots, n$) for every $e(x) \in \perp(M)$,
- (2) $C_0(L_p) = 1$ for every Morse function L_p on M^n .

Then M^n is diffeomorphic to n -sphere S^n .

PROOF. Let $e(x) \in \perp(M)$ be a unit vector at $x \in M^n$. We suppose that eigenvalues $\alpha_1, \dots, \alpha_n$ of $A_{e(x)}$ are non-positive. Let $t > 0$. Then x is a non-degenerate critical point of $L_{p(t,x)}$ of index zero. There exist sequences $\{x_\lambda\}$ of points in M^n and $\{p_\lambda\}$ of points in E^m as in Lemma of [2]. Since $C_0(L_{p_\lambda}) = 1$ from (2), Morse function L_{p_λ} attains to minimal value at exactly one point x_λ . Therefore

$$L_{p(t,x)}(x) = \lim_{\lambda} L_{p_\lambda}(x_\lambda) \leq \lim_{\lambda} L_{p_\lambda}(y) = L_{p(t,x)}(y)$$

for all $t > 0$. From (1), this implies one of the properties (a) and (b) in Lemma 4 holds for every unit vector $e(x)$ normal to M^n . According to Lemma 4, M^n belongs to a linear variety E^{n+1} . When $M^n \subset E^{n+1}$, the properties (a) and (b) for every unit vector $e(x) \in \perp(M)$ imply M^n is immersed into E^{n+1} as a convex hypersurface from Lemma 1. Thus, from Lemma 3, M^n is diffeomorphic to n -sphere S^n .

THEOREM 2. *Let M^n be a compact connected manifold immersed into E^m . Let $e(x) \in \perp(M)$ and let $\alpha_1, \dots, \alpha_n$ be eigenvalues of $A_{e(x)}$. We denote by N a set of $e(x) \in \perp(M)$ such that $\alpha_i \alpha_j > 0$ ($i, j = 1, \dots, n$). Assume that*

- (1) $\perp(M) = \bar{N}$,
- (2) $C_n(L_p) = 1$ for every Morse function L_p on M^n .

Then M^n is diffeomorphic to n -sphere S^n .

PROOF. Let $e(x) \in \perp(M)$ be a unit vector at $x \in M^n$. We suppose that eigenvalues $\alpha_1, \dots, \alpha_n$ of $A_{e(x)}$ are positive. Let α_1 be the smallest eigenvalue of $A_{e(x)}$. If $t > 1/\alpha_1$ then x is a non-degenerate critical point of $L_{p(t,x)}$ of index n . There exist sequences $\{x_\lambda\}$ of points in M^n and $\{p_\lambda\}$ of points in E^n as in Lemma of [2]. Since $C_n(L_{p_\lambda}) = 1$ from (2), Morse function L_{p_λ} attains to maximal value at exactly one point x_λ . Therefore

$$L_{p(t,x)}(x) = \lim_{\lambda} L_{p_\lambda}(x_\lambda) \geq \lim_{\lambda} L_{p_\lambda}(y) = L_{p(t,x)}(y)$$

and hence $\langle e(x), x - y \rangle \leq 0$ for all $y \in M^n$. If $s < 0$ then we obtain

$$L_{q(s,x)}(x) - L_{q(s,x)}(y) = -2s \langle e(x), x - y \rangle - \langle x - y, x - y \rangle \leq 0$$

for all $y \in M^n$. By the definition of N , this implies one of the properties (a) and (b) in Lemma 4 holds for every unit vector $e(x) \in N$.

When $e(z) \in \perp(M)$ is a unit vector at $z \in M^n$, there exists a sequence $\{e_\mu(x_\mu)\}$ of unit vectors in N which converges to $e(z)$. If (a) holds for all $e_\mu(x_\mu)$, then

$$\begin{aligned} L_{p(t,z)}(z) &= \lim_{\mu} L_{p_\mu(t,x_\mu)}(x_\mu) \\ &\leq \lim_{\mu} L_{p_\mu(t,x_\mu)}(y) = L_{p(t,z)}(y) \end{aligned}$$

for all $t > 0$ and $y \in M^n$ where $p_\mu(t, x_\mu) = x_\mu + te_\mu(x_\mu)$. Similarly, if (b) holds for all $e_\mu(x_\mu)$ then (b) holds for $e(z)$. Thus one of the properties (a) and (b) in Lemma 4 holds for every unit vector $e(x)$ normal to M^n . According to Lemmas 1 and 4, M^n belongs to a linear variety E^{n+1} , and it is immersed into E^{n+1} as a convex hypersurface. Consequently, from Lemma 3, M^n is diffeomorphic to n -sphere S^n .

References

- [1] J. Milnor, Morse theory, Ann. of Math. Studies, No. 51, Princeton University Press, 1963.
- [2] K. Nomizu and L. Rodríguez, Umbilical submanifolds and Morse functions, Nagoya Math. J., 48 (1972), 197-201.