Inverse Solution in Electrocardiography:
Determining Epicardial from Body Surface Maps by Using
the Finite Element Method

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A new method of determining epicardial potentials from body surface maps is presented. Epicardial potentials can be estimated via the forward transfer matrix computed by using the finite element method. Due to smoothing and decrease in value in potential distribution of the body surface, the inverse problem involved becomes, in nature, ill-conditioned and direct application of the usual inversion technique will give an extremely oscillatory solution. Therefore, in order to obtain a practically meaningful solution, an appropriate regularizing procedure must be developed and, in the present paper, an effective regularization based on the generalized inverse matrix is proposed and its usefulness is demonstrated.

Numerical experiments suggest that if the epicardial map includes high components of spatial frequency the inverted epicardial map will have poor resolution. This is especially true at the epicardial surface distant from the body surface, such as on the diaphragmatic side of the ventricle. If the epicardial maps have to be inverted over the entire region of the epicardium with a clinically allowable accuracy, about 180 body surface lead points and 3 significant figures in the measurements of body surface potentials will be needed.

THE inverse problem in electrocardiography is, when the electrical potentials at the body surface are given at all times during a cardiac cycle, to determine the electrical activities in the heart at each instant in time. The inverse problem does not have a unique solution in the sense that, with a known geometry and conductivity in the thoracic region, knowledge of the body surface maps does not determine the heart generators uniquely without any specification on the physiologic nature of them. Therefore most of the work on the inverse problem has resulted in the determination of equivalent point sources (dipoles and multipoles).

It seems more important and useful to determine the epicardial potential map (surface sources, that is double-layer or single-layer), rather than determine the generators within the heart, for the following two reasons. First, the uniqueness of the inverse solution will be mathematically guaranteed from the fact that, without any source specification, potential maps on a closed surface enclosing all the generators in the heart, such as the epicardium, can be determined uniquely from the body surface potentials. Second, the inverted epicardial map would be the same as the actual (in situ) map on the epicardium.

Key Words:
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if the geometry and conductivities of the torso were given precisely and if the measurements of body surface potentials had infinite accuracy.

Several attempts have been made to determine the epicardial from the body surface map. Martin and Pilkington derived an equation relating the single and double layers on the epicardium and the body surface and assessed the feasibility of the inverse problem eliminating the term of the single layer, derived the transfer matrix which includes only the potentials based on the geometry of the body and the heart. These methods are so-called "solid angle methods" because they result in the solid angle calculation.

The present paper gives a new method of solving the inverse problem by using the forward transfer coefficients determined by means of the finite element method, and simulates estimating epicardial from body surface maps, and also discusses the properties of the inverse solution. In the finite element method, since the electric field can be determined by minimizing an energy functional based on the variational principle, it is easy not only to compute the transfer coefficients of the electrocardiographic field having complex geometry but it is also easy to include inhomogeneities and/or anisotropies in conductivity.

GOVERNING EQUATION IN ELECTROCARDIOGRAPHY AND SOLUTION BY THE FINITE ELEMENT METHOD

Assuming that the electrical activity of the heart can be characterized as current generators, the governing equation for the electrocardiographic field within the human torso is given by

\[
\frac{\partial}{\partial x} \left( \sigma_x \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \sigma_y \frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left( \sigma_z \frac{\partial \phi}{\partial z} \right) + J = 0 \tag{1}
\]

In this equation, \(\phi(x, y, z)\) is the scalar electric potential, \(\sigma_x, \sigma_y, \) and \(\sigma_z\) are, respectively, conductivities in the three coordinate directions, and \(J(x, y, z)\) is the current-generating source lying in the region of the heart. \(J\) is taken as zero in source-free regions, such as those surrounded by the body surface and the epicardium.

The boundary condition will be such that, at the interface between different tissues, both the potential and the normal component of the current must be continuous:

\[ \phi_1 = \phi_2, \tag{2} \]

\[ -\sigma_1 \frac{\partial \phi_1}{\partial n} = -\sigma_2 \frac{\partial \phi_2}{\partial n} \]

where \(\phi_1\) and \(\phi_2\) are the potentials in regions 1 and 2, respectively, and \(\sigma_1\) and \(\sigma_2\) are the respective conductivities in the normal direction \(n\) to the interface. Especially at the body surface, the normal potential gradient must be zero:

\[ \frac{\partial \phi}{\partial n} = 0 \tag{3} \]

The problem solving the equation (1) with the boundary conditions (2) and (3), when the electrical generators in the heart are given, is called the forward problem. Taking into consideration complex geometry and different conductivities within the human thorax, it will be convenient and practical to utilize the finite element method. Hence the procedure to solve the forward problem using the finite element method will be briefly stated as follows.

According to the variational principle, the solution of Eq. (1) along with the boundary conditions (2) and (3) is equivalent to finding the potential function \(\phi\) which minimizes the energy functional which is given below:

\[
\int_{\Omega} \left[ \frac{1}{2} (\sigma_x \frac{\partial \phi}{\partial x})^2 + \sigma_y \left( \frac{\partial \phi}{\partial y} \right)^2 \right.
+ \sigma_z \left( \frac{\partial \phi}{\partial z} \right)^2 - J \phi \right] \, dx \, dy \, dz \tag{4}
\]

In the finite element method, the region is discretized into a number of subdivisions called elements and the equation involved is transformed from an infinite degree of freedom into a finite one. The unknown function \(\phi\) is defined, element by element, using the nodal parameters and the shape functions of element, and is then substituted into the functional of Eq. (4). Thereafter, the minimization of the functional can be carried out. The potential function can then be obtained in the sense of a discretizing approximation.

METHOD OF SOLVING INVERSE PROBLEM

The inverse problem is the determination of the heart activities from the measured potential maps on the body surface, under the non-con-
dunding condition on the same surface. However, since Eq. (1) is the partial differential equation of elliptic type without the time-varying term, both the potentials and the potential gradients cannot be specified independently of each other on the same surface. This means that the inverse problem cannot be solved directly by the finite element method. Therefore, we will solve the inverse problem through the forward transfer functions which can be determined by the use of the finite element method.

In the following, although we will state the method of determining the epicardial potentials from the body surface map because of the uniqueness of the inverse solution, this method can be applied to the more general case where the electrophysiologic nature of generators is specified more precisely and it follows that the electrical activity within the heart has to be determined.

1. Determination of the Forward Transfer Function through the Finite Element Method

The nodes located on the epicardium, by discretizing into finite elements, are assigned numbers 1, 2, ..., n. As a base, a set

\[ G = (g_1, g_2, ..., g_n) \]  \hspace{1cm} (5)

of n linear independent voltage sources was chosen, where \( g_k \), called k'th fundamental potential function, is any B spline function with two spatial variables. A letter in bold face represents a function or a vector. The base G must be a set of functions by which the epicardial potential distribution can be expressed uniquely. The following were used here as the base:

\[ g_k = \begin{cases} \phi_k = 1 \\ \phi_j = 0 \text{ for all } j \neq k \end{cases} \]  \hspace{1cm} (6)

where \( \phi_k \) is the potential at the node marked k on the epicardium. \( g_k \) is a piece-wise linear function.

The fundamental potential function given at the epicardium, together with Eq. (2) at the interface between different tissues and Eq. (3) at the body surface, has to be considered the boundary conditions. They determine uniquely the electric field in the region between the body surface and the epicardium through minimizing Eq. (4), where J has to be zero because there is no current-generating source in that region.

Thus, it is possible to obtain the potentials of all nodes in the thorax using the finite element method, under the boundary conditions given by Eqs. (2), (3) and (6). Choosing an arbitrary but fixed point at the body surface as a reference, the potentials at the nodes on the body surface and the epicardium are expressed as the potential differences from the reference, respectively, by the following vector notations:

\[ b_k = (b_{k,1}, b_{k,2}, ..., b_{k,m})' \]  \hspace{1cm} (7.1)

\[ e_k = (e_{k,1}, e_{k,2}, ..., e_{k,n})' \]  \hspace{1cm} (7.2)

Here \( b_{k,i} \) is the potential on the body surface at node i where the lead point is set, \( (i = 1, 2, ..., m) \), and \( e_{k,j} \) is the potential at node j on the epicardium, \( (j = 1, 2, ..., n) \), and ' (prime) represents transposition. It should be noted that \( e_k \) differs from \( g_k \) by the potential of the reference. After the potential distributions of Eq. (7.1) are obtained for all \( k (= 1, 2, ..., n) \), the forward transfer matrix can be written in the form of Eq. (8):

\[ B = (b_1, b_2, ..., b_m) \]  \hspace{1cm} (8)

which relates epicardial to body surface potentials under the boundary conditions (2), (3) and (6), and which can be determined from a knowledge of the geometry and conductivity within the thorax.

2. Estimating Epicardial from Body Surface Maps

A measured potential map on m lead points of the body surface at one instant during the cardiac cycles is denoted as a vector:

\[ v = (v_1, v_2, ..., v_m)' \]  \hspace{1cm} (9)

Each \( v_j \) represents the potential difference from the reference. Since the basic equation of Eq.(1) is linear, vector \( v \) will be representable by a linear combination of vectors \( b_k \), \( k = 1, 2, ..., n \). Here, it is assumed that \( m \) is greater than or equal to \( n \), although this assumption can be ignored in Eq. (15). The coefficients, \( a_k 's \), of the linear combination can be determined through minimizing the following criterion:

\[ |v - Ba|^2 \]  \hspace{1cm} (10)

Thus, we have

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\[ B'Ba = B'v \]  \hspace{1cm} (11)\\

where \( a = (a_1, a_2, \ldots, a_n)' \). Solving Eq. (11) for \( a \), the epicardial potentials \( e \) can be estimated by the following:

\[ e = Ea = E(B'B)^{-1} B'v = Z^{-1} v \]  \hspace{1cm} (12)\\

where \( E \) and \( Z^{-1} \) will be defined by Eqs. (13) and (14), respectively:

\[ E = (e_1, e_2, \ldots, e_n), \]  \hspace{1cm} (13)\\

\[ Z^{-1} = E(B'B)^{-1} B' \]  \hspace{1cm} (14)\\

\( Z^{-1} \) is the inverse transfer matrix which relates body surface maps to epicardial potential distributions.

3. Regularizing the Inverse Problem

For the solution of the inverse problem, Eq. (12) is in the desired form since this equation expresses the vector of potentials on the epicardium as the product of a matrix \( Z^{-1} \) times the vector of potentials measured on the body surface. However, as we shall see later, the inverse solution computed from Eq. (12) will be unstable and will contain a considerably large oscillation. Such instability is due to some eigenvalues of the matrix \( B'B \) which have very small values, or in other words, it is due to the weakness of linear independence among the \( b_k \)'s in the matrix \( B \). This kind of problem is generally called the "ill-conditioned" or "incorrectly-posed" problem, wherein, a slight error in measurement and/or in modeling will reduce the stability of the solution.

In the ill-conditioned problem some regularizing technique has to be incorporated to obtain a practically meaningful solution. One of the most simplified and useful ways of regularization is to replace Eq. (12) by

\[ e = E(B'B + \gamma I)^{-1} B'v \]  \hspace{1cm} (15)\\

where \( \gamma \) is the regularizing parameter and \( I \) the identify matrix, and \((B'B + \gamma I)^{-1}\) is called Moore-Penrose generalized inverse of the matrix \( B \). The effect of the modification is the relative neglect of the small eigenvalues below noise level included in the measurements of body surface potentials; the result of which is the stabilization of the estimation procedure.

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A TORSO MODEL AND NUMERICAL EXPERIMENTS OF THE INVERSE PROBLEM

The geometry of the torso model being considered is shown in Fig. 1. It shows the human thorax from the vertebral level T1 to the vertebral body of L5 and contains broadly three regions: the heart, the lungs and the thoracic wall. Each region has been assumed to be isotropic but different in conductivity: lungs = 0.08 mho/m, thoracic wall = 0.17 mho/m and heart = 0.53 mho/m².

In order to apply the general finite element procedure to fully three-dimensional problems of electric field analysis, the torso has been discretized into finite elements. A convenient subdivision of volume conductor is into six-cornered triangular prisms. In the present study the torso model has been divided into 15 units with 16 sections and each section has been discretized into 205 triangles with 117 nodes as shown in Fig. 2. The triangular prisms were made from two corresponding triangles in two adjacent sections. Based on these triangular prisms, a quite useful series of tetrahedra has been created automatically with a simple logical program. The advantage of having arbitrary tetrahedron element is clear when approximating any boundary shape of the heart, the lungs and the body.

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Fig.1. Schematic diagram of the torso model from the vertebral level T1 to the vertebral body of L5. In the present study, the lungs as well as the heart are considered as inhomogeneous regions. The top figure shows a cross-section of the fifth intercostal space which is indicated by the dashed line in the bottom.
Fig. 2. Finite element discretization of the torso model. In order to compute the electrocardiographic field by the use of the finite element method, the torso model was divided, in the final stage, into a large number of tetrahedra.

surface. Thus, the surface of the heart has been composed of a large number of tetrahedra, where 112 nodes were assigned on the epicardium and the distance between two adjacent nodes was from 0.8 cm to 2.8 cm with the average being 2 cm.

Using the torso model discretized into tetrahedra elements as mentioned above and furthermore assuming that, for simplicity, the value of the potential at an arbitrary point in a tetrahedron element can be defined by the 4 nodal values, the numerical simulations of the inverse problem have been carried out in the following sequences:

(1) Potential distribution in the thorax is computed for the case where hypothetical generators exist within the heart and/or on the epicardium. Namely, a solution for the forward problem in electrocardiography is computed.

(2) The inverse solution is determined from the potential distribution at the body surface obtained in the forward solution. Namely, the body surface potential distributions in the forward solution are used instead of the measured potential maps for the inverse problem.

RESULTS

Firstly we will show a solution of the forward problem in ECG. Figure 3 shows an example of the electric field induced by current-generating sources within the heart. These figures were obtained from a numerical computation using the finite element method and the discretized model of the torso mentioned earlier. Figure 3-A shows the distribution of isopotential lines on a cross-section near the fifth intercostal space where one hypothetical dipole was located. Figures 3-B and C show the isopotential map on the body surface and the perspective plot of it. Figures 3-D and E show the isopotential map on the epicardial surface and its perspective view. The epicardial geometric model used to draw the epicardial map is shown in Fig. 4.

In these and the following figures, potential patterns are piecewiselinear because the potential, in the elements consisting of the torso, was expressed by the linear function of the nodal potentials and no smoothing technique was applied before plotting. Furthermore, the isopotential lines are drawn equi-distant in value from
Fig. 3. An example of the electrocardiographic field induced by a current-generating dipole. The dipole was located near the fifth intercostal region, which is indicated by the dashed line in Fig. 1. (A) Isopotential map in the corresponding section. (B) Isopotential lines of the body surface potentials and (C) its perspective view. (D) Isopotential lines of the epicardial potentials and (E) its perspective view.

Careful investigation of the maps in Fig. 3 will reveal that considerable smoothing of isopotential lines as well as a strong decrease in values of the potentials occur at points distant from the electric sources. This fact suggests inherent difficulties with the inverse problem because, from such a body surface map with smoothed and decreased values, a detailed structure of generators in the heart must be recovered.

Figure 5 shows the perspective plot of the epicardial potential distribution inverted from the body surface map shown in Fig. 3-B, by the use of Eq. (12) which is not regularized. It is quite apparent that there is considerable oscillation with extremely large amplitudes and the "true" map may become completely buried. Thus, direct application of usual inversion procedures to an ill-conditioned problem, such as the inverse problem in electrocardiography, will result in an oscillatory solution. Therefore, it is important to develop an appropriate procedure to eliminate the instability caused by the inversion of the higher components of spatial fre-
Fig. 4. Diagram of the epicardial surface with the associated anatomical features, which was used to draw the epicardial map. The left portion is an anterior view and the right portion is an unfolded posterior (diaphragmatic) view. Symbols RV, LV and ATR identify the right and left ventricular region and atrial region, respectively.

Fig. 5. An example of the epicardial map without a regularizing procedure. This inverse solution was computed from the body surface map shown in Fig. 3-B. The amplitude of the plotted potentials shown is one-fifth of the original map. There is considerable oscillation with great amplitude, due to the fact that the inverse problem involved is ill-conditioned.

Fig. 6. The inverted epicardial map with an appropriate regularizing procedure. (A) Isopotential lines and (B) its perspective view. Compare this with Fig. 5. This inverse solution was computed from the same body surface map. This agrees well with the true epicardial map shown in Fig. 3-D or E.

The epicardial potentials, shown in Fig. 6, were inverted from the same body surface map by the use of Eq. (15) which involves a regularizing parameter $\gamma$, where $\gamma$ was adjusted by the iterative optimization method that has been specially developed for the inverse problem in electrocardiography. A comparison of these two figures (5 and 6), referring to the "true" map shown in Fig. 3-D or E, makes clear that an application of appropriate regularization procedures can suppress oscillatory errors without appreciably affecting the shape of the "true" potential distribution.

With the regularizing procedure presented above, the oscillatory solution, due to the ill-condition of the matrix $B' B$, can be excluded by an appropriate choice of value of the regularizing parameter $\gamma$, and the epicardial map in good agreement with the "true" one can be recovered. However, it can be expected that, if the "true" potential distributions contain higher components of spatial frequency or sharply inflected patterns, the error between the true and estimat-

ed epicardial potentials will increase considerably.

Figure 7 shows another example of the electrocardiographic field. Figures 7-A, B, C and D are the isopotential lines on the body surface, the perspective plot of it, the isopotential lines on the epicardium and its perspective plot, respectively. In this case, hypothetical generators exist on the epicardium and there are 2 positive and 3 negative peaks on the anterior and posterior (diaphragmatic) side of the ventricle, respectively. Comparing this with Fig. 3, it was found that, in Fig. 7, both the body surface map and the epicardial map involve more complex patterns and therefore contain higher components of spatial frequency.

Figure 8 shows the epicardial map inverted, from the body surface map given in Fig. 7-A, by the use of Eq. (15). The oscillatory error can be eliminated through an appropriate choice of the regularizing parameter. However, the estimated shape of the epicardial map is appreciably affected, especially at the posterior region. This will be because the posterior (diaphragmatic) region is distant from the body surface and, therefore, the potential distribution induced by the generators existing on the posterior side becomes smaller and smoother. Thus, the resolving power in the inverse estimation will be considerably low in epicardial region distant from the body surface such as in the diaphragmatic side.

The epicardial maps in Figs. 6 and 8 are inverted from their related body surface maps, under the conditions that the total number of body surface lead points is 96 and the significant figures in the measurements of body surface potentials are three. These conditions can be considered suitable in the conventional technique of measuring body surface potentials. However, since the inverse problem involved, as mentioned above, is ill-conditioned, the degree to which the resolving power can be obtained in the inverse estimation, depends greatly upon both the amount and quality of the information obtained on the body surface.

Figure 9 shows the epicardial potential
distribution inverted from the body surface map shown in Fig. 7-A, where the number of body surface lead points is assumed to be 180. A comparison of Figs. 7-C, D, 8 and 9 makes clear that an increase in number of lead points improves the resolving power of estimating epicardial potentials. The potential peaks on the anterior side of the epicardium are almost recovered, and even on the posterior side the potential peaks are considerably recovered although the height of the peaks is still small.

Similar effects on improving the resolving power were observed when the significant figures in measurements of body surface potentials were increased. Figure 10 shows, the two examples mentioned above, the relation of the relative error in the inverse estimation to the number of lead points with differences in the significant digits in the measurements of potentials. From this figure, it can be pointed out that, if the epicardial potentials have to be recovered on the entire epicardium, especially on the diaphragmatic side, about 180 body surface lead points and 3 significant digits will be needed.

**DISCUSSION**

Recently, the determination of epicardial potential maps from body surface measurements has been attempted by several researchers.\(^5\) The epicardial potential map is a theoretically reasonable solution for the ECG inverse problem because it has its own uniqueness and also it corresponds with the actual (*in situ*) potential distribution on the epicardium. Although the epicardial maps may not be the final goal, it will enhance clinical diagnosis from the tacit assum-
tion that the inverted epicardial map will reflect the electrophysiologic events which take place within the heart.

Since the equation governing the electrocardiographic field is linear, the inverse solution can be obtained by superposing the forward transfer functions which relate the source parameters to the body surface potential maps. Hence, there is a strong relationship between the forward and inverse problems and most of the properties of the inverse solution can be accounted for by that of the forward transfer functions.

A new method in estimating epicardial potentials from body surface maps using the finite element method was presented earlier by the author in two dimensions. In this paper, it was extended to three dimensions and the properties of the inverse solution was discussed in more detail. The use of the finite element method enables any inhomogeneities and anisotropies in conductivity to be readily incorporated into the equation, thus allowing the transfer coefficients relating body surface maps to epicardial potentials to be calculated with a high degree of accuracy.

The inverse problem in electrocardiography is inherently ill-conditioned due to the smoothing and decrease in value of potentials in the electric field. Therefore, direct application of usual inversion techniques will result in an oscillatory solution. In the present paper, in order to obtain a clinically meaningful solution, an effective regularizing procedure was proposed and its effectiveness was demonstrated. The two examples presented here are assumed to show typical electric fields induced by generators within the heart. The results obtained from these numerical experiments will give general properties of the inverse solution in electrocardiography, because of the linearity of the governing equation.

The following, about the inverse solution, should be noted:

1. If the epicardial potential distribution is extremely inflected or includes high components of spatial frequency, the inverted epicardial map will have poor accuracy.

2. At points deeper within the body (further from the body surface) such as the posterior (diaphragmatic) side of the heart, the resolving power in estimating epicardial potential is relatively low.

3. In general, the degree to which the resolving power can be obtained in the inverse estimation, depends greatly upon both the amount and quality of the information measured on the body surface.

4. The increase in number of body surface lead points and/or the increase in accuracy in measuring body surface potentials improves the resolving power of the inverse estimation.

In the numerical experiments presented, there seems to be a saturated level which will show a limit to the resolving power of the inverse estimation, even if the number of lead points and the significant figures are infinitely increased (see Fig. 10). Although this is probably because the transfer matrix was determined by the use of the computer at hand which has a limit of 7 significant digits, it may be suggested that the error in computing the transfer matrix, due to the variation in torso geometry, will have an influence on the accuracy of the inverse solution.

It may be appreciable that, even with the commonly used techniques of body surface mapping, the epicardial potentials inverted from these body surface maps are accurate enough on the anterior side. Hence, they might be useful for the limited purpose of, for example, diagnosing the location of a myocardial infarction and/or a cardiac arrhythmia in the anterior side.
of the heart.

However, in order to recover the clinically meaningful epicardial maps over the entire region of the epicardium, body surface lead points must number about 180 and significant figures in measurements of potentials should be three. Such requirements appear too difficult to perform as a clinical routine procedure even with the newest techniques in the measurements of body surface potentials.

A possible alternative to obtain more information about heart activities may be the sampling of body surface maps at a higher frequency in time, for example, at the frequency of 4 kHz, since this high-frequency mapping of the body surface potentials will give the same effect as the increase of body surface lead points.

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