Expression to predict the chromatic dispersion of a high-order mode in few-mode fibers

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Abstract: A general expression for the waveguide dispersion of a propagation mode is derived from the wave equation. Furthermore, a simple expression for the waveguide dispersion of high-order modes is derived as a function of the mode field radius by substituting an approximated function of the electrical field distribution of the mode into the general expression. The chromatic dispersions of the LP₀₁, LP₁₁, LP₀₂, and LP₂₁ modes in a four-mode fiber are estimated using the proposed expression, and their accuracies are evaluated through simulations.

Keywords: chromatic dispersion, waveguide dispersion, high-order mode, mode field radius, few-mode fiber

Classification: Optical Fiber for Communications

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1 Introduction

Internet traffic has continued to increase in recent years owing to the rapid proliferation of internet services. Recently, a large-capacity transmission of 100 Tbps has been realized using standard single-mode fibers (SSMFs) [1]. To extend the limit of the transmission capacity of SSMFs, the use of a spatial division multiplex (SDM) system using multi-core fibers and few-mode fibers (FMFs) has been extensively investigated [2]. Knowledge of the modal dispersion of high-order modes in the FMFs is of significant interest in developing SDM systems. It has been reported that the modal dispersion of the fundamental mode in a single-mode fiber can be simply predicted based on the wavelength dependence of the mode-field radius (MFR) [3]. However, the prediction of the waveguide dispersion of a high-order mode in the FMFs and its simple expression have not been adequately considered.

This paper proposes a general expression for waveguide dispersion, which can be applied to the high-order linearly polarized LP_{lm} mode in the FMFs. Furthermore, the general expression is simplified as a function of the MFR by approximating the electrical field as a higher-order Gaussian function [4]. A simulation was performed to verify the accuracy of the proposed simple expression of chromatic dispersion for the LP_{01} , LP_{11} , LP_{02} , and LP_{21} modes in a four-mode fiber with a step-index profile. While part of this work has been reported [5], this paper presents more details regarding the previous investigations.

2 Derivation of a general expression for waveguide dispersion

The wave equation in a fiber can be expressed in cylindrical coordinates as

$$\frac{\partial^2 \phi_{lm}}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_{lm}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi_{lm}}{\partial \theta^2} + (k^2 n^2(r) - \beta^2) \phi_{lm} = 0. \quad (1)$$

Considering the weakly guiding approximation, the electric component $\phi_{lm}(r, \theta)$ of the linearly polarized LP_{lm} mode (where l and m denote the azimuthal and radial mode numbers, respectively) in cylindrical coordinates can be expressed as follows [6]:

$$\phi_{lm}(r, \theta) = R_{lm}(r) \cos(l\theta) \text{ or } R_{lm}(r) \sin(l\theta), \quad (2)$$

where r and θ denote the radial distance and azimuthal angle, respectively.

The index profile of the fiber can be defined as

$$n^2(r) = \begin{cases} n_1^2(1 - 2\Delta(r)) & r \leq a \\ n_1^2(1 - 2\Delta_0) = n_2^2 & r \geq a \end{cases}, \quad (3)$$

where a and Δ denote the core radius and relative-index difference between the core and cladding, respectively. n_1 and n_2 denote the refractive indices of the core and cladding, respectively.

Substituting Eq. (2) into Eq. (1) yields

$$\frac{1}{r} \frac{d}{dr} \left[r \frac{dR_{lm}}{dr} \right] + (k^2 n^2(r) - \beta^2 - \frac{l^2}{r^2}) R_{lm} = 0. \quad (4)$$

The normalized propagation constant b and normalized frequency v are defined as

$$b = \frac{(\beta/k)^2 - n_2^2}{n_1^2 - n_2^2} = \frac{a^2}{v^2} (\beta^2 - k^2 n_2^2), \quad (5)$$

$$v = a k n_1 \sqrt{2\Delta}, \quad (6)$$

where k is the wave number.

Multiplying Eq. (1) by R_{lm} and integrating over the cross-section A of a fiber, b can be expressed as

$$b = -\frac{a^2}{v^2} \iint_A \left(\frac{dR_{lm}}{dr} \right)^2 dA + \iint_{core} R_{lm}^2 dA - \iint_{core} \frac{\Delta(r)}{\Delta_0} R_{lm}^2 dA - \frac{a^2}{v^2} \iint_A \frac{l^2}{r^2} R_{lm}^2 dA. \quad (7)$$

After some calculations, the normalized group delay $d(vb)/dv$ can be obtained as

$$\frac{d(vb)}{dv} = \iint_{core} R_{lm}^2 dA + \frac{a^2}{v^2} \iint_A \left(\frac{dR_{lm}}{dr} \right)^2 dA - \iint_{core} \frac{\Delta(r)}{\Delta_0} R_{lm}^2 dA + \frac{a^2}{v^2} \iint_A \frac{l^2}{r^2} R_{lm}^2 dA. \quad (8)$$

Differentiating Eq. (8) with respect to v yields

$$\begin{aligned} \frac{d^2(vb)}{dv^2} = & \iint_{core} \left(1 - \frac{\Delta(r)}{\Delta_0} \right) 2R_{lm} \frac{dR_{lm}}{dv} dA - \frac{2a^2}{v^3} \iint_A \dot{R}_{lm}^2 dA + \frac{a^2}{v^2} \iint_A 2\dot{R}_{lm} \frac{d\dot{R}_{lm}}{dv} dA \\ & + \frac{a^2}{v^2} \iint_A \frac{l^2}{r^2} 2R_{lm} \frac{dR_{lm}}{dv} dA - \frac{2a^2}{v^3} \iint_A \frac{l^2}{r^2} R_{lm}^2 dA. \end{aligned} \quad (9)$$

where $\dot{R}_{lm} = dR_{lm}/dr$ and $d\dot{R}_{lm}/dv = d/dr(dR_{lm}/dv)$.

Multiplying Eq. (4) by dR_{lm}/dv and integrating over A yields

$$\frac{a^2}{v^2} \iint_A \frac{d\dot{R}_{lm}}{dv} \dot{R}_{lm} dA = \iint_{core} \left(1 - \frac{\Delta(r)}{\Delta_0} \right) R_{lm} \frac{dR_{lm}}{dv} dA + \frac{a^2}{v^2} \iint_A \frac{l^2}{r^2} \frac{dR_{lm}}{dv} R_{lm} dA. \quad (10)$$

Substituting Eq. (10) into Eq. (9), the normalized waveguide dispersion $vd^2(vb)/dv^2$ can be expressed as

$$v \frac{d^2(vB)}{dv^2} = 2a^2 \frac{d}{dv} \left[\frac{1}{v} \left\{ \iint_A \dot{R}_{lm}^2 dA + \iint_A \frac{l^2}{r^2} R_{lm}^2 dA \right\} \right]. \quad (11)$$

When $[(\lambda/n_2)(dn_2/d\lambda)] \ll 1$, the waveguide contribution to the dispersion of the group delay τ per unit length of a fiber is [6]

$$D_w = \frac{d\tau}{d\lambda} = -\frac{n_2\Delta_0}{c\lambda} v \frac{d^2(vB)}{dv^2}, \quad (12)$$

where c is the velocity of light in vacuum.

Therefore, the waveguide dispersion D_w can be expressed as

$$D_w = \frac{d\tau}{d\lambda} = -\frac{n_2\Delta_0}{c\lambda} v \frac{d^2(vB)}{dv^2} = -\frac{n_2\Delta_0}{c\lambda} 2a^2 \frac{d}{dv} \left[\frac{1}{v} \left\{ \iint_A \dot{R}_{lm}^2 dA + \iint_A \frac{l^2}{r^2} R_{lm}^2 dA \right\} \right]. \quad (13)$$

3 Simple expression for chromatic dispersion

3.1 Waveguide dispersion

To obtain a simple expression for the waveguide dispersion of the LP_{lm} mode using Eq. (13), the expression for the electric field of the LP_{lm} mode is required. Here, we approximated the electric field $R_{lm}(r)$ of a given LP_{lm} mode in the FMFs as follows [4]:

$$R_{lm}(r) = \frac{2}{r_{lm}} \sqrt{\frac{(m-1)!}{(l+m-1)!}} \left(\frac{\sqrt{2}r}{r_{lm}} \right)^{(l)} \times L_{m-1}^{(l)} \left(\frac{2r^2}{r_{lm}^2} \right) \exp \left(-\frac{r^2}{r_{lm}^2} \right), \quad (14)$$

where r_{lm} is the spread of the electric field in the LP_{lm} mode, which corresponds to the higher-order Gaussian spot size [4]. $L_{m-1}^{(l)}(x)$, which is the associated Laguerre polynomial, can be defined as

$$L_{m-1}^{(l)}(x) = \sum_{i=0}^{m-1} (-1)^i \frac{(l+m-1)!}{i!(m-1-i)!(l+i)!} x^i. \quad (15)$$

It should be noted that Eq. (14) is an exact solution for a fiber with an infinite parabolic profile. It has been reported that the measured electric fields are in good agreement with the electric field approximated by Eq. (14) in several FMFs [7].

The approximated expression for the electric field of each LP_{lm} mode is expressed in terms of r_{lm} using Eqs. (14) and (15). The following definition for the mode field diameter (MFD) is considered a means to evaluate the spread of the electric field [8]:

$$2w_{lm} = 2 \left[\frac{\int_0^\infty R_{lm}^2(r) r dr}{\int_0^\infty \left(\frac{dR_{lm}}{dr} \right)^2 r dr} \right]^{1/2}. \quad (16)$$

The relationship between r_{lm} and the MFR w_{lm} can be obtained by substituting the approximated electric field R_{lm} of each mode into the MFD definition in Eq. (16). By substituting Eq. (14) into Eq. (13) and using the relationship between r_{lm} and w_{lm} , a simple expression for the waveguide dispersion of each mode as a function of MFR can be obtained. The simple expressions for the lowest four LP modes are summarized in Table I.

Table I. Waveguide dispersion expression for each mode in FMFs

Mode	Waveguide dispersion expression
LP ₀₁	$D_w = \frac{\lambda}{2\pi^2 n_1 c w_{01}^2} \left(1 - \frac{2\lambda}{w_{01}} \frac{dw_{01}}{d\lambda} \right)$
LP ₁₁	$D_w = \frac{\lambda}{\pi^2 n_1 c w_{11}^2} \left(1 - \frac{2\lambda}{w_{11}} \frac{dw_{11}}{d\lambda} \right)$
LP ₀₂	$D_w = \frac{\lambda}{2\pi^2 n_1 c w_{02}^2} \left(1 - \frac{2\lambda}{w_{02}} \frac{dw_{02}}{d\lambda} \right)$
LP ₂₁	$D_w = \frac{3\lambda}{2\pi^2 n_1 c w_{21}^2} \left(1 - \frac{2\lambda}{w_{21}} \frac{dw_{21}}{d\lambda} \right)$

3.2 Chromatic dispersion

The chromatic dispersion D of each mode in the FMFs can be expressed as [9]

$$D = -\frac{1}{c\lambda} \left[k \frac{dN_2}{dk} + k \frac{d(N_1 - N_2)}{dk} \frac{d(vb)}{dv} + (N_1 - N_2)v \frac{d^2(vb)}{dv^2} \right]$$

$$\simeq -\frac{1}{c\lambda} \left[k \frac{dN_1}{dk} + (N_1 - N_2)v \frac{d^2(vb)}{dv^2} \right] = D_m + D_w \quad (\text{when } \frac{d(vb)}{dv} \simeq 1) \quad (17)$$

where N_1 and N_2 denote the group indices of the core and cladding, respectively.

In this case, D can be approximately expressed as the sum of the dispersions of the waveguide (D_w) and material (D_m) when $d(vb)/d\lambda$ is approximately 1.

D_m can be defined as

$$D_m = -\frac{\lambda}{c} \frac{d^2 n}{d\lambda^2}. \quad (18)$$

In particular, the material dispersion is estimated from the Sellmeier formula in which the coefficients correspond to the relative-index difference Δ .

3.3 Simulations

Numerical simulations were performed to verify the accuracy of the simple expression in Table I to estimate the chromatic dispersion of each mode in an FMF. A four-mode fiber with a of 12 μm and Δ of 0.32% was considered, along with a step-index profile.

Figures 1(a)–1(d) show the estimated and calculated chromatic dispersions as a function of the wavelength for the LP₀₁, LP₁₁, LP₀₂, and LP₂₁ modes. Furthermore, to evaluate the accuracy of the proposed expression, we estimated the error of the chromatic dispersion for each mode, as shown in Fig. 1. It can be observed that the estimated chromatic dispersions for the LP₀₁, LP₁₁, and LP₀₂ modes are in good agreement with the calculated values. The estimated accuracy was within 0.8 ps/km/nm in the wavelength region between 1.2 μm and 1.6 μm . Although the estimated error for the LP₂₁ mode was determined to be less than 4 ps/km/nm, it was larger than those of the other modes. This error is probably due to the difference between the actual electric field of the LP₂₁ mode and the electric field approximated using Eqs. (14) and (16). It is known that the MFD values of higher-order modes depend on the definition for obtaining the MFD [7]; therefore, we need to clarify the appropriate definition in the near future to improve the estimation accuracy. Nevertheless, these results indicate that the proposed technique can estimate the

chromatic dispersion of a high order mode in a four-mode fiber with an accuracy of less than 4 ps/km/nm.

Fig. 1. Comparison of estimated and calculated chromatic dispersions and the estimated error for each mode in a four-mode fiber

4 Conclusion

A general expression for the waveguide dispersion was derived, which was applied to the high-order LP_{lm} mode in FMFs. Furthermore, a simple expression for chromatic dispersion, which is composed of waveguide and material dispersions, was derived from the general expression. The chromatic dispersions of the LP_{01} , LP_{11} , LP_{02} , and LP_{21} modes in a four-mode fiber were estimated using the proposed expressions. These chromatic dispersions could be successfully estimated with an accuracy of less than 1 ps/km/nm, except in the case of LP_{21} mode in which the estimated accuracy was less than 4 ps/km/nm. Future work will focus on improving the proposed technique.

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