Revisiting the sensitivity analysis of Google’s PageRank

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Abstract: PageRank is widely known as a prominent method used for indexing web pages and has significantly contributed to the development of Google’s search engine. Significant contributions have been made to the analyses of PageRank, in particular, its sensitivity analyses. However, the arguments presented thus far are within the range of the matrix perturbation theory or the analyses of Markov chains. This paper explores the application of Kato’s perturbation theory to study the sensitivity analyses of PageRank, which enables the consideration of a larger number of general situations than those in established arguments.

Keywords: PageRank, perturbation, linear operator, spectrum.

Classification: Fundamental theories for communications.

References

1 Introduction

The PageRank algorithm was originally proposed by authors Brin and Page [1]. Although the sensitivity analyses of the PageRank vector have been conducted thoroughly [2], we consider it worthy of further discussion, since more general situations are expected to appear in practical situations. Our contribution includes the following: (i) we consider more general situations than past arguments, (ii) we limit ourselves to the case of sufficiently small perturbation, but precisely discuss the dependency of the PageRank vector on the perturbation, and (iii) we consider the dependency of the power method on the perturbation. For instance, they obtained the estimate

\[ \| \pi(\chi) - \pi \|_1 \leq \frac{\alpha}{1 - \alpha} \sum_{i \in U_i} \pi_i \]

[2]. However, this estimate does not provide us with sufficient information regarding how the PageRank vector differs under the perturbation, nor how the perturbed vector is asymptotically denoted.

2 Terms and notations

2.1 Notations of norms

Hereafter, \( i = \sqrt{-1} \) is the imaginary unit. For a vector \( u = (u_1, u_2, \ldots, u_N)^T \) in general, we define the \( p \)-norm of the form as

\[ \| u \|_p \equiv \left( \sum_{j=1}^{N} |x_j|^p \right)^{\frac{1}{p}} \quad (p \in [1, \infty)), \]

and \( \| u \|_\infty \equiv \max_{1 \leq j \leq N} |u_j| \). For a squared matrix \( M = [m_{ij}] \in M_{N \times N} \) (hereafter, a set of \( N \) times \( N \) matrices is denoted as \( M_{N \times N} \)), we define

\[ \| M \|_\infty = \max_{1 \leq i \leq N} \sum_{j=1}^{N} |m_{ij}|, \quad \| M \|_1 = \max_{1 \leq j \leq N} \sum_{i=1}^{N} |m_{ij}|. \]

We also define the operator norm, denoted as \( \| M \| = \sup_{x \neq 0} \frac{\| Mx \|}{\| x \|} \) where the norm of the right-hand side is the usual norms of vector spaces: \( \| x \|_p \). Additionally, we denote the vector norm as \( \| \cdot \|_2 \) by \( \| \cdot \| \), and apply the corresponding operator norms. For a matrix \( M \), we use notations \( \sigma(M), \text{spr}(M) \) and \( \text{tr}(M) \) to denote the set of the eigenvalues of \( M \), the spectral radius, and the trace, respectively (See [3] for the definition of spectral radius). Hereafter, \( c \)'s with some indices stand for positive constants.

Remark 2.1. It is often useful to use \( \| \pi \|_1 = 1 \) for a PageRank vector \( \pi \), but Tikhonov’s theorem [4] states that \( 2 \)-norm is equivalent to \( 1 \)-norm.

2.2 Notations of Google matrices

The Google matrix \( G \) is defined by \( G = \alpha W + \frac{(1-\alpha)}{N} e v^T \), where \( e \) is a column vector whose elements are all 1, and \( v \) is a non-negative vector satisfying \( \| v \|_1 = 1 \). The constant \( \alpha \in (0,1) \), is called the damping factor, which
represents the probability that an internet surfer navigates from one web page \((i)\) to another \((j)\).

The row stochastic matrix \( \mathbf{W} \) is defined by \( \mathbf{W} = \mathbf{H} + \frac{\alpha}{N} \mathbf{e}^T \), where \( \alpha = (a_j) \), which satisfies \( a_j = 1 \) if web page \( j \) is a dangling node and \( a_j = 0 \) otherwise \cite{2}. The hyperlink matrix, \( \mathbf{H} = [h_{ij}] \), is defined by \( h_{ij} = 1/|P_i| \) if there is a link from web page \( i \) to web page \( j \) and \( h_{ij} = 0 \) otherwise; it is a type of weighted adjacency matrix. The following lemma explains the foundation of the theory of the Google matrix.

**Lemma 2.1.** The Google matrix, \( \mathbf{G} \), has a simple principal eigenvalue \( \lambda_1 = 1 \). In addition, the eigenvalues of \( \mathbf{G} \) satisfy \( \lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \ldots \).

Additionally, there exists a non-negative left-eigenvector \( \pi \) that corresponds to the principal eigenvalue \( \lambda_1 = 1 \), which is known as the PageRank vector.

### 3 Formulation

#### 3.1 Formulation and settings

In this paper, we consider a situation in which a small perturbation is imposed on the Google matrix \( \mathbf{G} \in \mathbb{M}_{N \times N} \). The Google matrix under this perturbation is denoted as \( \mathbf{G}(\chi) \), where \( \chi \in \mathbb{C} \). Likewise, \( \mathbf{W} \) under the perturbation is denoted as \( \mathbf{W}(\chi) \). Note that this perturbation is imposed so that \( \mathbf{G}(\chi) \) and \( \mathbf{W}(\chi) \) preserve their original characteristics. (both matrices are row stochastic, \( \mathbf{G}(\chi) \) is irreducible and primitive.)

#### 3.2 Perturbation theory in finite dimensional space

We denote the resolvent of \( \mathbf{G}(\chi) \) by \( R(\zeta, \chi) \equiv (\mathbf{G}(\chi) - \zeta \mathbf{I})^{-1} \), where \( \zeta \in \mathbb{C} \), and \( \mathbf{I} \) is the \( N \)-dimensional squared unit matrix. \( R(\zeta, \chi) \) is defined for all \( \zeta \) that are not equal to any eigenvalue of \( \mathbf{G}(\chi) \). We know that \( R(\zeta, \chi) \) is holomorphic with respect to \( \zeta \) and \( \chi \) for such \( \zeta \) (see Theorem II.1.5 in \cite{3}), which can be expanded as follows:

\[
R(\zeta, \chi) = R(\zeta) \left[ \mathbf{I} + \mathbf{W}(\chi)R(\zeta) \right]^{-1} = R(\zeta) + \sum_{l=1}^{\infty} \chi^l R^{(l)}(\zeta). \quad (1)
\]

By integrating the infinite series (1) over a closed loop \( \Gamma_h \) that encloses a single eigenvalue \( \lambda_h \) of \( \mathbf{G} \), we render the perturbation of the projection operator \( P_h(\chi) \):

\[
P_h(\chi) = -\frac{1}{2\pi i} \int_{\Gamma_h} R(\zeta, \chi) \, d\zeta = P_h(\chi) + \sum_{l=1}^{\infty} \chi^l P_h^{(l)}(\chi).
\]

When \( \chi = 0 \), it is equal to the eigenprojection \( P_h(0) = P_h = -\frac{1}{2\pi i} \int_{\Gamma_h} R(\zeta) \, d\zeta \), where \( R(\zeta) = (\mathbf{G} - \zeta \mathbf{I})^{-1} \). We also use

\[
S_h = -\frac{1}{2\pi i} \int_{\Gamma_h} (\zeta - \lambda_h)^{-1} R(\zeta) \, d\zeta,
\]

which satisfies \( P_h S_h = S_h P_h = 0 \). Additionally, we have \( G P_h = \lambda_h P_h + D_h \), where \( D_h \) is referenced as the "eigennilpotent" operator of \( \mathbf{G} \). For later use, for \( q \in \mathbb{N} \) we define:

\[
S_h^{(0)} = -P_h, \quad S_h^{(q)} = S_h^q (q > 0), \quad S_h^{(q)} = D_h^q (q < 0).
\]
For simplicity, we omit the subindex $h$ hereafter.

4 Sensitivity of PageRank vector

First, we consider the case of the analytic perturbation:

$$G(\chi) = \alpha(\chi)W(\chi) + \frac{(1 - \alpha(\chi))}{N}ev(\chi)^T$$

$$= G + \chi G^{(1)} + \chi^2 G^{(2)} + \ldots = G + \tilde{G}(\chi). \quad (2)$$

**Theorem 4.1.** For $\chi \in \mathbb{C}$ with sufficiently small $|\chi|$, the PageRank vector $\pi(\chi)$ under the perturbation of the form (2) is represented as

$$\pi(\chi)^T = \pi^T - \pi^T \tilde{G}(\chi)S[I + \tilde{G}(\chi)S]^{-1}, \quad (3)$$

where $\tilde{G}(\chi) = G(\chi) - G$. We also have the estimate:

$$\|\pi(\chi) - \pi\| \leq \|S\|\Phi_3(\chi) \frac{1}{1 - \Phi_2(\chi)},$$

where $\Phi_2(\chi)$ and $\Phi_3(\chi)$ are superior series of $\tilde{G}(\chi)S$ and $\tilde{G}(\chi)\pi$, respectively.

**Proof.** In virtue of Lemma 2.1, the principal eigenvalue of the Google matrix is $\lambda_1 = 1$ and simple. Thus, we have $\pi(\chi)^T = ([\pi^T P(\chi)]^T, \pi)^{-1}\pi^T P(\chi)$, where $(\cdot, \cdot)$ denotes the inner product of the vectors. Starting from $\pi(\chi)^T(G(\chi) - I) = 0$ and by using $(G - I)S = I - P$, we have

$$(\pi(\chi) - \pi)^T(G - I)S + \pi(\chi)^T \tilde{G}(\chi)S = 0. \quad (4)$$

By noting $\pi(\chi) = (\pi(\chi) - \pi) + \pi$ in (4), we arrive at

$$(\pi(\chi) - \pi)^T = -\pi^T \tilde{G}(\chi)S[I + \tilde{G}(\chi)S]^{-1}.$$

This proves the first part. The latter part is verified by applying the method of the superior series.

We next consider a special case, $G(\chi) = G + \chi G^{(1)}$.

**Theorem 4.2.** For sufficiently small $|\chi| > 0$, the PageRank vector under the perturbation above is represented as

$$\pi(\chi)^T = \pi^T - \sum_{j=1}^{\infty} \pi^T(\chi \alpha G^{(1)}S)^j. \quad (5)$$

We also have the estimate:

$$\|\pi(\chi) - \pi\| \leq \frac{\chi\alpha c_4\|G^{(1)}\|}{(1 - \alpha) - \chi\alpha c_4\|G^{(1)}\|}.$$
Proof. Let us substitute \( \tilde{G} = \chi G^{(1)} \) into (3). By expanding the resultant equality with respect to \( \chi \) and estimating its norm, we have
\[
\| \pi(\chi) - \pi \| \leq \frac{\chi \| G^{(1)} S \| \| \pi \|}{1 - \chi \| G^{(1)} S \|}.
\]
From the definition of \( S \), we then have
\[
\| S \| \leq \frac{1}{2\pi} \int_{\Gamma_1} \left\| R(\zeta) \right\| |d\zeta|,
\] (6)
where \( \Gamma_1 \) is a closed loop that encloses \( \lambda_1 = 1 \) with a sufficiently small radius \( \delta > 0 \). (Since \( |\lambda_2| \leq \alpha [2] \), it suffices to take \( \delta = \frac{1-\alpha}{2} \).) Thus, \( \zeta = 1 + \delta e^{i\theta} \) on \( \Gamma_1 \). Note that the eigenvalues of \( R(\zeta) \) are \( \left\{ \frac{1}{\lambda_0 - \zeta} \right\} \) [3]. For \( \lambda_1 = 1 \), the corresponding eigenvalue of \( R(\zeta) \) is \( \frac{1}{1-\zeta} = -\delta^{-1}e^{-i\theta} \) and simple, which is the only eigenvalue of \( R(\zeta) \) located on its spectral radius. Therefore, we can apply the following inequality [5]:
\[
\| R(\zeta) \| \leq c_{43} \text{spr}(R(\zeta)).
\]
(note that \( c_{43} \) is independent of \( \zeta \).) Noting that \( \text{spr}(R(\zeta)) = \min_{\lambda \in \sigma(G)} \frac{1}{|\lambda_0 - \lambda|} = \frac{1}{1-\alpha} \), and estimating the right-hand side of (6) from above, we estimate
\[
\| S \| \leq \frac{2c_{43}}{1 - \alpha}.
\]
Consequently, we acquire the desired result.

Remark 4.1. If we assume that \( H \) is normal, in addition to the assumptions in Theorem 4.2, then we have a sharper estimate: \( H \) is normal, then, we have a sharper estimate:
\[
\| S \| = \min_{\lambda \in \sigma(G), \lambda \neq 1} \frac{1}{|\lambda_0 - \lambda|} = \frac{1}{1 - \alpha}.
\]
If \( H \) is normal, then so are \( W \) and \( G \). Therefore, we obtain the equality above. In addition, the constant \( c_{43} \) is larger than 1 [5].

5 Sensitivity of power method convergence rate

Next, we estimate the sensitivity of the convergence rate of the power method. This is a concept similar to running-time stability [8]. The power method is a well-known approach used to find the PageRank vector \( \pi \) numerically through iterative calculations. Here, we consider the analytic perturbation of the form of (2) again. It is understood that the convergence rate of the power method for \( G \) is \( \frac{|\lambda_2|}{|\lambda_1|} \) [1]. As we have mentioned, \( \lambda_1 = 1 \). If \( W \) is reducible, \( |\lambda_2| = \alpha \), otherwise, \( |\lambda_2| < \alpha [6, 7] \). Therefore, if the perturbation is imposed by preserving the reducibility of \( W(\chi) \) for all \( \chi \) under consideration, we state that the convergence rate under the perturbation is \( \alpha(\chi) \). Otherwise, we need more careful discussions.

In the case that \( \lambda_2 \) forms a \( \lambda \)-group of cycle \( p \) and \( \chi = 0 \) is an exceptional point, we have the following Puiseux series in general [3]:
\[
\lambda_{2h}(\chi) = \lambda_2 + a_1 \omega^h \chi^{1/p} + a_2 \omega^{2h} \chi^{2/p} + \ldots \quad (h = 0, 1, 2, \ldots, p - 1) \quad (7)
\]
for \( p \geq 2 \), where \( \omega = e^{\frac{2\pi i}{p}} \).
Remark 5.1. For the definitions of the exceptional point, $\lambda$-group and the cycle of the $\lambda$-group, see [3].

Theorem 5.1. Let $\lambda_2$ be a subdominant eigenvalue of $W$. Let us impose the perturbation of the form (2) on $G$, and let $\chi = 0$ be an exceptional point. We also assume that its perturbed value, $\lambda_2(\chi)$, forms a $\lambda$-group of cycle $p$, each of which takes the form of the Puiseux series (7). Then, the convergence rate of the power method under the smallness of the perturbation lies in the interval

$$
\left(|\lambda_2| - \sum_{l=1}^{\infty} |a_l| |\chi|^{1/p}, |\lambda_2| + \sum_{l=1}^{\infty} |a_l| |\chi|^{1/p}\right),
$$

where $a_l$ ($l = 1, 2, \ldots$) are provided in (7).

Theorem 5.1 is a direct consequence of the facts stated above.

Corollary 5.1. Let us assume the same assumptions as in Theorem 5.1, and let $\lambda_2 \in \sigma(W)$ be simple. Then, the convergence rate of the power method under the smallness of the perturbation lies in the interval

$$
\left(|\lambda_2| - \sum_{l=1}^{\infty} |\hat{\lambda}(l)| |\chi|^l, |\lambda_2| + \sum_{l=1}^{\infty} |\hat{\lambda}(l)| |\chi|^l\right),
$$

where $\lambda_l$ ($l = 1, 2, \ldots$) take the following form:

$$
\hat{\lambda}(l) = \frac{1}{N_l} \text{tr} \left( G^{(l)} P \right),
$$

In this case, we have simpler representations for lower $l$’s; for instance,

$$
\hat{\lambda}(1) = \frac{1}{N} \text{tr} \left( G^{(1)} P \right),
$$

Corollary 5.2. Let us assume the same assumptions as in Theorem 5.1, and let $\lambda_2 \in \sigma(W)$ be simple. The small perturbation $\chi$ is imposed on $G$ as the linear form: $G(\chi) = G + \chi G^{(1)}$. We then have the same form of the convergence rate of the power method as in Corollary 5.1, with a simpler form of $\hat{\lambda}(l)$:

$$
\hat{\lambda}(l) = \frac{1}{N_l} \text{tr} \left( G^{(l)} P^{(l-1)} \right), \quad l = 1, 2, 3, \ldots.
$$

We limit ourselves to mentioning that if $\lambda_2$ is simple, the asymptotic representation of the weighted mean of the $\lambda$-group matches with that of $\lambda_2(\chi)$. In the case that the multiplicity of $\lambda$ is $m$ in general, the weighted mean of the $\lambda$-group is denoted as $\hat{\lambda}(\chi) = \frac{1}{m} \text{tr}(G(\chi)P(\chi))$. This yields the Corollaries 5.1 and 5.2.

6 Conclusion

In this paper, we discussed the PageRank vector under the small perturbation in the presence of the general form of the perturbation on the Google matrix. We also discussed the effect of the perturbation on the convergence rate of the power method.