Proposal of Zero-Padded CAZAC sequence with discrete chirp signal

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Abstract: This paper proposes a new Zero-Padded CAZAC sequence by using a discrete chirp signal and proves mathematically that the proposed Zero-Padded CAZAC sequence satisfies the original properties of constant amplitude and zero-autocorrelation in the time domain. This paper also demonstrates the effectiveness of proposed Zero-Padded CAZAC sequence by using computer simulation results when employing it as a preamble symbol in the channel estimation method for the OFDM signal with a non-Nyquist sampling rate in the non-linear and multipath fading channels.

Keywords: CAZAC, chirp signal, OFDM, channel estimation

Classification: Wireless Communication Technologies

References


1 Introduction

Recently the constant amplitude and zero-autocorrelation (CAZAC) sequence has been considered in various purposes for the Orthogonal Frequency Division Multiplexing (OFDM) system such as a channel estimation and synchronization of symbol timing under multipath fading environments [1, 2]. The reason is that the CAZAC sequence has good properties of constant amplitude in the time domain and zero-autocorrelation both in the frequency and time domains. In 2007, Kim and Song proposed the Zero-Padded CAZAC sequence which is generated from the classical 4-phase CAZAC sequence with a length of 16 [3]. The feature of Zero-Padded CAZAC sequence is to enable the extension of its length by inserting zero sequence with keeping the properties of original CAZAC sequence. However the generation of Zero-Padded CAZAC sequence was limited only from the classical 4-phase CAZAC sequence. From this fact, it is required to establish a generalization method for the Zero-Padded CAZAC sequence which can be used flexibly in the wireless communication systems.

To respond to the above requirement, this paper proposes a new Zero-Padded CAZAC sequence which employs a discrete chirp signal instead of using the classical poly-phase CAZAC sequence. This paper proves the properties of constant amplitude and zero-autocorrelation for the proposed Zero-Padded CAZAC sequence mathematically. This paper also demonstrates its effectiveness by using computer simulation results when applying it as a preamble symbol in the channel estimation method for the OFDM signal with a non-Nyquist sampling rate under the non-linear and multipath fading environments.

2 Proposal of Zero-Padded CAZAC sequence

This section proposes a new Zero-Padded CAZAC sequence and proves mathematically that it satisfies the original properties of CAZAC sequence. Let \( K, L \) and \( M \) be positive integers, and let \( N \) be the product of \( K, L \) and \( M \) \((N = KLM)\). Fig. 1 shows a structure of proposed Zero-Padded CAZAC sequence in the frequency domain. In Fig. 1, a frequency domain signal \( A_n \) \((0 \leq n \leq N - 1)\) when \( N \) is an even number is generated under the following two conditions (i) and (ii).

(i) \( A_n = 0 \) if \( mLK + L \leq n \leq (m + 1)KL - 1 \), \( 0 \leq m \leq M - 1 \)

(ii) Otherwise, \( A_n = e^{j\frac{2\pi n}{N}} \) where \( A_n \) is a discrete chirp signal with a constant amplitude \(|A_n| = |A|\).

Under the above two conditions, the discrete chirp signal in the frequency domain is converted to the time domain signal by N-point DFT which is given by,

\[
a_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} A_n e^{j\frac{2\pi nk}{N}}
\]  

(1)

This paper proves mathematically that the following Theorem 1 can be established.
From Eq. (4), Eq. (3) can be expressed by,

$$
\sum_{k=0}^{N-1} a_k^* \cdot a_{k+l} = 0 \quad \text{for} \ l \nmid M,
$$

where $*$ represents the complex conjugate.

**Proof.**

The autocorrelation of time domain signal $a_k$ is given by,

$$
\sum_{k=0}^{N-1} a_k^* \cdot a_{k+l} = \frac{1}{N} \sum_{k=0}^{N-1} \left( \sum_{n=0}^{N-1} A_n^* e^{-j \frac{2\pi nk}{N}} \cdot \sum_{m=0}^{N-1} A_m e^{j \frac{2\pi nm}{N}} \right)
$$

(3)

The last term in Eq. (3) is given by,

$$
\sum_{k=0}^{N-1} e^{j \frac{2\pi nk}{N}} = \begin{cases} N & \text{for} \ m = n \\ 0 & \text{for} \ m \neq n \end{cases}
$$

(4)

From Eq. (4), Eq. (3) can be expressed by,

$$
\sum_{k=0}^{N-1} a_k^* \cdot a_{k+l} = \sum_{n=0}^{N-1} |A_n|^2 e^{j \frac{2\pi nl}{M}}
$$

(5)

Under the above conditions (i) and (ii), Eq. (5) is given by,

$$
\sum_{k=0}^{N-1} a_k^* \cdot a_{k+l} = |A|^2 \left\{ \sum_{n=0}^{K-1} e^{j \frac{2\pi n}{M}} + \sum_{n=KL}^{(K+1)L-1} e^{j \frac{2\pi n}{M}} + \cdots + \sum_{n=(M-1)KL}^{(M-1)KL+L-1} e^{j \frac{2\pi n}{M}} \right\}
$$

$$
= |A|^2 \left\{ \frac{1 - e^{j \frac{2\pi l}{M}}}{1 - e^{j \frac{2\pi M}{M}}} + e^{j \frac{2\pi l}{M}} \left( \frac{1 - e^{j \frac{2\pi l}{M}}}{1 - e^{j \frac{2\pi M}{M}}} \right) + \cdots + e^{j \frac{2\pi l}{M} (M-1)KL} \left( \frac{1 - e^{j \frac{2\pi l}{M}}}{1 - e^{j \frac{2\pi M}{M}}} \right) \right\}
$$

$$
= |A|^2 \left( \frac{1 - e^{j \frac{2\pi l}{M}}}{1 - e^{j \frac{2\pi M}{M}}} \right) \sum_{m=0}^{M-1} e^{j \frac{2\pi lm}{M}}
$$

(6)

Since $lKL/N = l/M$ from $N = KLM$, the last term in Eq. (6) can be expressed by,

$$
\sum_{m=0}^{M-1} e^{j \frac{2\pi lm}{M}} = \begin{cases} M & \text{for} \ l \mid M \\ 0 & \text{for} \ l \nmid M \end{cases}
$$

(7)

Accordingly, the autocorrelation for the proposed Zero-Padded CAZAC sequence can be given by,

$$
\sum_{k=0}^{N-1} a_k^* \cdot a_{k+l} = \begin{cases} M |A|^2 \left( \frac{1 - e^{j \frac{2\pi l}{M}}}{1 - e^{j \frac{2\pi M}{M}}} \right) & \text{for} \ l \mid M \\ 0 & \text{for} \ l \nmid M \end{cases}
$$

(8)

From Eq. (8), we proved the **Theorem 1** that the proposed Zero-Padded CAZAC sequence satisfies the property of zero-autocorrelation in the time domain.

Under the above conditions (i) and (ii), the following **Theorem 2** can be also established.

**Theorem 2.** If $M = L$ and $K$ is even, the amplitude of time domain signal $|a_k|$ is constant for $k$. 
Proof.
From Eq. (1) and the above conditions (i) and (ii), the amplitude of time domain signal \(|a_k|\) can be given by,

\[
|a_k| = \frac{|A|}{\sqrt{N}} \left| \sum_{m=0}^{M-1} \sum_{n=0}^{L-1} e^{j\frac{2\pi m K L (n+k)}{N}} e^{j\frac{2\pi m K L a_k}{N}} \right|
\]

\[
= \frac{|A|}{\sqrt{N}} \left| \sum_{m=0}^{M-1} \left( \sum_{n=0}^{L-1} e^{j\frac{2\pi m K L (n+k)}{N}} \right) \right|
\]

(9)

The last term in Eq. (9) can be expressed by,

\[
e^{j\frac{2\pi m K L (n+k)}{N}} = e^{j\frac{2\pi m K L \frac{1}{M}(n+k)}{N}} e^{j\frac{\pi (n+k)^2}{N}}
\]

(10)

From the conditions that \(N = KLM, M = L, K\) is even, \((KL)^2/N = K\) and \(KL/N = 1/M\), Eq. (10) is given by,

\[
e^{j\frac{2\pi m K L (n+k)}{N}} = e^{j\frac{2\pi m (n+k) L}{N}} e^{j\frac{\pi (n+k)^2}{N}}
\]

(11)

By using Eq. (11), Eq. (9) is rewritten by,

\[
|a_k| = \frac{|A|}{\sqrt{N}} \left| \sum_{n=0}^{L-1} e^{j\frac{2\pi m (n+k) L}{N}} \sum_{m=0}^{M-1} e^{j\frac{\pi (n+k)^2}{N}} \right|
\]

(12)

The last term in Eq. (12) can be given by,

\[
\sum_{m=0}^{M-1} e^{j\frac{2\pi m (n+k) L}{N}} = \begin{cases} M & \text{for } (n+k) \mid M \\ 0 & \text{for } (n+k) \nmid M \end{cases}
\]

(13)

From Eq. (13) and \(M = L\), it implies that there exists unique \(n\) \((0 \leq n \leq L - 1)\) which satisfies \((n+k)\mid M\) for each \(k\). From this fact, \((n+k)\) is given by \(M \cdot r_k\) where \(r_k\) is an integer number for \(k\). Accordingly Eq. (12) has a constant amplitude for all \(k\) as given by,

\[
|a_k| = \frac{|A|}{\sqrt{N}} \cdot |e^{j\frac{\pi (r_k)^2}{N}}| = \frac{|A|}{\sqrt{N}}
\]

(14)

From Theorems 1 and 2, it can be concluded that the proposed Zero-Padded CAZAC sequence can keep the original properties of CAZAC sequence for the zero-autocorrelation and the constant amplitude in the time domain.

![Fig. 1. Structure of proposed Zero-Padded CAZAC sequence.](image)

3 Application of proposed Zero-Padded CAZAC sequence

This section presents computer simulation results to demonstrate the effectiveness of proposed Zero-Padded CAZAC sequence when employing it as a preamble symbol in the channel estimation method for OFDM system under the non-linear and multipath fading environments. In the real OFDM system, the sampling rate of transmission OFDM signal is usually taken by a non-Nyquist rate to reject the...
aliasing occurring at the output of digital to analogue (D/A) converter by using an analogue filter [4]. In other words, the number of subcarriers $D$ within the allocated OFDM bandwidth is smaller than the number of IFFT/FFT points $N$ in which the null subcarriers are added in the frequency domain at the both ends of OFDM symbol. The proposed Zero-Padded CAZAC sequence can satisfy the above requirements by shifting $S = (K - 1)L/2$ samples cyclically. The shifted Zero-Padded CAZAC sequence has $S$ null subcarriers at the both end of CAZAC sequence in the frequency domain which is given by,

$$A_n = \begin{cases} 
0 & 0 \leq n \leq S - 1 \text{ and } N - S \leq n \leq N - 1 \\
0 & mKL + L + S \leq n \leq (m + 1)KL - 1 + S \\
\exp(j\pi n^2/N) & \text{otherwise} 
\end{cases}$$

(15)

The time domain preamble symbol which is converted from Eq. (15) can satisfy the properties of constant amplitude and zero-autocorrelation even at the non-Nyquist sampling rate as proved in Section 2. In the performance evaluations, the parameters $L$, $M$ and $K$ for the proposed Zero-Padded CAZAC sequence as shown in Fig. 1 are taken by 8, 8 and 4, respectively. The numbers of IFFT/FFT points $N$ ($=LMK$), pilot subcarriers $D$ and null subcarriers $S$ are 256, 232 and 12, respectively. The Guard Interval length $N_g$ is 8 and the number of delay paths in the Rayleigh multipath fading channel is 6. The Rap model of solid state power amplifier (SSPA) with the non-linear coefficient $r = 1$ and operating input back-off 0 dB is employed as the non-linear amplifier at the transmitter [5].

Fig. 2 shows the time domain preamble symbols at the input of SSPA for the proposed CAZAC, conventional CAZAC and pilot subcarriers with QPSK. In the conventional two methods, $(N - D)/2$ null subcarriers are added at the both ends of $D$ subcarriers in the frequency domain. From the figure, it can be seen that the amplitude of proposed preamble symbol is a constant over $N$ samples even at the non-Nyquist sampling rate. From this fact, the proposed preamble symbol can keep the original signal at the output of SSPA which could achieve higher channel estimation accuracy in the multipath fading channels. While the amplitude of pilot subcarrier with QPSK has larger fluctuation which causes larger non-linear distortion to the preamble symbol at the output of SSPA. The amplitude of conventional CAZAC is no more constant and also the property of zero-autocorrelation in the time domain is no more satisfied at the non-Nyquist sampling rate which would lead the degradation of channel estimation accuracy in the non-linear channel. Fig. 3 shows the channel estimation accuracy of using the proposed preamble symbol and two conventional preamble symbols. The channel estimation accuracy is evaluated by the normalized mean square error (NMSE). The channel estimation method of taking the autocorrelation in the time domain is employed for the proposed Zero-Padded CAZAC sequence. From the reason that the property of zero-autocorrelation is no more satisfied for the conventional CAZAC at the non-Nyquist sampling rate, two conventional methods including the pilot subcarriers with QPSK employ the maximum likelihood (ML) channel estimation method which can achieve higher channel estimation accuracy although the computation complexity becomes relatively higher [6]. From Fig. 3, it can be observed that the proposed Zero-Padded CAZAC shows much higher channel estimation accuracy.
than the conventional CAZAC and the pilot subcarriers with QPSK. This is the reason that the time domain signals for the conventional preamble symbols include larger non-linear distortion at the output of SSPA due to higher amplitude fluctuation in the time domain signal as shown in Fig. 2. From Fig. 3, it can be concluded that the proposed Zero-Padded CAZAC can achieve higher channel estimation accuracy in the non-linear and multipath fading channels even at the non-Nyquist sampling rate.

Fig. 2. Amplitude of time domain preamble symbol at the input of SSPA.

Fig. 3. Performance of channel estimation accuracy.

4 Conclusions

This paper proposed the generalization method for the new Zero-Padded CAZAC sequence by inserting zero sequence into the discrete chirp signal. This paper proved mathematically that the proposed Zero-Padded CAZAC sequence satisfies the original properties of constant amplitude in the time domain and zero-autocorrelation both in the frequency and time domains. As an example of applications for the proposed Zero-Padded CAZAC sequence, this paper employed it as a preamble symbol in the channel estimation method for the OFDM system. From the computer simulation results, it was confirmed that the proposed CAZAC sequence can achieve higher channel estimation accuracy even at the non-Nyquist sampling rate in the non-linear and multipath fading channels.