EVALUATION OF THE DISTRIBUTION OF POWER
IN A WEIGHTED VOTING SYSTEM

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A distribution of weights in a weighted voting system sometimes yields quite undesirable effects on the outcome of voting, which have not been foreseen until such outcomes actually occur. Examples are the existence of so-called "dummies," unexpected power of veto, the failure of reflecting the dominance order of weights proportionally on the order of the effective power. The present paper introduce a number of criteria to distinguish such undesirable aspects which potentially exist in some particular configurations of the weight distribution. The criteria were stated in simple relations among weights which can be checked quite easily. Mathematical relations among the criteria were also presented in theorems with their empirical interpretations. Using the proposed criteria, the recent results of the general election for the House of Representatives in Japan was analyzed.

1. Introduction

In any existing society of democracy, there are some allowance for the unequal distribution of power, based upon a variety of socially accepted grounds, such as differences in the availability of natural resources, initial investments, and the population of supporting bodies.

Once a particular distribution of power is admitted, the members in a voting system are usually allocated differential votes so that they can cast different numbers of votes, either for the support or for the rejection of a proposed bill. Such a voting system is called, "a weighted voting system." Examples of voters of this kind are; the countries in the European Communities, stockholders in their general meeting, and representative voters in the legislature. Obviously, members having a large number of votes can easily pass the bill to which they support, by calling upon only a few more members, if necessary. By the same token, they can also block the passage of the bill with no difficulty. On the other hand, members having a small number of votes must persuade a great number of other members in order to pass what he believes to be the best.

Usually, a member's power of voting in a weighted voting system is evaluated in terms of how often he can be pivotal, i.e., taking the critical role for the passage of bills. In other words, if a member can transform the voting result from the passage to the blockage, by simply defecting himself or shifting from being the supporter to the rejector, while no other members change their votes, then we regard that he has a pivotal power on the outcome in this particular case. The concept of pivot as the basis...
of voting power has been first introduced by Shapley and Shubik (1954); they proposed the power index of a member as the proportion of permutations of all the members in which the member becomes pivotal. Banzhaf (1965) and Coleman (1971) both considered the set of "winning coalitions," i.e., successful groups which have enough supporters to pass the bill. Then, Banzhaf computed for each member the number of the winning coalitions which include the member in a pivotal position, then defined the power index for the member by the proportion of the above measure against the total of the same measures computed for all members. Coleman, on the other hand, defined the power index by the proportion of the winning coalitions in which the member takes a pivotal role against the total number of all winning coalitions.

Unfortunately, any of the power indices introduced above does not necessarily correspond to the distribution of power expressed in terms of the number of votes allocated to each member. For example, if a member holds more than one half of the total number of votes for all members, while the decision rule being simple majority, then he can act like a "dictator," i.e., his power measured by any of the above indices becomes 1.00, and those for the other members become 0.00, in spite of the fact that they have some votes to participate in voting.

There may exist some members who can exert some strong power under certain circumstances. For example, suppose that three members, Mr. A, Miss B, and Mrs. C had been allocated votes, 4, 3, and 1, correspondingly. Assume that the decision rule was a qualified majority of 6 out of 8 votes. Then it is obvious that, although none of the members is a "dictator," Mr. A alone or Miss B alone can block whatever the other members have proposed, and Mrs. C has no power to reflect her opinion, unless other member(s) would agree with her. In this case, we call Mr. A and Miss B as "veto-holders," and Mrs. C as a "dummy." Unfortunately, the indices proposed by Banzhaf (1965) and Coleman (1971) could not distinguish between "dictators" and "veto-holders" in voting bodies, because they yield the same 1.00 power.

It should be noted that, in spite of having been allocated some votes, Mrs. C was in fact completely powerless: she could never be in the pivotal position, and any decision can be made excluding her. (In this case, any of the power indices for Mrs. C would be zero.) Existence of such a dummy may not be desirable in most democratic societies.

There are some other cases in which the distribution of votes initially allocated to the members yields some unexpected effects on the outcomes of votes. For example, suppose that Mr. A has 1 vote, Miss B has 2 votes, and Mrs. C has 3 votes, decision rule being simple majority. In this case, there are exactly three possibilities of forming a "winning coalition": coalitions (A, C), (B, C), and (A, B, C). Therefore, there should be no difference as to the chance of being pivotal between Mr. A and Miss B, thus the two members having the same power, in spite of the fact that they had been allocated different numbers of votes. In such a case, we shall say that the weights of voting between the two members violate monotonicity, i.e., the condition that greater weights must yield greater voting powers, and vice versa.

The emergence of dummies and the unexpected violation of the monotonicity have been analyzed for some real cases, such as the New Jersey Senate, and Nassau County
in New York State (Banzhaf, 1965), and the European Communities in 1958 (Brams and Affuso, 1976). However, it is not an easy task to find whether there is a dictator, veto-holders, dummies, and so on, especially for a large size of voting body. Therefore it would be desirable to have some easy ways to distinguish important aspects of voting rule, such as the existence of a dictator, veto-holders, dummies, and to identify the members whose weights (numbers of votes allocated) violate the monotonicity. In the present paper, we shall propose simple criteria to find these aspects underlying a given voting body and the decision rule. Using the same method, we may also find a number of other aspects which any member of voting body may be concerned with, e.g., the number of other members to persuade in order to pass (or block) what he supports (or rejects).

2. Basic Definitions

Let \( I = \{1, 2, \cdots, n\} \) be a set of members in a voting body. Let \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \) be a vector of weights, where \( \alpha_i \) is the weight (usually, a certain number of votes) allocated to member \( i \).

Let \( L \) be a set of bills, and suppose that the voting body is deciding upon whether or not to pass a bill, \( \lambda \) in \( L \). Let \( D_i \) be a preference function such that \( D_i = 1 \) if and only if member \( i \) supported \( \lambda \), and \( D_i = 0 \) if and only if member \( i \) did not support (or rejected) \( \lambda \).

Now, suppose the values of the preference functions \( \{D_i\} \) are determined for the given \( \lambda \). The ordered set of preference values, the profile of members' preferences, will be denoted \( d_{\lambda} \). That is, \( d_{\lambda} = (D_1, D_2, \cdots, D_n) \).

Let us assume here that the set of bills, \( L \), is rich enough to yield all possible distributions of individual choices, i.e., for every element, \( r \), in \( \{1, 0\}^n \), there is the corresponding \( \lambda \) in \( L \), such that \( d_{\lambda} = r \).

Here, it will be convenient for the later use to introduce some notations for the case that a certain member \( i \) has not decided his or her choice yet. The unknown choice by member \( i \) will be denoted \( \tilde{D}_i \). The profile of preferences containing \( \tilde{D}_i \) will be denoted \( d_{\lambda}(i) \). That is, \( d_{\lambda}(i) = (D_1, D_2, \cdots, D_{i-1}, \tilde{D}_i, D_{i+1}, \cdots, D_n) \).

If the present member, \( i \), decided to "support" \( \lambda \), then \( D_i = 1 \) in \( d_{\lambda}(i) \), thus obtaining the profile to be denoted \( d_{\lambda}(i^+) \). Similarly, if the member \( i \) decided to "reject" \( \lambda \), then \( D_i = 0 \) in \( d_{\lambda}(i) \), thus obtaining the profile denoted \( d_{\lambda}(i^-) \). That is,

\[
\begin{align*}
    d_{\lambda}(i^+) &= (D_1, D_2, \cdots, D_{i-1}, 1, D_{i+1}, \cdots, D_n) , \\
    d_{\lambda}(i^-) &= (D_1, D_2, \cdots, D_{i-1}, 0, D_{i+1}, \cdots, D_n) .
\end{align*}
\]

For given bill \( \lambda \) in \( L \), the sum of weights for the supporters will be denoted \( s[d_{\lambda}] \); obviously, \( s[d_{\lambda}] = \alpha \cdot d_{\lambda} \).

Let \( K \) be a fixed real-value defining the decision rule such that the bill, \( \lambda \), is passed if and only if \( \alpha \cdot d_{\lambda} \geq K \). We may say that \( \lambda \) is blocked if and only if \( \alpha \cdot d_{\lambda} > N - K \). In order to avoid undecided cases, assume that \( K > N/2 \), where \( N \) is the total sum of allocated weights, i.e., \( N = \sum_{i \in I} \alpha_i \).

Any profile which satisfies \( \alpha \cdot d_{\lambda} \geq K \) will be called a winning coalition. The set of
Winning coalitions will be denoted $A_a$, i.e.,

$$A_a = \{ d_\mu | \alpha \cdot d_\mu \geq K, \mu \in L \} .$$

Similarly, the set of non-winning coalitions, $\overline{A}_a$, is defined by

$$\overline{A}_a = \{ d_\mu | \alpha \cdot d_\mu < K, \mu \in L \} .$$

Let us define a winning coalition for member $i$, $A_a(i)$, by the set of all winning coalitions which include member $i$ as a supporter;

$$A_a(i) = \{ d_\mu | d_\mu \leq i+1 \in A_a, \mu \in L \} .$$

In fact, the power index, called "Zipke's index," proposed by Nevison et al. (1978) corresponds to the notion of winning coalitions including member $i$. Zipke's index, $P_z^i$, can be expressed as;

$$P_z^i = \frac{\|A_a(i)\|}{\|A_a\|} .$$

However, Nevison et al. have shown that Zipke's index $P_z^i$ is essentially the same as Coleman's index $P_c^i$. That is, $P_z^i = 2P_c^i - 1$. Coleman's index will be introduced later in this paper (Section 4), since it exclusively uses the concept of "pivot" to express the power of voting.

3. Dictators and Veto-Holders

In this paragraph, we shall characterize dictators and veto-holders by the minimum size of coalitions to guarantee either the passage or the blockage of the bill.

Let $H_i = (\alpha_i(1), \alpha_i(2), \ldots, \alpha_i(n))$ be a permutation of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, such that $\alpha_i(1) = \alpha_i$, and $\alpha_i(j) \geq \alpha_i(j+1)$ for all $j = 2, 3, \ldots, n-1$. Then we can obtain integers $g_i, h_i$ in $1, 2, \ldots, n$ satisfying the inequalities;

$$\alpha_i(1) + \alpha_i(2) + \cdots + \alpha_i(g_i-1) < K,$$

$$\alpha_i(1) + \alpha_i(2) + \cdots + \alpha_i(g_i) \leq N - K,$$

$$\alpha_i(1) + \alpha_i(2) + \cdots + \alpha_i(h_i-1) < K,$$

$$\alpha_i(1) + \alpha_i(2) + \cdots + \alpha_i(h_i) \leq N - K,$$

where $\alpha_i(0) = 0$.

The $g_i$ is interpreted as the minimum size of supporters' coalitions with member $i$ to guarantee the passage of the bill, and $h_i$ is to be interpreted as the minimum size of rejectors' coalitions with member $i$ to guarantee the blockage of the bill. Since $K > N/2$, it always holds that $g_i \geq h_i$ for all $i$ in $I$.

**Dictator and Veto-holder.** Using the $g_i$ and $h_i$, we may define dictators and veto-holders: A **dictator** is the member $\delta$ such that $g_\delta = 1$, i.e., $\alpha_\delta \geq K$; A **veto-holder** is the member $\theta$ such that $h_\theta = 1$, i.e., $\alpha_\theta > N - K$.

We have the following theorems.

**Theorem 3.1** Any dictators are always veto-holders, but not vice versa.

(Proof) Since $g_i \geq h_i$ for all $i$, if $g_i = 1$, then $h_i = 1$. Q.E.D.

**Theorem 3.2** A member $\theta$ is a veto-holder if and only if member $\theta$ is always a supporter in every winning coalition.
Proof 

(i) Suppose that member \( \theta \) is a veto-holder, i.e., \( \alpha_\theta > N - K \), therefore, \( K > N - \alpha_\theta \). Then if member \( \theta \) is a rejector in some winning coalition, \( d_\lambda \), 

\[
 s \left[ d_\lambda < \theta^- > \right] \geq K .
\]  

(3.3)

Notice that it always holds that, 

\[
 s \left[ d_\lambda < \theta^- > \right] \leq N - \alpha_\theta .
\]  

(3.4)

From (3.3) and (3.4), we have \( K \leq N - \alpha_\theta \), which contradicts with the initial supposition that member \( \theta \) is a veto-holder. Therefore, it must be concluded that if member \( \theta \) is a veto-holder, then he must be a supporter in every winning coalition.

(ii) Suppose that member \( \theta \) is not a veto-holder, i.e., \( \alpha_\theta \leq N - K \). Then there exists a winning coalition in which member \( \theta \) is a rejector: For example, consider a profile \( d'_\lambda \) in which only member \( \theta \) is a rejector, i.e., \( D_\theta = 0 \), \( D_i = 1 \) for all \( i \neq \theta \). Then \( s[ d'_\lambda ] = N - \alpha_\theta \geq K \) (\( \therefore \alpha_\theta \leq N - K \)). Therefore, \( d'_\lambda \) is a winning coalition in which member \( \theta \) is a rejector. This argument concludes that if a member \( \theta \) is a supporter in every winning coalition, then member \( \theta \) is a veto-holder.

Therefore, (i) and (ii) conclude the proof of Theorem 3.2. Q.E.D.

Almost-Dictator and Almost-Veto-Holder. Even if a member \( i \) is neither dictator nor veto-holder, he (or she) may hold quite a large power of voting: For example, a member may be powerful enough to pass any bill if only another member, possibly the most powerless one, joined the member in support of the bill. Let us call member \( i \) an almost-dictator if he (or she) can block the bill by making a coalition with any other member in the body. Similarly, member \( j \) is called an almost-veto-holder if he (or she) can block the bill by making a coalition with any other member. More formally, member \( i \) is an almost-dictator if and only if \( a_i + a_{\min} \geq K \), where \( a_{\min} \) is the smallest weight in the body, and member \( j \) is an almost-veto-holder if and only if \( a_j + a_{\min} > N - K \). It would be obvious that any almost-dictators are also almost-veto-holders, since \( K > N - K \), but not vice versa.

4. Power Indices

Pivotal Sets and Pivotal Power. Let us define a pivotal set for member \( i \), \( P_a(i) \), by the set of all supporters’ coalitions in which member \( i \) takes the pivotal role, i.e.,

\[
P_a(i) = \{ d_\mu | d_\mu < i^+ > \in A_a, \ d_\mu < i^- > \in \bar{A}_a, \ \mu \in L \} .
\]

The cardinality of the set \( P_a(i) \), denoted \( \| P_a(i) \| \), is called the pivotal power.

In fact, the power indices by Banzhaf (1965) and Coleman (1971) correspond to the present notion of pivotal power. Using the present notations.

Banzhaf’s index, \( P_b^i \), can be expressed as:

\[
P_b^i = \| P_a(i) \| / \sum_{i \in I} \| P_a(i) \| .
\]

Coleman’s index, \( P_c^i \), can be expressed as:

\[
P_c^i = \| P_a(i) \| / \| A_a \| .
\]
5. Dummies

We may define a dummy as a member whose pivotal power is zero. In other words, member $i$ is dummy if and only if there is no $\lambda \in L$ such that $\bar{d}_\lambda(i^+) \in A_\lambda$ and $\bar{d}_\lambda(i^-) \in \bar{A}_\lambda$.

Before introducing a series of theorems concerning dummies, let us obtain some relations on the sum of supporters' weights.

It would be obvious that the sum of supporters' weights for $\bar{d}_\lambda(i)$, denoted $s[\bar{d}_\lambda(i)]$, must be the same as that for $\bar{d}_\lambda(i^-)$, that is,

$$s[\bar{d}_\lambda(i)] = s[\bar{d}_\lambda(i^-)]. \quad (5.1)$$

Obviously, the following relations hold:

$$s[\bar{d}_\lambda(i^+)] = s[\bar{d}_\lambda(i^-)] + \alpha_i = s[\bar{d}_\lambda(i)] + \alpha_i. \quad (5.2)$$

From the definition of dummy introduced above, the following theorems obtain;

**Theorem 5.1** Member $i$ is dummy if and only if every $\lambda \in L$ satisfies;

$$s[\bar{d}_\lambda(i)] \geq K, \quad \text{or}$$

$$s[\bar{d}_\lambda(i)] < K - \alpha_i. \quad (5.3) \quad (5.4)$$

*(Proof)* By definition, member $i$ is not dummy if there exists some $\lambda \in L$ such that $s[\bar{d}_\lambda(i^+)] > K$ and $s[\bar{d}_\lambda(i^-)] < K$. (5.5)

Using (5.2), (5.5) is equivalent to

$$K - \alpha_i < s[\bar{d}_\lambda(i)] < K. \quad (5.6)$$

From the denial of (5.6), we obtain (5.3) and (5.4). Q.E.D.

**Theorem 5.2** Any veto-holder is not a dummy.

*(Proof)* If member $0$ is a veto-holder, i.e., $\alpha_0 > N - K$, then

$$N - \alpha_0 < K. \quad (5.7)$$

Consider a profile, $\bar{d}_\lambda(0) = (1, 1, \ldots, 1, \bar{D}_\theta, 1, \ldots, 1)$. Then it holds that $s[\bar{d}_\lambda(0^+)] = N \geq K$, and that $s[\bar{d}_\lambda(0^-)] = N - \alpha_0 < K$, because of (5.7). Therefore, $s[\bar{d}_\lambda(0^+)] \geq K$ and $s[\bar{d}_\lambda(0^-)] < K$. That is, member $0$ is not a dummy. Q.E.D.

**Theorem 5.3** If there is a dictator, then all the other members are dummy.

*(Proof)* Suppose that the dictator $\delta$ in $I$ "supported" $\lambda$. Then, for any $i \neq \delta$, $s[\bar{d}_\lambda(i)] \geq K$, thus satisfying (5.3). Suppose that the dictator "rejected" $\lambda$. Then, $s[\bar{d}_\lambda(i)] \leq N - K - \alpha_i < K - \alpha_i$, because of $N - K < K$, then satisfying (5.4). That is, any member $i$, if not a dictator, must be dummy. Q.E.D.

6. Monotonicity

If member $i$ has been allocated the greater weight than member $j$, i.e., $\alpha_i > \alpha_j$, then we may naturally expect that the power of member $i$ would be greater than that of member $j$. If we take the pivotal power, then we expect that $\alpha_i > \alpha_j$ implies $\|P_\delta(i)\| > \|P_\delta(j)\|$. Unfortunately, however, the above expectation often fails.

In the present section, we shall be concerned with the condition, called *monotonicity,*
that if $\alpha_i > \alpha_j$, then $\|P_a(i)\| > \|P_a(j)\|$.

In what follows, we shall introduce some new notations. The $\bar{d}_3(i, j)$ indicates that members $i$ and $j$ have not decided whether to support or to reject the bill, $\lambda$. After they made up their minds, we have the following cases:

1. $dA(i, j) < i^+, j^+>$ if $D_i = 1$, $D_j = 1$,
2. $dA(i, j) = i^+, j^->$ if $D_i = 1$, $D_j = 0$,
3. $dA(i, j) = i^-, j^+>$ if $D_i = 0$, $D_j = 1$,
4. $dA(i, j) = i^-, j^->$ if $D_i = 0$, $D_j = 0$.

As we may see easily, the following relations always hold:

\[ s[\bar{d}_3(i, j)] = s[\bar{d}_3(i, j) - \alpha_i] \]
\[ s[\bar{d}_3(i, j) - \alpha_i] = s[\bar{d}_3(i, j)] - \alpha_i. \]

**Theorem 6.1** For any $i, j \in I, \alpha_i > \alpha_j$ implies $\|P_a(i)\| \geq \|P_a(j)\|$.

*(Proof)* For any $d_3 \in P_a(j)$, one of the following relations holds because member $j$ is pivot in $d_3$.

1. $s[\bar{d}_3(i, j)] > K$ and $s[\bar{d}_3(i, j) - \alpha_i] < K$.
2. $s[\bar{d}_3(i, j) - \alpha_i] < K$ and $s[\bar{d}_3(i, j)] < K$.

Using (6.2), we obtain,

\[ s[\bar{d}_3(i, j)] = s[\bar{d}_3(i, j)] + \alpha_i < s[\bar{d}_3(i, j) + \alpha_i = s[\bar{d}_3(i, j) + \alpha_i. \]

Therefore,

\[ s[\bar{d}_3(i, j)] < s[\bar{d}_3(i, j)]. \]

Therefore, (1) and (2) above imply the following conditions;

1. $s[\bar{d}_3(i, j)] > K$ and $s[\bar{d}_3(i, j)] < K$.
2. $s[\bar{d}_3(i, j)] > K$ and $s[\bar{d}_3(i, j)] < K$.

In other words, every $d_3 \in P_a(j)$ satisfies $d_3 \in P_a(i)$. That is, $\|P_a(i)\| \geq \|P_a(j)\|$.

**Theorem 6.1** tells us that pivotal powers would not reverse the order of the allocated weights.

**Theorem 6.2** For every $i, j \in I, \alpha_i > \alpha_j$ implies $\|P_a(i)\| > \|P_a(j)\|$ if and only if there is some $\lambda$ in $L$ such that

\[ K - \alpha_i \leq s[\bar{d}_3(i, j)] < K - \alpha_j. \]

*(Proof)* In order to obtain that $\|P_a(i)\| > \|P_a(j)\|$, it suffices to show that there is some $\lambda$ in $L$ such that $d_3 \in P_a(i)$ and $d_3 \notin P_a(j)$.

Here, $d_3 \in P_a(i)$ implies either one of the following (7) and (8):

1. $s[\bar{d}_3(i, j)] \geq K$ and $s[\bar{d}_3(i, j)] < K$.
2. $s[\bar{d}_3(i, j)] \geq K$ and $s[\bar{d}_3(i, j)] < K$. 


On the other hand, $d_A \notin P_a(j)$ implies that one of the following three conditions holds.

(9) \(s[d_A< j^+>] < K\) and \(s[d_A< j^->] < K\).

(10) \(s[d_A< j^+>] \geq K\) and \(s[d_A< j^->] \geq K\).

(11) \(s[d_A< j^+>] < K\) and \(s[d_A< j^->] \geq K\).

Each of the above conditions can be analysed further into two conditions depending upon $D_i=1$ or 0.

(9.1) \(s[d_A< j^+, i^+>] < K\) and \(s[d_A< j^-, i^+>] < K\).

(9.2) \(s[d_A< j^+, i^->] < K\) and \(s[d_A< j^-, i^->] < K\).

(10.1) \(s[d_A< j^+, i^+>] > K\) and \(s[d_A< j^-, i^+>] > K\).

(10.2) \(s[d_A< j^+, i^->] > K\) and \(s[d_A< j^-, i^->] > K\).

(11.1) \(s[d_A< j^+, i^+>] < K\) and \(s[d_A< j^+, i^+>] > K\).

(11.2) \(s[d_A< j^+, i^->] < K\) and \(s[d_A< j^- , i^- >] \geq K\).

Here, however, (9.1), (10.2), and (11.1) are incompatible with (7), while (9.1), (10.2), and (11.2) are incompatible with (8). Thus only (9.2) and (10.1) remain. Combining each of these with (7) and (8), we obtain the desired relation,

\[K-\alpha_i \leq s[d_A(i, j)] < K-\alpha_j.\]

**Q.E.D.**

**Theorem 6.3** Suppose that there is a veto-holder \(\theta\) in \(I\), i.e., \(\alpha_\theta > N-K\). Then for every member \(i\) in \(I\) such that \(\alpha_i \geq \alpha_\theta\), it holds that \(\|P_a(i)\| = \|P_a(\theta)\|\).

(Proof) Since member \(\theta\) is a veto-holder, \(\alpha_\theta > N-K\), i.e., \(N-\alpha_\theta < K\). On the other hand, for every member \(i \in I\),

\[s[d_A(i, \theta)] \leq N-\alpha_\theta - \alpha_i < K-\alpha_i.\]

Therefore, \(s[d_A(i, \theta)] < K-\alpha_i\). From Theorem 6.1 and Theorem 6.2, we obtain \(\|P_a(i)\| = \|P_a(\theta)\|\). Q.E.D.

**Theorem 6.4** Suppose that there is a veto-holder \(\theta\) in \(I\). Then for every member \(i \in I\), such that \(\alpha_i \leq N-K\) (i.e., member \(i\) is non-veto-holder), then \(\|P_a(\theta)\| > \|P_a(i)\|\).

(Proof) Consider a profile \(d_A^*\) such that

\[d_A^*(\theta, i) = (1, 1, \ldots, 1, D_\theta, 1, \ldots, 1, D_i, 1, \ldots, 1).\]

Then,

\[s[d_A^*(\theta, i)] = N-\alpha_i-\alpha_\theta \geq K-\alpha_\theta \quad (\therefore \alpha_i \leq N-K).\]  \((6.4)\)

On the other hand, since \(\alpha_\theta > N-K\),

\[s[d_A^*(\theta, i)] = N-\alpha_\theta - \alpha_i < K-\alpha_i.\]  \((6.5)\)

From (6.4) and (6.5), it holds that

\[K-\alpha_\theta \leq s[d_A^*(\theta, i)] < K-\alpha_i.\]

From Theorem 6.2, we obtain \(\|P_a(\theta)\| > \|P_a(i)\|\). Q.E.D.
7. Discussion and Conclusions

Using the results of the analyses in Sections 3-6, we may propose six conditions desirable for a weighted voting system to be "democratic" in some broad sense. The conditions are also summarized in terms of simple relations among allocated weights.

**C1 (Non-Dictatorship):** There should be no dictator in the voting body. That is, $\alpha_i < K$ for every $i$ in $I$.

**C2 (Non-Dummy):** There should be no dummy in the voting body. That is, $K - \alpha_i \leq s[\lambda(i)] < K$ for some $\lambda$ in $L$.

**C3 (Monotonicity):** The weights of voting should satisfy monotonicity. That is, there must be at least one possibility such that $K - \alpha_i \leq s[\lambda(i, j)] < K - \alpha_j$ for every $\alpha_i$ and $\alpha_j$ such that $\alpha_i > \alpha_j$.

**C4 (Non-almost-dictator):** It may not be desirable to have any almost-dictators. In this case, it must hold that $\alpha_i \leq K - \alpha_{\text{min}}$ for all $i$ in $I$.

**C5 (Non-Almost-Veto-Holder):** It may not be desirable to have any veto-holders. In this case, $\alpha_i \leq N - K - \alpha_{\text{min}}$ for all $i$ in $I$.

**C6 (Non-Veto-Holder):** It may not be desirable to have veto-holders in the voting body. In this case, $\alpha_i \leq N - K$ for all $i$ in $I$.

The first three, C1, C2, and C3, seem to be important requirements for any democratic voting. Violations of C4, C5, and C6 are often admitted in some social situations.

We have also derived a number of relations among voting characteristics.

**R1.** Any dictator is also a veto-holder, but not vice versa.

**R2.** A member is a veto-holder if and only if he (or she) is a supporter in every winning coalition.

**R3.** Veto-holders could never be dummies.

**R4.** If there is a dictator, then all the other members are dummies, having no power at all.

**R5.** The weak monotonicity holds between any weights of voting; that is, $\alpha_i > \alpha_j$ implies $\|P_\lambda(i)\| \geq \|P_\lambda(j)\|$.

**R6.** Any weights that are greater than the weight of a veto-holder (if it existed) would have the same power regardless of the differences in the weights.

**R7.** If a veto-holder exists, the weight of the veto-holder and that of any non-veto-holder should be monotonic.

**Analysis of Election Data.** Using the above criteria for democratic voting system, it would be interesting to analyze the result of general election for the House of Representatives in Japan, polled on October 7, 1979. At the time of the Lower House dissolution, the Liberal Democratic Party had been ruling so-called a "stable majority," occupying more than one half of the total seats in the House. According to our criteria, it simply implies that LDP had entertained the dictatorship, also the power of vetoing, while all the other parties had been dummies under simple majority.

As shown in Table 1, the result of the election was quite drastic. The LDP could not keep "stable majority", and lost not only the dictatorship, but also the veto-power. In the new distribution of seat numbers, there is only one dummy, the Social Demo-
ocratic Federation (Shaminren), which has only two seats in the new House. It should be noted here, however, how the monotonicity criterion is violated, under simple majority, even in the new distribution of seats. For example, the Japan Socialist Party, which occupies 107 seats, holds the pivotal power of only 4, which happened to be the same power as the New Liberal Club, which has only 4 seats. It should also be noted that if the decision rule is the qualified majority of the two third of total seats, which is supposed to be introduced for important issues, such as the amendments of the National Constitution, the situation will be much improved: Although LDP still has veto-power, it has no longer the dictatorship. Moreover, no party will be dummy, and the monotonicity holds everywhere.

**Final Remarks.** Readers may wonder if the “pivotal power” introduced in the present paper surely represents what we ordinarily evaluate as the political power or the social influence. It is true that the present notion of power reflects a very limited aspect of voting power, namely, the chances of being a pivot in all possible distribution of individual choices.

In the calculation of pivotal power, we have implicitly assumed the unrealistic conditions as follows: (i) There should be all possible distribution of individual choices, to occur all equally likely. (ii) There should be no difference as to which member to make a coalition with. Both (i) and (ii) ignore all aspects from ideological considerations, similarity or dissimilarity among members, bargaining possibilities, and so on.

Therefore, the evaluation of power in the present model should not be taken too seriously. Nevertheless, the present authors consider that the analyses introduced in the paper may serve for checking possibilities of extreme cases, violating important criteria for the democracy, often introduced without being noticed by most people because of the complexity of “number games” in politics.
REFERENCES


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