A short proof for the multi-deletion error correction property of Helberg codes

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Abstract: A short proof for the s -deletion error-correction property of Helberg codes is given.

Keywords: Helberg codes, deletion error, Levenshtein codes

Classification: Fundamental Theories for Communications

References

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1 Introduction

The deletion error model is a synchronization error model over communication and storage channels and is associated with phenomena that lose a part of transmitted symbols. An s -deletion code C is an error-correcting code that for any codeword $x \in C$ and for any set D of at most s -indices, we can uniquely find the codeword from a deleted sequence x^D . Here x^D is a sequence obtained from x by deleting symbols with index included in D . Hence the length of x^D is the length of x minus the cardinality $|D|$. Another model, the insertion error model, corresponds with phenomena that receive specious symbols. Levenshtein showed s -deletion error-correcting codes, s -insertion error-correcting codes, and s -insertion/deletion (i.e., combination) error-correcting codes are all equivalent [1]. Note that a substitution error, e.g., 0 is flipped to 1, is a combination of insertion and deletion errors, e.g., 0 is changed to 01 and then subsequently changed to 1. Therefore multi deletion error correcting code is regarded as more general than a multi bit-flip error-correcting code which is often discussed for communication theory.

Helberg codes $C^{s,n}(a)$ of length n and of parameter a [2] are a class of s -deletion error correction codes and are known as a generalization of Levenshtein codes [1]. The code space $C^{s,n}(a)$ is defined as

$$C^{s,n}(a) := \{x \in \{0, 1\}^n \mid \langle x, \lambda^{(s)} \rangle \equiv a \pmod{\lambda_{n+1}^{(s)}}\},$$

where $x = (x_1, x_2, \dots, x_n)$, $\lambda^{(s)} = (\lambda_1^{(s)}, \lambda_2^{(s)}, \dots, \lambda_n^{(s)})$, and $\langle \cdot, \cdot \rangle$ is a standard inner product. In the definition of $C^{s,n}(a)$, $\{\lambda_i^{(s)}\}$ is an integer sequence defined as $\lambda_i^{(s)} := 0$ for $i \leq 0$ and $\lambda_i^{(s)} := 1 + \sum_{1 \leq j \leq s} \lambda_{i-j}^{(s)}$. Therefore, for $i > 0$, we have positivity, i.e., $\lambda_i^{(s)} > 0$ and the strictly increasing property $\lambda_i^{(s)} < \lambda_{i+1}^{(s)}$. Note that $C^{s,n}(a)$ is an integer sequence but not a binary field one.

Abdel-Ghaffar et al. first proved that Helberg codes were s -deletion error-correcting codes [3]. To prove that, they showed two following key lemmas:

Lemma 1.1 (Lemma 2 in [3]). *For any binary sequence x, y of length n and any index set $D, D' \subset \{1, 2, \dots, n\}$, if $|D|, |D'| \leq s$ and $x^D = y^{D'}$, we obtain*

$$-\lambda_{n+1}^{(s)} < \langle x, \lambda^{(s)} \rangle - \langle y, \lambda^{(s)} \rangle < \lambda_{n+1}^{(s)}.$$

Lemma 1.2 (Lemma 3 in [3]). *For any codeword $x, y \in C^{s,n}(a)$, and any index set $D, D' \subset \{1, 2, \dots, n\}$, if $|D|, |D'| \leq s$ and $x^D = y^{D'}$, we obtain*

$$\langle x, \lambda^{(s)} \rangle \neq \langle y, \lambda^{(s)} \rangle.$$

By combining the lemmas above, it directly follows that Helberg codes are s -deletion error-correcting codes.

This letter gives a short proof of the lemmas above based on the simple observations: a) for any $x \in \{0, 1\}^n$ let x' be a sequence obtained by replacing some entries of x with 0 (resp. with 1). This implies $\langle x, \lambda^{(s)} \rangle \geq \langle x', \lambda^{(s)} \rangle$ (resp. $\langle x, \lambda^{(s)} \rangle \leq \langle x', \lambda^{(s)} \rangle$). b) for any $x \in \{0, 1\}^n$ and a sequence x' obtained by permuting the i th element x_i and the $(i+1)$ th element of x , if $x_i = 0$ (resp. $x_i = 1$), we have $\langle x, \lambda^{(s)} \rangle \geq \langle x', \lambda^{(s)} \rangle$ (resp. $\langle x, \lambda^{(s)} \rangle \leq \langle x', \lambda^{(s)} \rangle$).

Repeatedly using observations a) and b) implies

$$\langle x^D 0^{|D|}, \lambda^{(s)} \rangle \leq \langle x, \lambda^{(s)} \rangle \leq \langle x^D 1^{|D|}, \lambda^{(s)} \rangle, \quad (1)$$

where $x^D 0^{|D|}$ (resp. $x^D 1^{|D|}$) is a concatenated sequence of x^D and all 0 (resp. all 1) sequence of length $|D|$.

New Proof of Lemma 1.1. $\langle x, \lambda^{(s)} \rangle - \langle y, \lambda^{(s)} \rangle \leq \langle x^D 1^{|D|}, \lambda^{(s)} \rangle - \langle y^{D'} 0^{|D'|}, \lambda^{(s)} \rangle = \langle 0^{n-|D|} 1^{|D|}, \lambda^{(s)} \rangle \leq \sum_{1 \leq j \leq n} \lambda_{n+1-j}^{(s)} < \lambda_{n+1}^{(s)}$. The 1st inequality is from (1). The next equality is from the assumption $x^D = y^{D'}$. The next inequality is from the assumption $|D| \leq s$. The last is from the definition of $\lambda^{(s)}$. $\square$

New Proof of Lemma 1.2. For $1 \leq i \leq n$, let $x_{|I}$ denote a subsequence of x which consists of the first i entries. By the assumption $x \neq y$, we can assume $x = x_{|I} 1 w$ and $y = y_{|I} 0 w$ for some $w \in \{0, 1\}^{n-I-1}$ and some $1 \leq I < n$, without loss of generality. It follows that $\langle x, \lambda^{(s)} \rangle - \langle y, \lambda^{(s)} \rangle = \lambda_{I+1}^{(s)} + \langle x_{|I}, \lambda_{|I}^{(s)} \rangle - \langle y_{|I}, \lambda_{|I}^{(s)} \rangle$.

On the other hand, we have $\langle x_{|I}, \lambda_{|I}^{(s)} \rangle - \langle y_{|I}, \lambda_{|I}^{(s)} \rangle \geq \langle z_{|I-e} 0^e, \lambda_{|I}^{(s)} \rangle - \langle z_{|I-f} 1^f, \lambda_{|I}^{(s)} \rangle \geq \langle z_{|I-s} 0^s, \lambda_{|I}^{(s)} \rangle - \langle z_{|I-s} 1^s, \lambda_{|I}^{(s)} \rangle$, where $z = x^D (= y^{D'})$, $e = |\{1 \leq i \leq I \mid i \in D\}|$ and $f = |\{1 \leq i \leq I \mid i \in D'\}|$. The first inequality is from (1) and the second one is from the assumption $|D|, |D'| \leq s$.

Furthermore we have $\langle z_{|I-s} 0^s, \lambda_{|I}^{(s)} \rangle - \langle z_{|I-s} 1^s, \lambda_{|I}^{(s)} \rangle = -\langle 0^{I-s} 1^s, \lambda_{|I}^{(s)} \rangle = -\sum_{i=1}^s \lambda_{I+1-i}^{(s)} > -\lambda_{I+1}^{(s)}$. The last inequality is from the definition of $\lambda_{I+1}^{(s)}$. It follows that $\langle x, \lambda^{(s)} \rangle > \langle y, \lambda^{(s)} \rangle$. $\square$

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