Optimal multiple assignment scheme for strongly secure ramp secret sharing schemes with general access structures*

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Abstract: Iwamoto et al. proposed the strong security definition for ramp secret sharing schemes with general access structures, and also proposed an integer optimization approach to construct a secret sharing with the smallest share size and general access structure, based on a threshold secret sharing scheme. This note proves that if the underlying threshold scheme is strongly secure then the constructed multiple assignment scheme gives a strongly secure secret sharing scheme.

Keywords: secret sharing, access structure, strong security

Classification: Fundamental Theories for Communications

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1 Introduction

Secret sharing (SS) [1] is a cryptographic scheme to encode a secret to multiple shares being distributed to participants, so that only *qualified* sets of participants can reconstruct the original secret from their shares. A set of participants is called *forbidden* if the set has absolutely no information about the secret. A secret sharing scheme is called *perfect* [2] if every set of participants is always qualified or forbidden. If a set is neither qualified or forbidden in a secret sharing scheme, the scheme is said to be *ramp* or *non-perfect*. A set is said to be *intermediate* if it is neither qualified nor forbidden. A merit of the ramp schemes is to reduce share size (the number of bits) while keeping the secret size [3, 4, 5]. A drawback of ramp schemes is that part of the secret can be known to an intermediate set of participants. To exclude such a possibility, Yamamoto and Iwamoto [4, 6] defined the notion of strong security for the ramp schemes.

On the other hand, traditionally, the access structure called the threshold structure has been the most focused one, where a set of participants is qualified if and only if the number of participants is $\geq t$. A scheme with a threshold structure is called a threshold scheme. A well-known method to realize an arbitrary access structure is the multiple assignment scheme proposed by Shamir [1] and named by Ito et al. [7]. It assigns multiple shares of a threshold scheme to single participant, and a single share can be assigned to multiple participants.

Ito et al. [7] did not consider to minimize the share size. Later, Iwamoto et al. [8] proposed an integer optimization approach to construct a secret sharing scheme with the minimum share size based on a threshold scheme. Despite they proposed the strong security criterion, they did not considered the strong security property of their proposal [8]. This note shows that if the underlying threshold scheme is strongly secure in the sense of [6] then the scheme constructed by [8] is also strongly secure.

2 Notation and the main lemma

Let q be a prime power, and $\mathbf{F}_q$ the finite field with q elements. In this note we assume that secret and all shares are vectors over $\mathbf{F}_q$. We will define the access structure when there are n participants and secret consists of L symbols in $\mathbf{F}_q$. Let V_i

be the random variable of the i -th share for $i = 1, \dots, n$, and S the random variable of the secret. We assume the uniform probability distribution of S . The encoding rule of a secret sharing scheme is a conditional joint probability distribution of $(V_1, \dots, V_n)$ given secret S . The encoding rule determines a joint probability distribution of $(S, V_1, \dots, V_n)$. For a set T , 2^T denotes its power set $\{T_0 \mid T_0 \text{ is a subset of } T\}$, and we have $|2^T| = 2^{|T|}$. For $i = 0, \dots, L$, we define $\mathcal{A}_i \subset 2^{\{1, \dots, n\}}$ by

$$\mathcal{A}_i = \{A \subset \{1, \dots, n\} \mid I(S, (V_j)_{j \in A}) = i\},$$

where $I(\cdot, \cdot)$ is the mutual information in $\log_q$. We assume

$$2^{\{1, \dots, n\}} = \bigcup_{i=0}^L \mathcal{A}_i.$$

The collection $\mathcal{A}_0, \dots, \mathcal{A}_L$ is called the access structure. Recall that secret S is a random variable on $\mathbb{F}_q^L$. Let S_i be the i -th component of S . A given secret sharing scheme is said to be *strongly secure* [6] when $A \in \mathcal{A}_i$, $B \subset \{1, \dots, L\}$ and $|B| = L - i$ implies $I((S_j)_{j \in B}; (V_k)_{k \in A}) = 0$.

The multiple assignment map was introduced in [1] and named in [7], which is reviewed in the following. It constructs a secret sharing scheme with n participants from that with m participants. Usually the underlying secret sharing scheme is a threshold scheme. The original scheme and the constructed scheme have the same secret denoted by a uniform random variable S on $\mathbb{F}_q^L$. Let $W_1, \dots, W_m$ be the shares of the original scheme. A multiple assignment map is a map Φ from $\{1, \dots, n\}$ to $2^{\{1, \dots, m\}}$. In the new constructed secret sharing scheme, the i -th participant receives $\{W_j \mid j \in \Phi(i)\}$ as his/her share, which is denoted by V_i . Denote the access structure of the original scheme by $\mathcal{A}_0^{\text{old}}, \dots, \mathcal{A}_L^{\text{old}}$, and that of the new constructed scheme by $\mathcal{A}_0^{\text{new}}, \dots, \mathcal{A}_L^{\text{new}}$.

Main Lemma: If the original secret sharing scheme generating $W_1, \dots, W_m$ is strongly secure then the constructed scheme is also strongly secure.

Proof: Let V_i be the i -th share of the new constructed scheme, and $A^{\text{new}} \subset \{1, \dots, n\}$ represent a set of shares V_i . Since V_i consists of $W_1, \dots, W_m$, the set

$$A^{\text{old}} = \bigcup_{i \in A^{\text{new}}} \Phi(i)$$

is the set of original shares corresponding to the set A^{new} of the new shares. For a set $B \subseteq \{1, \dots, L\}$, we have the equality

$$I((V_i)_{i \in A^{\text{new}}}; (S_j)_{j \in B}) = I((W_i)_{i \in A^{\text{old}}}; (S_j)_{j \in B}),$$

which implies

$$A^{\text{new}} \in \mathcal{A}_i^{\text{new}} \Leftrightarrow A^{\text{old}} \in \mathcal{A}_i^{\text{old}}. \quad (1)$$

Equation (1) and the strong security of the original scheme generating $W_1, \dots, W_m$ imply the strong security of the new constructed scheme. ■

3 How to construct strongly secure optimal multiple assignment scheme

Nishiara and Takizawa [9] proved that the McEliece-Sarwate ramp secret sharing scheme [10] is strongly secure. The McEliece-Sarwate scheme satisfies the defi-

inition of (t, L, m) -ramp secret sharing scheme required in [8 Section 5]. Therefore one can apply Iwamoto et al.'s proposal [8] to the McEliece-Sarwate scheme. Iwamoto et al.'s proposal [8] generates a multiple assignment scheme that is optimal in the sense that it minimizes the share sizes. By the main lemma proved in this note, one can see that the constructed ramp secret sharing scheme is strongly secure.

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