A MIXING INDEX BASED ON A MARKOV MODEL

Akihiro Nozaki*

A new mixing index, a criterion of degree of mixedness, is proposed for one-dimensional mixtures of two constituents on the basis of a Markov model. Homogeneous, random and segregated mixtures are clearly discriminated by the values of their mixing index. Practical way for evaluating the value of the index of a mixture from the variance of spot samples is discussed. Some applications to the three-dimensional mixtures are also considered.

Key Words and Phrases; Markov model, homogeneous mixture, random mixture, segregated mixture, index of mixture, spot samples

* International Christian University Osawa, Mitaka-shi, Tokyo, Japan

1. Introduction

Solid mixing or solid blending is an operation to mix two or more solid materials in particulate form so that the compositions of constituents might be homogeneous throughout the whole mixture. It is very often required in daily life, for instance in cooking foods and in shuffling cards, as well as in industrial, agricultural and pharmaceutical processes. However, unlike a liquid mixture, the homogeneity cannot be realized spontaneously. So we are often asked if a solid mixture is well mixed. To answer this question properly, we need some "mixing index", that is, a criterion of degree of mixedness.

Let us consider the ordering of ten boys and ten girls in a line. We can easily distinguish the following two extreme cases as following:

(1) B G B G B G B G B G B G B G G
and

(2) G G G G G G G G G G B B B B B B B B B B,

where the letter B stands for a boy and G for a girl. The line (1) is a "completely homogeneous mixture", while the line (2) is completely segregated. Between them, we may distinguish a random mixture as follows.

(3) G G B B G B G B G B B B B B B G G.

Obviously, a completely homogeneous arrangement of type (1) is an ideal mixture, although it cannot be obtained by any conventional mixing devices. A random mixture will be obtainable more efficiently and useful enough in many cases. So many mixing indices so far proposed are essentially criteria of degrees of randomness, which are usually defined in terms of statistical parameters such as "variance of spot samples taken from a mixture". A comprehensive survey on mixing indices is found in L.T. Fan, S.J. Chen and C.A. Watson (1960). A different way of measuring randomness is found in Nozaki (1976).

An interesting exception is the contact number proposed by Y. Akao [1974], which means the number of positions where two particles of different kinds contact to each
other. He defined a mixing index $M$ of a mixture of two constituents as follows:

$$M = \left( \frac{C}{N\check{p}} \right) / (n^*q) .$$

where $\check{p}$ represents the population concentration of a constituent, $q=1-\check{p}$, $N$ denotes the total number of particles and $n^*$ denotes the average number of adjacent particles of each particle. Since the value

$$Nn^*\check{p}q$$

is equal to the theoretical expectation $C_\gamma$ of the contact number of a completely random mixture, $M$ can be written in the following form:

$$M = C/C_\gamma .$$

Unfortunately, it is often impossible to count the contact number of a mixture. So Akao suggested to estimate the contact number from concentrations of a constituent in samples, in assuming that every sample is randomly mixed (see T. Yoshizawa and H. Shindo (1978) another model is found in Akao et al. (1976)).

This assumption may be quite reasonable in many practical cases. However, it is not at all safe for a homogeneous mixture, since the homogeneity of of its samples is disregarded and replaced by randomness.

In this paper, we propose a new mixing index $T$ on the basis of Markov model for one-dimensional mixture consisting of two constituents. It can discriminate clearly homogeneous, random and segregated mixtures. We shall show that the value of our mixing index $T$ can be estimated by the variance of spot samples taken from the mixture.

We shall see also that a better estimation of the value of $M$ can be obtained through our mixing index $T$.

2. Basic Concepts and Notations

We consider a sequence $X_1X_2X_3\ldots$ of particles of two types 0 and 1 such as

0 0 1 0 1 1 0 ...

We regard the sequence as a stationary two-state Markov chain which is determined by the transition probability matrix as follows:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} ,$$

where $P_{ij}$ represents the probability that the particle following to a particle $i-1$ is $j-1$. Obviously, $P_{ij}$'s are non-negative and

$$P_{11} + P_{12} = P_{21} + P_{22} = 1 .$$

We define mixing index $T$ of the sequence in the following manner:

$$T = 1 - \text{trace } P = 1 - P_{11} - P_{22} \quad (1)$$

The value of $T$ ranges from -1 to +1. When $T$ is equal to one, both $P_{11}$ and $P_{22}$ should be equal to zero, that is,
This implies that the sequence is completely homogeneous:

\[ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ldots \]

If \( T \) is equal to zero, then

\[
P = \begin{bmatrix}
\hat{p} & q \\
\hat{p} & q
\end{bmatrix},
\]

where \( \hat{p} \) is the population concentration of the particle 0. In this case, the sequence is completely random. Now if \( T \) is equal to -1, then the matrix \( P \) becomes the unit matrix. So the sequence is completely segregated. Thus we can separate homogeneous, random and segregated sequences by the values of our mixing index \( T \).

An experimental comparison between the indices \( T \) and \( M \) was done by J. Fukazawa: she showed that the values of \( M \) for homogeneous mixtures, estimated under Akao’s assumption, are almost same with those for random mixtures (Y. Hashiguchi and J. Fukazawa, 1979). On the other hand, the values of \( T \) are near to zero for random mixtures and near to one for homogeneous ones. Thus our mixing index \( T \) is advantageous to distinguish homogeneous mixtures from random ones.

3. Mixing Index \( T \) and Contact Number

Now let us consider the relations among the mixing index \( T \), the contact number \( C \) and other parameters related to them. In what follows, we represent the transition probability matrix \( P \) as follows:

\[
P = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}.
\]

Obviously,

\[0 \leq a, b, c, d \leq 1\]

and

\[a + b = c + d = 1.
\]

A. Transition matrix \( P \) and population concentration \( \hat{p} \).

If both \( b \) and \( c \) are positive, then the summation

\[(P + P^2 + \ldots + P^n)/n\]

converges to the matrix of the form:

\[
A = \begin{bmatrix}
\hat{p} & q \\
\hat{p} & q
\end{bmatrix},
\]

where \( \hat{p} \) is the population concentration of the particle 0 and \( q = 1 - \hat{p} \) (see, for instance, J.G. Kemeny and J.L. Snell, 1960). Besides, we have the following equalities:
PA = AP = A,

which is equivalent to the following equality:

\[qc = \frac{c}{b + c}.

(1)

If the matrix \(P\) is known, then we can determine the population concentration \(\rho\) by solving the equation (1):

\[\rho = \frac{c}{b + c}.

It should be noted that the matrix \(P\) is symmetric if and only if \(\rho = 1/2\).

B. **Mixing index \(T\) and transition matrix \(P\).**

If the mixing index \(T\) and the population concentration \(\rho\) are known, then we can find the transition matrix \(P\) by solving the equation (1) together with the equality:

\[T = 1 - a - d.

In fact, we have:

\[a = 1 - q(T + 1), \quad b = q(T + 1), \quad c = p(T + 1), \quad d = 1 - p(T + 1),

where \(q = 1 - p\).

C. **Matrix \(P\) and contact number \(C\)**

We assume that the population concentration \(\rho\) is known to us and satisfies the following inequality:

\[0 < \rho < 1 \quad \text{(2)}

Let \(N\) be the total number of particles. Then the number of the particle 0 is \(\rho N\), which can have at most \(2\rho N\) contact points with particles 1. Therefore the contact number \(C\) cannot exceed \(2\rho N\). By a similar reasoning, we have:

\[C \leq 2\rho N, \quad 2qN. \quad \text{(3)}

Now if the matrix \(P\) is known, then the contact number \(C\) can be estimated as follows:

\[C = N(pd + qc) \quad \text{(4)}

For instance, if \(T\) is equal to one, then

\[a = d = 0, \quad b = c = 1, \quad \rho = 1/2

and hence

\[C = N.

If \(T\) is equal to zero, then

\[a = b = c = d = 1/2, \quad \rho = 1/2

and therefore
In both cases, the values of \( C \) estimated from spot samples by Akao’s method are approximately \( N/2 \).

Now suppose that \( C \) is known. Then an estimation of the matrix \( P \) can be obtained by solving the equation (1) and (4), provided that the inequalities (2) and (3) are valid.

\[
\begin{align*}
  a &= 1-C/2pN, \\
  b &= C/2pN, \\
  c &= C/2qN \\
  d &= 1-C/2qN.
\end{align*}
\]

For the matrix \( P \) to be a probability matrix, we need the inequality (3). We need also (2) for utilizing the equation (1).

4. Practical Estimation of Mixing Index \( T \)

In practice, it is quite desirable to estimate a mixing index from observable data, such as sample concentrations of the particles of a specified type.

Let us consider \( r \) spot samples each of which contains \( n \) particles. We denote by \( x_i \) the number of 1’s in the \( i \)-th sample, and by \( \theta \) the variance of \( x_i \)'s. We assume again that the population concentration \( \hat{p} \) is known and

\[
0 < \hat{p} < 1.
\]

What we want is to estimate our mixing index \( T \) using the variance \( v \). Such an estimation is possible, at least in principle, if the theoretical variance \( v \) is a monotone function of \( T \): if this is the case, then \( T \) will be determined by \( v \), which can be estimated by \( \theta \).

Suppose that \( \hat{p}=1/2 \). Then the matrix \( P \) is symmetric, and hence

\[
c = b, \quad d = 1-c = a.
\]

We shall show that the theoretical variance \( v \) is a monotone increasing function of the component \( a \) when the sample size \( n \) is even.

**Theorem** Let us consider a 0-1 Markov chain with the transition probability matrix of the form:

\[
P = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.
\]

We take a sample of \( n \) consecutive elements from the chain and consider the variance \( v(a, n) \) of the number of 1’s in the sample. Then for any even number \( n \), the value of \( v(a, n) \) is a monotone increasing function of the diagonal element \( a \) of \( P \).

**Proof** Since \( P \) is symmetric, the population concentration \( \hat{p} \) is equal to 1/2. Hence theoretical mean of the number of 1’s is \( n/2 \).

Let us denote a spot sample of size \( n \) by

\[X_1 \cdots X_n .\]

Then the variance, hereafter abbreviated as \( v(n) \), can be represented as follows.
\[ v(n) = \sum_{i=1}^{n} (\sum_{j=1}^{n} a_{i-j/2})^2 P(X_i = a_i; \ 1 \leq i \leq n), \quad (1) \]

where the leftmost sigma means the summation for all binary \( n \)-tuples \((a_i)'s). In particular,
\[ v(2) = a/2 + a/2 = a. \quad (2) \]

In what follows we shall utilize the following quantities.

(a) The average number \( M(n) \) of 1's in the sample of size \( n \):
\[ M(n) = n/2. \]

(b) The average number \( I(n) \) of 1's in a sample starting from 1:
\[ I(n) = \sum (1 + \sum_{i=2}^{n} a_i) P(X_i = a_i; \ 2 \leq i \leq n | X_1 = 1). \quad (3) \]

(c) The average number \( O(n) \) of 1's in a sample starting from 0.

A. Properties of \( v(n) \)

We denote:
\[
w(x) = \begin{cases} 
  a & \text{if } x = 0, \\
  b & \text{if } x = 1.
\end{cases}
\]

then for any binary \( n \)-tuple \((a_i)\),
\[ P(X_i = a_i) = (1/2) w(a_1 \oplus a_2) \cdots w(a_{n-1} \oplus a_n), \]
where \( \oplus \) denotes the addition in modulo 2. Hereafter we denote:
\[ u(a_1, \ldots, a_n) = w(a_1 \oplus a_2) \cdots w(a_{n-1} \oplus a_n). \]

Then we have
\[ u(s, \ldots, t) = u(s^*, \ldots, t^*), \]
where \( x^* \) stands for \( 1-x \), and
\[ (\sum a_i - n/2)^2 = (\sum (1-a_i^*) - n/2)^2 \]
\[ = (n/2 - \sum a_i^*)^2 \]
\[ = (\sum a_i^* - n/2)^2. \]

Consequently,
\[ (\sum a_i - n/2)^2 u(a_1, \ldots, a_n) = (\sum a_i^* - n/2)^2 u(a_1^*, \ldots, a_n^*). \]

Now let us divide the leftmost summation of (1) into two parts as following:
\[ A = \sum a_1 = 1 + \sum_{i=2}^{n} a_{i-n/2}^2 (1/2) u(1, a_2, \ldots, a_n) \]
and
\[ B = \sum a_1 = 0 \sum_{i=2}^{n} a_{i-n/2}^2 (1/2) u(0, a_2, \ldots, a_n). \]
The latter part $B$ can be written as follows:

$$\sum a_i^* = 1 \left(1 + \sum_{i=2}^{n} a_i^* - n/2 \right)^2 \left(1/2 \right) u(1, a_2^*, \ldots, a_n^*) .$$

Since $a_i^*$'s are arbitrary, this is identical with $A$. Thus we have:

$$v(n) = A + B = 2A = \sum (1 + \sum_{i=2}^{n} a_i^* - n/2)^2 u(1, a_2, \ldots, a_n) . \tag{5}$$

Now let us divide again the leftmost summation of (5) into two parts as following:

$$C = \sum (2 + \sum_{i=3}^{n} a_i - n/2)^2 u(1, 1, a_3, \ldots, a_n) ,$$

and

$$D = \sum (1 + \sum_{i=3}^{n} a_i - n/2)^2 u(1, 0, a_3, \ldots, a_n) \nonumber$$

$$= \sum (1 + \sum_{i=3}^{n} a_i^* - n/2)^2 u(0, 1, a_3, \ldots, a_n^*) .$$

Then

$$C = \sum \left[ (1 + \sum a_i - (n-1)/2 + 1/2)^2 w(0) u(1, a_3, \ldots, a_n) \right]$$

$$= a \left[ \sum (1 + \sum a_i - (n-1)/2)^2 u(1, a_3, \ldots, a_n) \right]$$

$$+ \sum (1 + \sum a_i - (n-1)/2) u(1, a_3, \ldots, a_n) + 1/4 \nonumber$$

$$= a \left[ v(n-1) + (I(n-1) - (n-1)/2) + 1/4 \right] .$$

In a similar manner, we have

$$D = b \left[ v(n-1) - (I(n-1) - (n-1)/2) + 1/4 \right] .$$

Hence

$$v(n) = C + D = v(n-1) + (a-b) \left( I(n-1) - (n-1)/2 \right) + 1/4 . \tag{6}$$

B. Properties of $I(n)$ and $O(n)$

Let us denote

$$D(n) = I(n) - O(n) .$$

[1] By definition,

$$I(2) = 1 + a \quad \text{and} \quad O(2) = b = 1 - a$$

and therefore

$$I(2) + O(2) = 2 , \quad D(2) = I(2) - O(2) = 2a . \tag{7}$$

In general,

$$I(n) + O(n) = n$$

since

$$\left( I(n) + O(n) \right)/2 = M(n) = n/2 .$$
Consequently,

\[ I(n) - n/2 = D(n)/2. \] (8)

[2] For any integer \( n \) greater than one,

\[ I(n) = 1 + a \cdot I(n-1) + b \cdot O(n-1) \]

and

\[ O(n) = a \cdot O(n-1) + b \cdot I(n-1). \]

Hence

\[ D(n) = 1 + (a-b) I(n-1) + (b-a) O(n) \]

\[ = 1 + (a-b) D(n-1). \] (9)

Now let us consider the following inequality:

\[ D(n) \geq 0 \] (10)

By (7), this is the case for \( n=2 \). If this is true for some \( n \), then \( I(n) \) is not less than \( O(n) \) and therefore

\[ I(n+1) > I(n) > b I(n) + a O(n) = O(n+1). \]

Thus (10) is valid for any integer greater than one.

[3] If \( a \leq b \) and \( n \) is even, then

\[ (b-a) D(n) \leq 1/2. \] (11)

This is the case for \( n=2 \), since

\[ (b-a) D(2) = 2a(b-a) = 2a(1-2a) \leq 1/4. \]

Now suppose that \( b \geq a \) and the inequality (11) is valid for some \( n \). Then by (9)

\[ (b-a) D(n+2) = (b-a)(1-(b-a) D(n+1)) \]

\[ = (b-a)(1-(b-a)+\frac{(b-a)^2}{2} D(n)) \]

\[ \leq (b-a)(1+(b-a)/2) = 2x(1-x) \leq 1/2, \]

where \( x \) stands for \( (b-a)/2 \). Hence the inequality (11) is valid for any even integer \( n \) greater than one.

[4] Let us denote the differentiation by the diagonal element \( a \) by a prime ('). By the formula (9),

\[ D'(n) = 2D(n-1) + (a-b) D'(n-1). \] (12)

By (12) and (9), we obtain

\[ D'(n) = 2D(n-1) + (a-b) (2D(n-2) + (a-b) D'(n-2)) \]

\[ = 2 + 4(a-b) D(n-2) + (a-b)^2 D'(n-2). \] (13)

Now let us consider the following inequality:

\[ D'(n) \geq 0 \] (14)

This is the case for \( n=2 \), since \( D'(2) \) is equal to two.

If \( a \geq b \) and (14) is valid for some \( n \), then by (12)
\[ D'(n+1) = 2D(n) + (a-b) D'(n) \geq 0. \]

Thus (14) is valid for any integer greater than one, provided that \( a \geq b. \)

If \( a \leq b \) and (14) is valid for some \( n \), then by (13)
\[ D'(n+2) = 2 - 4(b-a) D(n) + (b-a)^2 D'(n) \geq 2 - 4/2 = 0. \]

Hence (14) is valid for any even integer \( n \), when \( a \leq b. \)

Now let us prove that
\[ a(b-a) D'(n) \leq 1/2 \quad (15) \]
for any even number \( n \). This is true for \( n=2 \), since
\[ a(b-a) D'(2) = 2a(1-2a) \leq 1/4. \]

Besides, if (15) is valid for some \( n \), then by (13)
\[ a(b-a) D'(n+2) = 2a(b-a) - 4a(b-a)^2 D(n) + a(b-a)^2 D'(n) \leq 2a(b-a) + (b-a)^2/2 = (1+2a)(1-2a)/2 \leq 1/2. \]

Hence the inequality (15) is valid for any even number \( n \).

C. Properties of \( v' (n) \)

As it has been shown in (6),
\[ v(n) = v(n-1) + (a-b)(I(n-1) - (n-1)/2) + 1/4 = v(n-1) + (a-b) D(n-1)/2 - 1/4 \quad (\text{see (8)}) \]
and hence
\[ v'(n) = v'(n-1) + D(n-1) + (a-b) D'(n-1)/2. \quad (16) \]

Now let us consider the following inequality:
\[ v'(n) > 0. \quad (17) \]

This is true for \( n=2 \), since
\[ v'(2) = [a]' = 1 > 0 \]
(see (2)). If \( a \geq b \) and (17) is valid for some \( n \), then \( D(n) \), \( (a-b) \) and \( D'(n) \) are all non-negative. Therefore by (16)
\[ v'(n+1) = v'(n) + D(n) + (a-b) D'(n)/2 > 0. \]

Hence (17) is valid for any integer \( n \) greater than one, provided that \( a \geq b. \)

Now suppose that \( a \leq b \) and \( n \) is even. Then by (16) and (12),
\[ v'(n) = v'(n-2) + D(n-2) - (b-a) D'(n-2)/2 \]
\[ + (1-(b-a) D(n-2)) - (b-a)2(D(n-2) - (b-a) D'(n-2))/2 \]
\[ \begin{align*}
\quad = v'(n-2) + (1-2(b-a)) D(n-2) \\
&+ ((b-a)^2/2-(b-a)/2) D'(n-2) + 1 \\
= v'(n-2) + 2aD(n-2) - (b-a) D(n-2) \\
&- a(b-a)D'(n-2) + 1.
\end{align*} \]

Since \( a \leq b \) and \( n \) is even, we can utilize the relations (14), (11) and (15). So we obtain

\[ \begin{align*}
v'(n) \geq v'(n-2) - 1/2 - 1/2 + 1 \\
\geq v'(n-2).
\end{align*} \]

Thus \( v'(n) \) is not less than \( v'(n-2) \). Therefore, if \( v'(n-2) \) is positive, then so is \( v'(n) \). Since \( v'(2) \) is positive, the inequality (17) holds for any even number \( n \).

After all, the theoretical variance \( v(n) \) is a monotone increasing function of the diagonal element \( a \) of the transition matrix \( P \) for any even number \( n \). This completes the proof of the theorem.

When \( P \) is symmetric,

\[ T = 1-2a, \]

that is,

\[ a = (1-T)/2. \]

So the theoretical variance \( v \) of spot samples is a monotone decreasing function of \( T \). Thus the value of \( T \) can be determined by \( v \), which can estimated by the observed variance \( \hat{v} \).

Some values of \( v(a, 30) \) are shown in Table 1.

It should be noted that the functions \( D(n) \) and \( v(n) \) are explicitly represented in the following forms.

\[ D(n) = (1-x^n)/(1-x), \]

and

\[ v(n) = a + [x/2(1-x) + 1/4](n-2) - x^2(1-x^n-2)/2(1-x)^2, \]

where

\[ x = a-b. \]

5. Markov Model for Three-dimensional Mixture

Let us consider the three-dimensional mixture of particles 0 and 1. For the simplicity, we assume that particles form a cubic mesh; in other words, the position of every particle can be represented by integer coordinates \((i, j, k)\): we denote the particle at the mesh point \((i, j, k)\) by \(x(i, j, k)\). We also assume that the transition probabilities of particles can be described by a 2X2 matrix \( P = [P_{ij}] \) in the following manner.

\[ \begin{align*}
P(x(i+1, j, k) = s \mid x(i, j, k) = t) \\
= P(x(i, j+1, k) = s \mid x(i, j, k) = t) \\
= P(x(i, j, k+1) = s \mid x(i, j, k) = t)
\end{align*} \]
In other words, we assume that the distribution of particles on any line parallel to an axis has the same transition matrix $P$, independent of the axis. This assumption will be acceptable for the models proposed in J.A. Dukes (1951) and F.S. Lai et al. (1974).

Under this assumption, the overall (three-dimensional) population concentration is identical with the concentration on an axis, provided that neither $b$ nor $c$ are equal to zero. We have therefore the same relations as in Section 3:

$$A = \lim_{n \to \infty} \frac{1}{n}(P + P^2 + \cdots + P^n)$$

and

$$AP = PA = A,$$

where

$$A = \begin{bmatrix} \hat{p} & 1 - \hat{p} \\ \hat{p} & 1 - \hat{p} \end{bmatrix}.$$

We shall summarize below what we can say about three-dimensional binary mixture (see Figure 1).

[1] If a three-dimensional binary mixture can suitably represented in the above mentioned manner, then its mixdness can well be represented by our mixing index $T$.

[2] By the value of $T$, we can discriminate clearly homogeneous, random and segregated mixtures.

[3] If the population concentration $\hat{p}$ is known, then we can calculate the matrix $P$ and estimate the contact number $C$ from the value of $T$.

[4] If the population concentration $\hat{p}$ is equal to $1/2$, then we can estimate the

---

**Fig. 1**

(a) $T=1$: a completely homogeneous mixture

(b) $T=-1$: a completely segregated mixture
value of $T$ from the variance of the numbers of 1's in spot samples.

Our assumption is not satisfied by striated mixtures considered by L.T. Fan et al. (1974). However, for striated mixtures, we can associate to each axis a transition matrix as follows:

$$P_X = P_Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$P_Z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$  

Hence the mixedness is determined by the third matrix $P_Z$, whose components represent the transition probabilities between $x(i, j, k)$ and $x(i, j, k+1)$. So our discussion can be applied also for striated mixtures.

6. Concluding Remarks

We have shown some basic properties of a new mixing index $T$, which is defined on the basis of a Markov model. Our index $T$ is quite convenient for distinguishing homogeneous mixtures from random ones, as well as random mixtures from segregated ones.

When the population concentration $\phi$ is equal to $1/2$, the value of $T$ can be estimated from the variance of spot samples. The estimation of $T$ in general seems rather hard. However, when the sample size $n$ is very large, we can utilize the limiting variance $\mathcal{V}$ of the number of 1's in a sample, which can be explicitly represented as follows (Kemmeny and Snell, 1960, Section 4.8):

$$\mathcal{V} = \lim_{n \to \infty} \frac{v(n, \phi)}{n} = \frac{bc(2-b-c)}{(b+c)^3},$$

where $a, b, c$ and $d$ are the components of the transition probability matrix $P$. 

<table>
<thead>
<tr>
<th>$T$</th>
<th>$a$</th>
<th>$\nu(a, 30)$</th>
<th>$\mathcal{V}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0.8</td>
<td>0.1</td>
<td>0.9566</td>
<td>1.8</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2</td>
<td>1.9922</td>
<td>1.6</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3</td>
<td>3.3163</td>
<td>1.4</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4</td>
<td>5.0694</td>
<td>1.2</td>
</tr>
<tr>
<td>0</td>
<td>0.5</td>
<td>7.5000</td>
<td>1</td>
</tr>
<tr>
<td>-0.2</td>
<td>0.6</td>
<td>11.0938</td>
<td>0.8</td>
</tr>
<tr>
<td>-0.4</td>
<td>0.7</td>
<td>16.9444</td>
<td>0.6</td>
</tr>
<tr>
<td>-0.6</td>
<td>0.8</td>
<td>28.1250</td>
<td>0.4</td>
</tr>
<tr>
<td>-0.8</td>
<td>0.9</td>
<td>57.5124</td>
<td>0.2</td>
</tr>
</tbody>
</table>

*) Estimation of the value $\mathcal{M}=C/C_r$ in our method (see Section 3, relations (4) and (5)).

If $\nu=6.28$, then the estimated value of $a$ would be 0.454.
Now let \( \rho \) be the population concentration of a constituent and \( q = 1 - \rho \). Then, as we have noted in Section 3,

\[
\rho = \frac{c}{b+c}
\]

and

\[
q = 1 - \rho = \frac{b}{b+c}.
\]

Therefore,

\[
V = \frac{bc}{(b+c)^2} \left( \frac{2}{b+c} - 1 \right)
\]

\[
= \rho q \left( \frac{2}{2-a-d} - 1 \right)
\]

\[
= \rho q \left( \frac{2}{1+T} - 1 \right).
\]

It can be easily seen that the limiting variance \( V \) is a monotone decreasing function of our index \( T \). Thus \( T \) can be estimated from the sample variance \( \hat{\theta} \), provided that the sample size \( n \) is sufficiently large and the variance \( \hat{\theta} \) can be utilized as an estimator of the limiting variance \( V \). In such a case, the size of each sample is not necessarily the same.

As for three-dimensional mixture, we considered only some particular cases. General discussion on three dimensional mixtures is highly desirable.

**References**


(Received October, 1979)