Cramer-Rao lower bound for wideband ToA estimation in the presence of RF nonideality

Tiandong Wang, Yukui Pei^{a)}, and Ning Ge

Tsinghua National Lab for Info. Science and Technology, Beijing, China

a) peiyk@mail.tsinghua.edu.cn

Abstract: In wideband localization systems based on time of arrival (ToA), RF nonideality, such as carrier phase noise, IQ imbalance, phase and frequency offset, plays an important role for ToA estimation accuracy. In this work, Cramer-Rao lower bound (CRB) for ToA estimation in the presence of RF nonideality is first proposed as criterion for ToA estimator performance in practical ToA localization system. Evaluation of CRB with and without phase noise has been performed, especially closed form of CRB without phase noise has been derived. Theoretical and numerical results show that the ToA accuracy presents *floor* phenomenon in the presence of RF nonideality, and phase noise plays the dominant role. The effect of record length has also been evaluated.

Keywords: Cramer-Rao lower bound, RF nonideality, phase noise (PN), time of arrival (ToA), wideband localization

Classification: Navigation, Guidance and Control Systems

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1 Introduction

The Cramer-Rao lower bound (CRB) is a well known lower bound on variance of unbiased estimation on any deterministic parameter, and serves as a benchmark of practical estimators [1]. Time of arrival (ToA) is one of the fundamental parameters of time-of-flight based wideband localization [2]. In previous papers, the CRB of ToA depends on the effective bandwidth of the transmitted signal, the signal to noise ratio (SNR) [2], and path overlap coefficient (in multipath environment) [3]. Based on these three factors, in [2], the root mean square error (RMSE, the square root of CRB) of ToA achieves 10^{-10} seconds when high SNR (above 13 dB), corresponding to less than 3 cm ranging error.¹ Moreover, when SNR goes higher, the ranging error goes to less than 1 cm.

However, in real scenarios [4, 5, 6], error of ranging (or localization) is often tens of centimeters, ten times above the CRB proposed by [2]. For in real scenarios, apart from multipath, there are a lot of non-ideal hardware factors that affect the practical system performance, such as clock drift and jitter, carrier frequency and phase offset, phase noise (PN) and IQ imbalance. If these factors are ignored, the CRB of ToA may not be functional to the practical systems. In [7], the effect of clock frequency offsets for ultra-wideband TOA estimation was investigated, and a novel algorithm was proposed to estimate and compensate the clock difference. For other factors, there are few literatures. Moreover, in real scenarios, the record length is finite, while the bandlimited signal is infinite long. Thus the record length greatly affects the performance of ToA localization systems. Meanwhile, many simplified ToA estimation methods [2] use finite samples to estimate ToA. Since long record length leads to the increase of calculation burden, it is of great significance to investigate the effect of record length on ToA estimation.

In this paper, we investigate an analytical expression of the CRB on ToA with carrier frequency and phase offset, IQ imbalance in AWGN channel, followed by CRB analysis with carrier phase noise in AWGN channel.

This paper is organized as follows: in Section II, the ToA system model will be introduced; analysis on CRB of ToA will be presented in Section III; in Section IV numerical results will be presented; and the conclusion goes to Section V.

Notations: $\mathbb{E}_{\mathbf{z}}\{\cdot\}$ is the notation of the expectation operator with respect to random vector $\mathbf{z}$; $(\cdot)^H$ is complex conjugate transpose of its argument; $\text{tr}\{\cdot\}$ is the trace of a square matrix; $\text{Re}\{\cdot\}$ is the real part of its argument; $[\cdot]_{i,j}$ is the element at the i th row and j th column of its argument; $\mathbb{Z}$ denotes the set of integer.

2 System model

We consider a scenario where a pulse² $s(t)$ of which the energy is denoted as E is first passed to an non-ideal radio front-end (RF) through both in-phase and

¹in the condition of UWB CM4 channel model and with 1 ns pulse width.

²In general, $s(t)$ can be a sequence modulated by shaping pulse, e.g. root raised cosine pulse. For bandlimited signal, we consider the sampling interval contains most energy of the signal.

quadrature channel, then transmitted through a channel with additive white Gaussian noise (AWGN). At last, it is received by another non-ideal RF and sampled with sampling period T_s . At receiver front-end, the arriving signal is

$$x(t) = \text{Re}\{(s(t) + js(t))c_u(t)\}, \quad (1)$$

where $c_u(t)$ is the up-conversion carrier expressed as

$$c_u(t) = E_u \left(\left(1 + \frac{\alpha_u}{2} \right) \cos \left(\omega t + \phi_u(t) + \frac{\theta_u}{2} \right) + j \left(1 - \frac{\alpha_u}{2} \right) \sin \left(\omega t + \phi_u(t) - \frac{\theta_u}{2} \right) \right). \quad (2)$$

In (2), E_u stands for the carrier amplitude; α_u and θ_u stand for IQ amplitude and phase imbalance; $\phi_u(t)$ is carrier phase noise and ω is the carrier angular frequency. For down-conversion carrier, the corresponding parameters are denoted as $c_d(t)$, E_d , α_d , $\phi_d(t)$, and θ_d . Moreover, the frequency and phase offset are denoted as ω_0 and ψ . That is, the I and Q channel down-converted signal $r'_I(t)$ and $r'_Q(t)$ can be acquired by $r_I(t)$ and $r_Q(t)$ in (3) with cancelling the term of passband signal at 2ω .

$$\begin{cases} r_I(t) = x(t - \tau) \cos \left((\omega + \omega_0)t + \phi_d(t) + \psi + \frac{\theta_d}{2} \right) \\ r_Q(t) = x(t - \tau) \sin \left((\omega + \omega_0)t + \phi_d(t) + \psi - \frac{\theta_d}{2} \right). \end{cases} \quad (3)$$

Since the phase noise is very small, i.e. $\phi(t)$ is concentrated in a very small range with large probability, we use the approximation with first order Taylor series expansion³ $\sin(\phi(t)) \approx \phi(t)$ and $\cos(\phi(t)) \approx 1$. When finished down-conversion, after some trigonometrical transformation and some algebra, the received signal becomes

$$\begin{aligned} r(t) &= r'_I(t) + jr'_Q(t) \\ &= \mu(t) + v(t)(\phi_u(t) - \phi_d(t)) + n(t). \end{aligned} \quad (4)$$

where $n(t)$ is the white Gaussian noise of which the variance is σ^2 and the power spectral density is N_0 , and the $\mu(t)$ and $v(t)$ are

$$\begin{aligned} \mu(t) &= s(t - \tau) E_\mu \left(\left(1 + \frac{\gamma_\mu}{2} \right) \cos \left(\omega_0 t + \Psi_\mu + \frac{\Phi_\mu}{2} \right) + j \left(1 - \frac{\gamma_\mu}{2} \right) \sin \left(\omega_0 t + \Psi_\mu - \frac{\Phi_\mu}{2} \right) \right) \\ v(t) &= s(t - \tau) E_v \left(\left(1 + \frac{\gamma_v}{2} \right) \cos \left(\omega_0 t + \Psi_v + \frac{\Phi_v}{2} \right) + j \left(1 - \frac{\gamma_v}{2} \right) \sin \left(\omega_0 t + \Psi_v - \frac{\Phi_v}{2} \right) \right). \end{aligned} \quad (5)$$

³In the following paragraph, we assume the distribution of $\phi(t)$ is Gaussian. It's true the range of $\phi(t)$ is from $-\infty$ to $+\infty$ while $\sin(\phi(t))$ is from -1 to $+1$, which makes this approximation seem unreasonable. However, Since $\phi(t)$ is very small, the probability of large $\phi(t)$ can be neglected in real scenario.

(5) shows a form of signal multiplied by non-ideal carrier, where the parameters $E_\mu, E_\nu, \gamma_\mu, \gamma_\nu, \Phi_\mu, \Phi_\nu, \Psi_\mu$ and Ψ_ν are the function of $E_u, E_d, \alpha_u, \alpha_d, \theta_u, \theta_d$ and ψ . See Appendix A for detailed derivation.

For phase noise, we assume $\phi_u(t)$ and $\phi_d(t)$ are i.i.d and generated by phase locked loop (PLL) which subject to zero mean Gaussian distribution, and use one-pole and one-zero behavioral model [8] of which the power spectrum density function $PSD(f)$ is shown as (6), where K is the low frequency phase noise, f_p denotes the pole frequency determined by the cutoff frequency of the high-pass filter given by the PLL, and f_z is the zero frequency determined by the noise floor of the VCO [8].

$$PSD(f) = K \frac{1 + (f/f_z)^2}{1 + (f/f_p)^2}. \quad (6)$$

After sampled with the sampling period T_s , the received signal is shown as (7). Rewritten in form of vector, the received signal becomes (8), where $\mathbf{r} = [r(0), \dots, r((N-1)T_s)]^T$, $\boldsymbol{\mu} = [\mu(0), \dots, \mu((N-1)T_s)]^T$, $\mathbf{v} = \text{diag}(v(0), \dots, v((N-1)T_s))$, $\boldsymbol{\phi} = [\phi_u(0) - \phi_d(0), \dots, \phi_u((N-1)T_s) - \phi_d((N-1)T_s)]$ and $\mathbf{n} = [n(0), \dots, n((N-1)T_s)]$. The covariance matrix of $\boldsymbol{\phi}$ is denoted as $\mathbf{R}$.

$$r(mT_s) = \mu(mT_s) + v(mT_s)(\phi_u(mT_s) - \phi_d(mT_s)) + n(mT_s), \quad m = 0, 1, \dots, N-1. \quad (7)$$

$$\mathbf{r} = \boldsymbol{\mu} + \mathbf{v}\boldsymbol{\phi} + \mathbf{n}. \quad (8)$$

Since $\boldsymbol{\phi}$ and $\mathbf{n}$ are subject to Gaussian distribution, $\mathbf{r}$ is still a Gaussian process. In this paper, we assume that the parameters excluding the time delay τ have been already known. Thus, for Gaussian process $\mathbf{r}$ [9], the CRB of τ is

$$CRB(\tau) = \left(\text{tr} \left[\left(\mathbf{C}^{-1} \frac{\partial \mathbf{C}}{\partial \tau} \right)^2 \right] + 2 \text{Re} \left[\frac{\partial \boldsymbol{\mu}^H}{\partial \tau} \mathbf{C}^{-1} \frac{\partial \boldsymbol{\mu}}{\partial \tau} \right] \right)^{-1}, \quad (9)$$

where $\mathbf{C}$ is the covariance matrix of $\mathbf{r}$.

3 Evaluation of CRB

3.1 CRB without carrier phase noise

If the carrier phase noise is small enough, i.e., K is small enough so that the term $v(t)(\phi_u(t) - \phi_d(t))$ can be ignored. Thus the $\mathbf{C}$ in (9) is $\sigma^2 \mathbf{I}$, and the CRB of τ is

$$\begin{aligned} CRB(\tau) &= \left(2 \text{Re} \left[\frac{\partial \boldsymbol{\mu}^H}{\partial \tau} \mathbf{C}^{-1} \frac{\partial \boldsymbol{\mu}}{\partial \tau} \right] \right)^{-1} \\ &= \frac{2\sigma^2}{\sum_{n=0}^{N-1} \left| \frac{\partial r(t-\tau)}{\partial \tau} \Big|_{t=nT_s} \right|^2}. \end{aligned} \quad (10)$$

If the sampling period is small enough, i.e., when $T_s \rightarrow 0$, using Fourier Transformation property and Parseval's Theorem [9] (Appendix B), the $CRB(\tau)$ becomes

$$CRB(\tau) = \frac{1}{\beta_{RF} E / N_0}. \quad (11)$$

(11) is the $CRB(\tau)$ in the condition of continuous-time waveform, E/N_0 is the signal to noise ratio (SNR), $S(\omega)$ is the Fourier Transform of $s(t)$, $E =$

$\int_{T_s} s(t)^2 dt = \int_{-\infty}^{\infty} |S(\omega)|^2 d\omega$, and the β_{RF} is the *effective bandwidth* in the condition of non-ideal RF in (12).

$$\beta_{RF} = \frac{\left(\frac{E_p^2}{\pi} \left(1 + \frac{\gamma_p}{2}\right) \int_{-\infty}^{\infty} \left| (\omega - \omega_0) S(\omega - \omega_0) e^{-j(\Psi_p + \frac{\phi_p}{2})} + (\omega + \omega_0) S(\omega + \omega_0) e^{j(\Psi_p + \frac{\phi_p}{2})} \right|^2 d\omega \right.}{4 \int_{-\infty}^{\infty} |S(\omega)|^2 d\omega} \left. + \frac{E_p^2}{\pi} \left(1 - \frac{\gamma_p}{2}\right) \int_{-\infty}^{\infty} \left| (\omega - \omega_0) S(\omega - \omega_0) e^{-j(\Psi_p - \frac{\phi_p}{2})} + (\omega + \omega_0) S(\omega + \omega_0) e^{j(\Psi_p - \frac{\phi_p}{2})} \right|^2 d\omega \right) . \quad (12)$$

Remark: Comparing to the effective bandwidth of ideal RF (13), the β_{RF} is the weighted average of frequency shifted version of $\omega S(\omega)$. The weighting factors depends on the non-ideal RF parameters.

$$\beta_{ideal} = \frac{\int_{-\infty}^{\infty} |\omega S(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |S(\omega)|^2 d\omega} . \quad (13)$$

3.2 CRB with carrier phase noise

If K in (6) is large enough, which makes phase noise non-negligible, the covariance matrix $\mathbf{C}$ in (9) will not be $\sigma^2 \mathbf{I}$ any more. Moreover, $\mathbf{C}$ will depend on τ due to $s(t - \tau)$ in term $\mathbf{v}$ in (8). And the combined noise $\mathbf{v}\phi + \mathbf{n}$ becomes colored noise. In this condition, the CRB can hardly be expressed in a closed form. Since ϕ and $\mathbf{n}$ are independent, $\mathbf{C}$ becomes

$$\begin{aligned} \mathbf{C} &= \mathbb{E}_{\phi, \mathbf{n}} \{ (\mathbf{v}\phi + \mathbf{n})(\mathbf{v}\phi + \mathbf{n})^H \} \\ &= \mathbf{v}\mathbf{R}\mathbf{v}^H + \sigma^2 \mathbf{I}. \end{aligned} \quad (14)$$

According to the one-zero and one-pole carrier phase noise model (6), the auto correlation function of ϕ will be (15), where $\delta(\cdot)$ is the delta function and $m \in \mathbb{Z}$. Thus $[\mathbf{R}]_{i,j} = R(i - j)$.

$$R(m) = K \left[\frac{f_p^2}{f_z^2} \delta(m) + \pi \left(\frac{f_p^2 f_z^2 - f_p^4}{f_p f_z^2} \right) e^{-2\pi f_p T_s |m|} \right]. \quad (15)$$

Therefore, the CRB of τ can be calculated using (9), (14) and (15).

4 Numerical results

We use root raised cosine (RRC) pulse as $s(t)$ to calculate the CRB.

Fig. 1 shows the relationship between SNR and RMSE in different scenarios. In this result, The RRC rolloff factor is 0.5; the symbol rate is 300 MHz; the sampling rate is 900 MHz; and the waveform record length is 2 symbols. The phase noise factor K is -40 dB; and the f_p is 10 kHz; f_z is 1 MHz. This result shows that by adding non-ideal RF without PN, the RMSE will raise up due to the change from β_{ideal} to β_{RF} . Moreover, when added PN, the RMSE will raise more in high SNR region, i.e., the system performance flattens. This is because in high SNR region, the effect of white Gaussian noise is much less than that of PN, and in (15), there is a term of $\delta(m)$, which is stands for white noise part of phase noise. Therefore, although white Gaussian noise is decreasing, phase noise limits the

Fig. 1. CRB of ToA in 3 different scenarios: CRB in the assumption of pure AWGN channel, CRB without phase noise, and CRB with affection of phase noise.

performance of the ToA system. Moreover, the exponential term in (15) is also a limitation of the performance.

In low SNR region RMSE with PN is smaller than that without PN. This is because in (14), ν carries the information of τ , which makes RMSE with PN a little smaller than that without PN. However, it is well-known that in low SNR region, CRB is not a tight bound [2], so it is not of great significance to pay attention on CRB of low SNR region.

Fig. 2 reveals in the condition of different K , the relationship between SNR and RMSE. When K decreases, the floor of CRB decreases. Thus the strength of phase noise K becomes a bottom neck that affects the performance of ToA localization system in high SNR region. The values of other parameters are same as Fig. 1.

Fig. 2. CRB of ToA with different phase noise level by adjusting K level. The floor of CRB drops off as K decreases.

Fig. 3 shows that long record length can increase the performance of ToA estimation. However, it is shown that increase from short record length (2 symbols) to middle one (4 symbols) leads to great performance improvement, while that from

middle one to long one (8 symbols) leads to little gain. This is because most (e.g. 95%) energy of the pulse is concentrated on a finite time interval. When the record length increases, the energy of the record signal does not increase so rapidly as the record length does.

Fig. 3. CRB of ToA with different record length.

These numerical results provide a reason why in real system, the error of ranging or localization system is often at the level of decimeter (10^{-9} seconds correspond to 30 cm). There certainly exist other factors that limit the performance, such as clock drift, ADC converter error and so on, but non-ideal RF and record length are significant factors.

5 Conclusion

In wideband ToA localization system, RF nonideality greatly affects ToA estimation performance. Evaluation of CRB on ToA estimation in the presence of RF nonideality provides criterion to ToA localization design and ToA estimator performance, which gives a reasonable way to improve the ranging accuracy for real localization systems.

A Received signal calculation

After expanding (3), cancelling 2ω term, and approximating using Taylor series expansion, $r(t)$ can be separated in three parts which are $\mu(t)$, $v(t)(\phi_u(t) - \phi_d(t))$, and $n(t)$. $\text{Re}\{\mu(t)\}$ is shown as (16) and (17).

$$\begin{aligned} \text{Re}\{\mu(t)\} = & \left[\frac{(\beta + 2)(\alpha + 2)}{8} \cos\left(\omega_0 t + \psi_0 - \frac{\theta_1 - \theta_2}{2}\right) \right. \\ & \left. + \frac{(\beta - 2)(\alpha + 2)}{8} \sin\left(\omega_0 t + \psi_0 + \frac{\theta_1 + \theta_2}{2}\right) \right] \\ & \cdot E_u E_d s(t - \tau). \end{aligned} \quad (16)$$

After reformulating $\text{Re}\{\mu(t)\}$ and $\text{Im}\{\mu(t)\}$ into form of $A \cos(a)$ and $B \cos(b)$, respectively, we have form of (5), which is the same form of IQ imbalance,

frequency and phase offset carrier form. And the $v(t)$ is derived as the same way as $\mu(t)$.

$$\begin{aligned} \text{Im}\{\mu(t)\} = & \left[\frac{(\beta-2)(\alpha-2)}{8} \cos\left(\omega_0 t + \psi_0 + \frac{\theta_1 - \theta_2}{2}\right) \right. \\ & \left. + \frac{(\beta-2)(\alpha+2)}{8} \sin\left(\omega_0 t + \psi_0 - \frac{\theta_1 + \theta_2}{2}\right) \right] \\ & \cdot E_u E_d s(t - \tau). \end{aligned} \quad (17)$$

B CRB of continuous-time without phase noise

According to (10):

$$\begin{aligned} \lim_{T_s \rightarrow 0} \text{CRB}(\tau) &= \lim_{T_s \rightarrow 0} \frac{2\sigma^2}{\sum_{n=0}^{N-1} \left| \frac{\partial r(t-\tau)}{\partial \tau} \right|_{t=nT_s}^2} \\ &= \lim_{T_s \rightarrow 0} \frac{2\sigma^2 T_s}{\sum_{n=0}^{N-1} T_s \left| \frac{\partial r(t-\tau)}{\partial \tau} \right|_{t=nT_s}^2} \\ &= \frac{2N_0}{\int_{T_r} \left| \frac{\partial r(t-\tau)}{\partial \tau} \right|^2 dt}. \end{aligned} \quad (18)$$

T_r stands for record length. Using the Parseval's Theorem and relation $\int \left| \frac{\partial s(t-\tau)}{\partial \tau} \right|^2 dt = \int |s'(t)|^2 dt$, we have:

$$\begin{aligned} \int_{T_r} \left| \frac{\partial r(t-\tau)}{\partial \tau} \right|^2 dt &= \int_{T_r} |s'(t) c_\mu(t)|^2 dt \\ &= \int_{-\infty}^{\infty} |(\omega S(\omega)) * C_\mu(\omega)|^2 d\omega. \end{aligned} \quad (19)$$

In (19) $c_\mu(t) = \mu(t)/s(t - \tau)$, and $*$ stands for convolution. Thus we have (12).

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