Calculation of Natural Frequencies of Teeth Supported with the Periodontal Ligament

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Natural frequencies and vibration modes of four kinds of teeth were calculated by using a mechanical model. The alveolar bone and the tooth were assumed as rigid bodies, while the periodontal ligament was assumed as an elastic spring. All the natural frequencies were within a range of 1 to 10 kHz. The first natural frequencies of four teeth were about 1.5 kHz, and decreased as the root length decreased. Their vibration modes were tipping movements of the root. The natural frequency of the twisting vibration mode, or rotating movement around the tooth axis, was affected by root configuration. When subjected to a periodic force, the tooth and periodontal ligament would vibrate with the corresponding resonance mode. This phenomenon may be used as a method for the diagnosis and the treatment of a periodontal tissue.

Keywords: Natural frequency, Periodontal ligament, Numerical analysis

INTRODUCTION

The movement of a gomphosis tooth produced by a dynamic force—that is, the dynamic response of tooth—is one of the mechanical characteristics of the periodontal tissue. Incidentally, this dynamic response of teeth has been leveraged and further investigated in many experimental studies to develop dental devices. Recently, Castellini and Scalise used a laser Doppler vibrometer to measure tooth mobility produced by an impact force. As for Huang et al., they used an impact force to measure the natural frequencies of human incisors, canines, first premolars, and first molars. In such experimental studies, movements inside the periodontal tissue were difficult to observe. Hence, analytical methods—based on mechanical models—are necessary to clarify the mechanisms of dynamic response. According to the fundamental principle of vibration, a dynamic response depends on the natural frequency of a mechanical system. Therefore, in the case of the gomphosis tooth, it is necessary to know the natural frequencies of the tooth supported by the periodontal tissue.

In an analytical study, Lee et al. used a two-dimensional finite element method to calculate the natural frequencies of incisors. They showed that the natural frequency decreased with an increase in resorption of the alveolar bone. If a three-dimensional finite element method were used, then the natural frequencies of a three-dimensional tooth could be calculated. However, such an analysis has not been carried out because of difficulties in preparing a three-dimensional finite element model.

The purpose of this study was to calculate the natural frequencies and vibration modes of three-dimensional teeth under a force level of a few newtons (N). If the vibration of tooth is utilized for the diagnosis and treatment of periodontal disease, such a light force will be applied to the tooth. With this degree of initial tooth mobility, the tooth and alveolar bone are hardly deformed—but any tooth movement that occurs will be due to the deformation of the periodontal ligament (PDL). For these conditions, a simple mechanical model of a rigid tooth supported by a spring-like action of the PDL was constructed. In this manner, the natural frequencies of a tooth with a complex root configuration could be easily calculated without employing the finite element method.

MATERIALS AND METHODS

Moments of inertia

Four kinds of teeth with different root configurations, i.e., upper right central incisor (UR1), upper right first premolar (UR4), upper right first molar (UR6), and lower left first molar (LR6), were selected to calculate the natural frequencies. Figure 1 shows how to make a three-dimensional tooth model. Cross-section of the human extracted tooth was scanned successively using micro X-ray CT (MCT-12505MF, Hitachi Co. Ltd., Tokyo, Japan) at 0.5-mm intervals. These images were then input into a three-dimensional CAD program (Mechanical Desktop, Autodesk Inc., CA, USA), whereby outlines
of the enamel, dentin, and pulp were traced to make a wire frame model. Using the pre-processor of a FEM (finite element method) software (ANSYS 5.6, ANSYS Inc., PA, USA), a surface model was created and divided into small triangular elements.

Assuming the densities of enamel, dentin, and pulp to be 3, 2, and 1 g/cm³ respectively, the location of the center of gravity, G, was calculated by numerical integration. Then, xyz-coordinates—of which the origin was located at G—were composed for each tooth as shown in Fig. 1. The x-axis, y-axis, and z-axis coincided with the mesiodistal, buccolingual, and tooth axis directions, respectively. Based on these coordinates, moments of inertia—\( I_x, I_y, \) and \( I_z \)—were calculated by numerical integration. Further, products of inertia—\( I_{xy}, I_{yz}, \) and \( I_{zx} \)—were also calculated.

**Spring constant of periodontal ligament (PDL)**

When the force acting on a tooth is less than 1 N, the tooth remains in proportion to the force. The PDL was assumed to be an elastic spring, while the tooth and alveolar bone were assumed to be rigid bodies. Forces and moments acting on the center of gravity, G, were denoted as \( F = [P_x, P_y, P_z, M_x, M_y, M_z] \), while displacements and rotations of tooth at G were denoted as \( U = [u_x, u_y, u_z, \theta_x, \theta_y, \theta_z] \), where the symbol \( T \) indicated a transpose matrix. Then, the relationship between \( F \) and \( U \) could be written as \( KU = F \), where the matrix \( K \) was the spring constant of the PDL.

To calculate the spring constant, the PDL was assumed to be a homogeneous linear elastic film with a uniform thickness of \( t = 0.2 \) mm. Young’s modulus and Poisson’s ratio of the PDL were assumed to be \( E = 0.2 \) MPa and \( \nu = 0.47 \), respectively. Under these assumptions, the spring constant of the PDL, \( K \), could be calculated using a surface model of the tooth. Details on the calculation method of \( K \) were given in a previous article.

**Natural frequency**

Mass of the tooth was denoted as \( m \), and elements of matrix \( K \) were denoted as \( K_{ij} \). Natural angular frequency, \( \omega \), was then calculated using the following equation:

\[
\begin{bmatrix}
K_{11}m\omega^2 & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\
K_{21} & K_{22}m\omega^2 & K_{23} & K_{24} & K_{25} & K_{26} \\
K_{31} & K_{32} & K_{33}m\omega^2 & K_{34} & K_{35} & K_{36} \\
K_{41} & K_{42} & K_{43} & K_{44}m\omega^2 & K_{45} & K_{46} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55}m\omega^2 & K_{56} \\
K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66}
\end{bmatrix} \begin{bmatrix}
\omega^2 \\
\omega^2 \\
\omega^2 \\
\omega^2 \\
\omega^2 \\
\omega^2
\end{bmatrix} = 0
\]

The natural frequency, \( f \), was obtained as \( f = \omega / (2\pi) \). Six natural frequencies were calculated corresponding to six degrees of freedom. Six vibration modes were then obtained from the eigen vectors for each natural frequency. Equation (1) shows that the natural frequency, \( f \), was proportional to a square root of the spring constant, \( K \). Then, all elements of \( K \) were proportional to \( E/t \) which is shown below:

\[
E/t = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)t}
\]

where \( E \) is Young’s modulus, \( \nu \) is Poisson’s ratio, and \( t \) is the thickness of the PDL. In other words, natural frequencies change depending on \( \sqrt{E/t} \). When Young’s modulus, Poisson’s ratio, and thickness of the PDL were different from the assumed values, changes in the natural frequency were calculated using this dependency.

**RESULTS**

The location of the center of gravity, G, and xyz-coordinates for each tooth are shown in Fig. 2. As shown, G was located near the cervical line for each tooth. Table 1 shows the mass, moments of inertia, and inertia products for each tooth. The moments of inertia \( I_x, I_y, \) and \( I_z \), which were calculated using the cross-sectional areas in the xy-plane, had about the same magnitude. The \( I_x \) and \( I_y \), for UR6 and LR6 were about twice those for UR1 and UR4. This was because the molars had large roots. For the same reason, the moments of inertia about the tooth axis (z-axis), \( I_z \), for UR6 and LR6 were markedly larger than those for UR1 and UR4. The inertia products were significantly smaller than the moments of inertia, because the xyz-coordinates for
Fig. 2 Tooth configurations and xyz-coordinates, of which the latter’s origin was located at the center of gravity, G.

### Table 1 Masses, moments and products of inertia calculated using wire frame models of teeth

<table>
<thead>
<tr>
<th>Tooth</th>
<th>Mass (g)</th>
<th>$I_{xx}$</th>
<th>$I_{yy}$</th>
<th>$I_{zz}$</th>
<th>$I_{xy}$</th>
<th>$I_{xz}$</th>
<th>$I_{yz}$</th>
<th>$I_{zx}$</th>
<th>$I_{zy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UR1</td>
<td>1.2</td>
<td>34.7</td>
<td>36.4</td>
<td>7.5</td>
<td>0.04</td>
<td>2.5</td>
<td>1.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>UR4</td>
<td>1.2</td>
<td>30.4</td>
<td>27.3</td>
<td>9.1</td>
<td>0.5</td>
<td>-0.4</td>
<td>-0.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>UR6</td>
<td>2.0</td>
<td>63.2</td>
<td>53.6</td>
<td>30.0</td>
<td>1.7</td>
<td>-1.6</td>
<td>-1.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>LR6</td>
<td>1.9</td>
<td>47.1</td>
<td>51.3</td>
<td>26.2</td>
<td>-1.2</td>
<td>1.4</td>
<td>-4.2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 2 Root surface areas $A$ and spring constants of the periodontal ligament (PDL), $K_s$, calculated using surface models of teeth. $K_{xx}$, $K_{xy}$, and $K_{xz}$ are spring constants for translation movement in $x$, $y$, and $z$-directions, respectively. $K_m$, $K_n$, and $K_s$ are spring constants for rotational movement about $x$, $y$, and $z$-axis, respectively.

<table>
<thead>
<tr>
<th>Tooth</th>
<th>Root surface area $A$ (mm²)</th>
<th>$K_{xx}$</th>
<th>$K_{xy}$</th>
<th>$K_{xz}$</th>
<th>$K_{yy}$</th>
<th>$K_{yx}$</th>
<th>$K_{yz}$</th>
<th>$K_{zx}$</th>
<th>$K_{zy}$</th>
<th>$K_{zz}$</th>
<th>$K_{sy}$</th>
<th>$K_{sz}$</th>
<th>$K_{nx}$</th>
<th>$K_{nz}$</th>
<th>$K_{sx}$</th>
<th>$K_{ny}$</th>
<th>$K_{ny}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UR1</td>
<td>196</td>
<td>0.58</td>
<td>0.60</td>
<td>0.13</td>
<td>0.60</td>
<td>0.08</td>
<td>0.19</td>
<td>0.10</td>
<td>0.05</td>
<td>0.05</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>UR4</td>
<td>237</td>
<td>0.80</td>
<td>0.59</td>
<td>0.19</td>
<td>0.60</td>
<td>0.06</td>
<td>0.19</td>
<td>0.11</td>
<td>0.07</td>
<td>0.07</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>UR6</td>
<td>440</td>
<td>1.49</td>
<td>1.04</td>
<td>0.41</td>
<td>1.04</td>
<td>0.84</td>
<td>0.39</td>
<td>0.30</td>
<td>0.23</td>
<td>0.23</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LR6</td>
<td>393</td>
<td>1.41</td>
<td>0.84</td>
<td>0.39</td>
<td>0.84</td>
<td>0.63</td>
<td>0.39</td>
<td>0.30</td>
<td>0.23</td>
<td>0.23</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Each tooth (Fig. 2) almost coincided with the principal axes of inertia.

Table 2 shows the root surface areas $A$ and spring constants $K_s$ of the PDL for the four kinds of teeth. The spring constants $K_s$ were the diagonal elements of the matrix $K$. The $K_{xx}$, $K_{xy}$, $K_{xz}$, $K_{yy}$, and $K_{zy}$ for UR1 and UR4 were about half of those for UR6 and LR6. These values changed depending on the root surface area $A$. The spring constant $K_s$ changed remarkably depending on root configuration. The smallest value of $K_s$ was in UR1 with a single root. Intermediate values of $K_s$ were found in UR4 and LR6 with two roots. As for the maximum value of $K_s$, it was found in UR6 with three roots.

Figures 3(a) to 3(d) show the natural frequencies and vibration modes for each tooth. All the natural frequencies ranged from 1 to 10 kHz (from 1,000 to 10,000 Hz). In these figures, distributions of mean stress, $\sigma_{xx} = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$, in the PDL are shown with color contours, whereby blue indicates tensile stress and red indicates compressive stress. Vibration modes could also be identified from the stress distribution in the PDL. The first and second vibration modes were tipping movements (rotation about $y$-axis or $x$-axis). The center of tipping was located at about half length of the root. Both tensile and compressive stresses acted on either the mesial or buccal surface of the root. These movements were the same as the tipping movement of orthodontic tooth movement. Moreover, the same movement was produced in measurements of tooth mobility. The third mode was an intrusive and eruptive vibration along the tooth axis. Near the root apex, tensile or compressive stress was observed. The fourth vibration mode was twisting (rotating) about the tooth axis. The fifth and sixth vibration modes were translational vibrations in mesiodistal or buccolingual direction. Tensile or compressive stress acted on either mesial or buccal surface of the root. This movement was the same as the bodily movement of orthodontic tooth movement.

Figure 4 shows a comparison of the natural fre-
Fig. 3  Six natural frequencies (from $f_1$ to $f_6$) and corresponding vibration modes. Stress distribution in the periodontal ligament (PDL) is indicated with color contours. (a) Upper central incisor (UR1); (b) Upper first premolar (UR4); (c) Upper first molar (UR6); (d) Lower first molar (LR6).
Natural frequencies of gomphosis teeth

![Graph showing natural frequencies of teeth](image)

**Fig. 4** Comparison of natural frequencies among the four teeth.

![Graph showing change in minimum natural frequency with root length](image)

**Fig. 5** Change in the minimum natural frequency $f_i$ with increase in root length $l$.

Among the four teeth were slight—despite the marked changes in root configuration. An explanation could be proffered by examining the case of $f_i$. When the three-dimensional tooth model was approximated with a one-dimensional mass-spring model, the natural frequency of tipping vibration, $f_i$, was proportional to $\sqrt{K_a/I_y}$ or $\sqrt{K_s/I_z}$, where $K_a$ and $K_s$ were the spring constants of the tipping movement (rotation about y-axis and x-axis respectively). The spring constants $K_a$ or $K_s$ for UR1 and UR4 were about half of those for UR6 and LR6 (Table 2). Similarly, for the moments of inertia $I_y$ or $I_z$, they were changed by about the same ratio as $K_a$ or $K_s$ (Table 1). As a result, the variation in $f_i$ was slight among the four teeth.

The natural frequency of the twisting (rotating) vibration about tooth axis, $f_t$, varied remarkably among the four teeth. In the one-dimensional vibration model, $f_t$ was proportional to $\sqrt{K_{st}/I_z}$, where $I_z$ was the moment of inertia about tooth axis (z-axis) and $K_{st}$ was the spring constant of the twisting (rotating) movement. The spring constant $K_{st}$ changed markedly depending on the root configuration (Table 2). In the same manner as $K_s$, the moment of inertia $I_z$ varied markedly too depending on the root configuration (Table 1). However, the amount of variation in $I_z$ was less than that in $K_{st}$. As a result, $f_t$ changed depending on the spring constant $K_{st}$.

Tooth mobility, a movement produced by a unit force applied to the crown, has been measured and used to diagnose the periodontal tissue. Tooth mobility changes with the condition of the periodontal tissue. The present results made it clear that the first vibration mode was a tipping movement observed in measurements of tooth mobility. Therefore the minimum natural frequency corresponded to the tooth mobility. Since tooth mobility was inversely proportional to the spring constant of the PDL, tooth mobility increased while the natural frequency decreased with an increase in thickness of the PDL, a decrease in Young’s modulus of the PDL, and an increase in alveolar bone loss (a decrease in root length). In other words, by measuring the natural frequency, the mechanical conditions of the periodontal tissue (Young’s modulus of the PDL, thickness of the PDL, and root length) could also be estimated quantitatively. It was thus a useful method for estimating tooth mobility. Resonance occurs when the frequency of the external force acting on a tooth is equal to its natural frequency, leading the tooth to vibrate in the corresponding mode. This phenomenon may be utilized for periodontal treatment.

In the present model, since viscosity of the PDL was not taken into account, vibration was not damped and amount of the tooth movement at reso-

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**DISCUSSION**

Except for $f_a$, variations in the natural frequencies...
nance became infinite. However, a fluid component in the PDL produces viscosity. If viscous damping were taken into consideration, then movement at resonance would become finite. Moreover, damping will increase with an increase in frequency of the external force. Whereas the PDL was assumed to be a linear elastic material, the stress-strain relation in the PDL will not be linear when tooth movement becomes pronounced. Calculation in the nonlinear regime is difficult work, but which should be the follow-up work to the present study. The present calculation was valid only for the range of initial tooth displacement, that is, tooth displacement within several tens of micron millimeters.

According to the result obtained by Huang et al., the values of \( f_i \) for UR1, UR4, UR6, and LR6 were \( 1.35 \pm 0.20 \), \( 1.32 \pm 0.17 \), \( 1.19 \pm 0.13 \), and \( 1.18 \pm 0.16 \) kHz, respectively. Taking into account the effect of measurement error, these frequencies were almost independent of the kind of tooth being measured. In the present study, the correspondent values of \( f_i \) were 1.35, 1.61, 1.63, and 1.78 kHz. These calculated values were in the same order as the measured values. However, the calculated frequencies for UR4, UR6, and LR6 were higher than the measured values. The underlying causes for this difference could not be fully and comprehensively explained in this study. One reason could be the difference in tooth configuration between the calculation models and the teeth used for measurement. In particular, root length significantly affected the natural frequency (Fig. 5). The other cause could arise from differences in the mechanical conditions of the PDL, that is, thickness \( t \), Young's modulus \( E \), and Poisson's ratio \( \nu \).

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REFERENCES