TRAFFIC DYNAMICS IN PURSUIT OF EQUILIBRIUM

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Abstract: This paper studies how equilibrium is achieved in day-to-day traffic dynamics. Trip-makers update their perceived cost on a daily basis and adjust their route choice accordingly. The limiting behavior of day-to-day dynamics is characterized by the notion of equilibrium, which forms a stationary state. The attractiveness of an equilibrium state is examined by stability and can be quantified by its attraction basin. This paper illustrates how instability, as well as the problem of non-convergence from states outside the attraction basin, can be removed by modifying network configuration. This paper further investigates other attractors including cycles and chaos that are associated with the dynamic process in the pursuit of traffic equilibrium.

Key Words: day-to-day traffic dynamics, equilibrium stability, attraction basin

1. INTRODUCTION

The concept of equilibrium plays an important role in traffic assignment analysis. Equilibrium is often used as a predictor for the long-term state of a traffic network. Past research has mainly focused on the existence and uniqueness of equilibrium solution, as well as various algorithms for finding such a solution (Sheffi, 1985). The underlying presumption is that if equilibrium exists, then it will also arise. The day-to-day adjustment process that must have preceded equilibrium is deemed consequential and as such has been neglected in the formulation of equilibrium models. This idealization on the universal attractiveness of equilibrium remains dubious. Indeed, even day-to-day dynamics with apparently reasonable adjustment behavior may well fail to converge to equilibrium (Horowitz, 1984). It is therefore necessary to conduct an elaborate study on the day-to-day dynamics in the pursuit of traffic equilibrium.

Equilibrium characterizes the limiting behavior of day-to-day dynamics and forms a stationary state that remains unchanged over time. By introducing the dynamic approach, studies on traffic assignment equilibrium is extended from the static state of equilibrium to the dynamic attainability of equilibrium, as shown in Figure 1. The properties of existence and uniqueness do not address the attractiveness of an equilibrium state. Stable equilibrium attracts all points in its neighborhood and therefore is immune to perturbation, while unstable equilibrium cannot sustain even with very small fluctuations. Unstable equilibrium is thus not likely to last.
As for convergence from a given initial point, the equilibrium’s attraction basin needs to be identified. The attraction basin quantifies the equilibrium’s attractiveness by establishing its domain of attraction. The equilibrium state with its attraction basin covering the whole state space is globally attractive. On the other hand, an equilibrium state that is only locally attractive cannot guarantee convergence from an arbitrary initial point. Points within the attraction basin are attracted to the equilibrium while points outside are not.

Various formulations of the day-to-day dynamic process have been studied by Cantarella and Cascetta (1995), while Watling (1999) has systematically studied the stability conditions of such dynamics. This paper intends to expand their results by shedding light on some of the previously unfamiliar aspects of equilibrium, including the study on attraction basin, alternative attraction through temporary network alternation, and other types of attractors. To more clearly show the broad view of equilibrium attraction from the global state space, we will focus on the concepts while avoiding complicated mathematic regarding the dynamical system.

The paper is organized as follows: Section 2 formulates a typical day-to-day dynamic model based on trip-makers’ learning process on travel cost. Stochastic equilibrium under the logit route choice model is reached when the mean perceived route cost equals the actual travel cost. Equilibrium stability is analyzed in Section 3. An example of instability is shown where measures can be taken to reconstruct a stable equilibrium. Section 4 examines properties of the equilibrium’s attraction basin. Temporary network alternation can be made to attract points outside the attraction basin to equilibrium. Two types of attractors other than equilibrium are investigated in Section 5. Section 6 ends the paper with conclusions and some discussions.

2. DAY-TO-DAY TRAFFIC DYNAMICS

In this section we consider the general case of traffic assignment with elastic demand and the logit route choice model. Traffic dynamics is modeled as the day-to-day learning process of
travel cost. Trip-makers are assumed to have knowledge of the costs on all routes, no matter whether they choose the route or not.

2.1. Traffic assignment
Consider a network with $N$ origin-destination (OD) pairs. Each OD pair $i$ ($i = 1, 2, \ldots, N$) is connected by a set of routes, denoted as $R_i$, with $m_i = |R_i|$ as the number of routes for OD pair $i$ and $M = \sum_{i=1}^{N} m_i$ as the number of routes for the whole network. Routes are numerated as 1, 2, $\ldots, m_i$ for routes in $R_1$, $m_1 + 1, m_1 + 2, \ldots, m_1 + m_2$ for routes in $R_2$, and so on. On day $n$, a demand of $d_i^{(n)}$ users on OD pair $i$ make their travel choices over the route set $R_i$. The $M$-vector $x^{(n)} = [x_1^{(n)}, x_2^{(n)}, \ldots, x_i^{(n)}, \ldots, x_M^{(n)}]^T \in F^{(n)}$ denotes a traffic assignment of route flow, where $F^{(n)}$ is the feasible set:

$$F^{(n)} = \{x^{(n)} \in \mathbb{R}^M : \sum_{r \in R_i} x_r^{(n)} = d_i^{(n)}, \forall i = 1, 2, \ldots, N\}. \quad (1)$$

While any assignment satisfying (1) is considered a feasible flow, only flow that follows trip-makers’ behavioral characteristics are realistic. The logit route choice model is the commonly used model where individual trip-makers choose the routes they perceive to have the least cost. The probability of choosing a route is a function of the mean perceived route costs and the dispersion parameter $\theta$ ($\theta > 0$). Denote $C^{(n)} = [C_1^{(n)}, C_2^{(n)}, \ldots, C_i^{(n)}, \ldots, C_M^{(n)}]^T$ as the vector of mean perceived route cost. For a trip-maker on the OD pair $i$ ($i = 1, 2, \ldots, N$), the probability of choosing route $r$ ($r \in R_i$) on day $n$ is given as

$$\Pr \{r, n\} = \frac{1}{1 + \sum_{r \in R_i, r \neq r} \exp[\theta(C_r^{(n)} - C_i^{(n)})]} \cdot \quad (2)$$

The corresponding flow assignment $x^{(n)}$ therefore is

$$x^{(n)} = \text{diag} \{d^{(n)}\} p(C^{(n)}), \quad (3)$$

where the $M$-vector $d^{(n)}$ is the transformed demand vector, given as

$$d^{(n)} = \left[\frac{d_1^{(n)}}{m_1}, \frac{d_2^{(n)}}{m_2}, \ldots, \frac{d_i^{(n)}}{m_i}, \ldots, \frac{d_M^{(n)}}{m_M}\right]^T, \quad (4)$$

and the $M$-vector $p(C^{(n)})$ is the choice probability vector, given as

$$p(C^{(n)}) = \left[\Pr \{1, n\}, \Pr \{2, n\}, \ldots, \Pr \{r, n\}, \ldots, \Pr \{M, n\}\right]^T. \quad (5)$$

Because we have that

$$\sum_{r \in R_i} \Pr \{r, n\} = 1, \forall i = 1, 2, \ldots, N, \quad (6)$$

it follows that the feasibility requirement is fulfilled here, i.e. $x^{(n)} \in F^{(n)}$.

2.2. Day-to-day traffic dynamics
The day-to-day variation of network flow is captured by trip-makers’ comprehension of travel cost, which is modeled as a learning process shown in Figure 2. Trip-makers’ perception of travel cost on day $n$ is a result of previous experience. The perceived travel costs regulate the demand and route choices on day $n$, leading to a logit assignment as in (3). The resulting network flow determines the actual travel cost, which is included in updating the perceived travel costs on day $n + 1$.

The learning process is modeled by a day-to-day updating of mean perceived costs:

$$C^{(n+1)} = \beta c(x^{(n)}) + (1 - \beta) C^{(n)}, \quad (7)$$
where the mean perceived cost on day \( n+1 \) is a weighted average of \( e(x^n) \), the actual encountered cost on day \( n \), and \( C^n \), the mean perceived cost on day \( n \). This is a typical learning process where new information is absorbed while previous information is also preserved. Parameter \( \beta (0 \leq \beta \leq 1) \) represents the forgetfulness of trip-makers. For the extreme case of \( \beta = 1 \), all past information is abandoned. On the other end, \( \beta = 0 \) represents the case where actual cost does not at all influence perceived cost. Realistic models usually have the value of \( \beta \) in between.

![Day (n)]

Figure 2 Day-to-day traffic dynamics

Travel demand is elastic. The change in the perceived cost from day to day influences the daily demand:

\[
d^{(n)} = d(C^n).
\] (8)

The network flow is then assigned as in (3). Given the actual network flow, the actual travel costs are determined by the performance functions:

\[
e(x^n) = [c_1(x^n), c_2(x^n), ..., c_r(x^n), ..., c_M(x^n)]^T.
\] (9)

The whole dynamical system is written as

\[
C^{(n+1)} = \beta e(x^n) + (1 - \beta)C^n,
\]

\[
x^{(n)} = \text{diag}(d^{(n)})p(C^n),
\]

\[
d^{(n)} = d(C^n),
\] (10)

which can be simplified and represented by a recurrence function of the mean perceived cost:

\[
C^{(n+1)} = \beta e[\text{diag}(d(C^n))p(C^n)] + (1 - \beta)C^n.
\] (11)

The mean perceived cost, \( C^n \), therefore gives a state in the dynamical system. Demand and network flow, which are absent in (11) but dependent on the perceived cost, can subsequently be determined through (8) and (3).

2.3. The corresponding equilibrium

In the day-to-day dynamics of traffic flow, equilibrium is observed when the perceived cost remains stationary in the updating process and therefore the network flow does not change over time. Equilibrium solution is obtained by finding a stationary point (i.e. fixed point) of the recurrence function (11):

\[
C^* = \beta e[\text{diag}(d(C^*))p(C^*)] + (1 - \beta)C^*,
\] (12)

which is equivalent to (assuming \( \beta \neq 0 \))

\[
C^* = e[\text{diag}(d(C^*))p(C^*)],
\] (13)
i.e. the mean perceived travel cost is equal to the actual travel cost. The equilibrium state of
the day-to-day dynamics is therefore identical to the stochastic user equilibrium (SUE) under
the logit choice model.

Despite having its fixed point identical to the SUE solution, the day-to-day model of traffic
dynamics allows investigation on how such equilibrium is obtained through the process of
cost learning. We note that the equilibrium state as in (13) is independent of parameter $\beta$.
Parameter $\beta$, which does not affect the very state of equilibrium, certainly affects the way
how equilibrium is obtained. Therefore a dynamic approach is suitable for studying the course
of reaching equilibrium, which is detailed in the following sections.

The existence of the SUE solution is generally assured as long as the feasible set is non-empty
(Bell and Iida, 1997). The uniqueness of the SUE solution usually requires the demand and
travel cost function to fulfill some conditions concerning monotonicity and separability (or
symmetry). When multiple equilibria exist, the static analysis focused on the static state of
equilibrium fails to inform which equilibrium state will arise eventually. The following
example (Watling, 1999) is intended to show the case of multiple equilibria and will be
examined again in the later sections.

Example 1—Multiple equilibria. Consider a three-route one-OD network with fixed demand
of 2 and cost functions as:
\[ c_1(x) = x_1 + 3x_2 + 1, \quad c_2(x) = 2x_1 + x_2 + 2, \quad c_3(x) = x_3 + 6. \]
Assuming a logit model with $\theta = 1$, there are totally three equilibria:
\[ x_i^* = [1.75, 0.15, 0.10]^T, \quad x_{ii}^* = [0.77, 1.03, 0.20]^T, \quad x_{iii}^* = [0.22, 1.59, 0.19]^T. \]
Because demand is fixed here, the dynamics is actually 2-dimensional. We shall represent the
system state by the cost differences $(g_1, g_2) = (C_1 - C_2, C_1 - C_3)$. The equilibria are therefore
given as, respectively,
\[ (g_1, g_2)_i^* = (-2.45, -2.89), \quad (g_1, g_2)_{ii}^* = (0.30, -1.34), \quad (g_1, g_2)_{iii}^* = (1.95, -0.19). \]
All three equilibria satisfy the principals of stochastic equilibria. It is impossible to tell which
one of them is going to arise.

3. STABILITY OF EQUILIBRIUM

Although the fixed point of the day-to-day traffic dynamics coincides with the stochastic user
equilibrium, its existence and uniqueness does not guarantee its attractiveness. A stable
equilibrium is one such that, by starting sufficiently close the equilibrium, the dynamical
system can be made to remain within an arbitrarily small distance from the equilibrium and to
converge to the equilibrium in the infinity of time. Stability signifies the equilibrium’s
immunity to small perturbations. Unstable equilibrium is not likely to last. It is therefore
important for traffic engineers to ensure that the designed equilibrium is indeed stable and
thus maintainable.

3.1. Perturbation stability

Consider for the dynamical system $x^{(n+1)} = f(x^{(n)})$, the fixed point $x^*$ ($x^* = f(x^*)$) is
(asymptotically) stable if, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that for any point
$x^{(0)} : |x^{(0)} - x^*| < \delta$ we have that $|x^{(n)} - x^*| < \varepsilon, \forall n \geq 1$ and $\lim_{n \to +\infty} x^{(n)} = x^*$. It should be noted that,
strictly speaking, future states should be written as \( x^{(n)}(n = 1, 2, \ldots) \) since they intrinsically depend on the initial condition \( x^{(0)} \). However, for notational simplicity, we denote them as \( x^{(n)} \) and omit the specification of the initial state unless where confusion is likely to arise.

This type of stability is also called \textit{perturbation stability} because it describes the system behavior under small perturbations from equilibrium. Stable equilibrium can sooner or later recover from perturbation whereas unstable equilibrium diverges quickly even under very small perturbation. In the day-to-day traffic dynamics, it then follows that unstable equilibrium, though satisfying the equilibrium criteria, is not maintainable. Traffic conditions continuously experience various kinds of fluctuations from day to day. Unstable equilibrium cannot recover from any such fluctuations. In other words, unstable equilibrium is also unobservable because it is transient and an invariant equilibrium flow over days cannot be observed.

The following theorem (Watling, 1999) provides a method for establishing the stability property of equilibrium. Stability is evaluated by studying a linear approximation of the dynamical system, linearized in the neighborhood of the equilibrium.

\textbf{Theorem 1.} Consider the dynamical system (11), simplified as

\[
\mathbf{C}^{(n+1)} = \mathbf{A} \mathbf{C}^{(n)},
\]

with \( \mathbf{A} \) everywhere differentiable, and let the \( M \times M \) matrix \( \mathbf{A} \) denote the Jacobian matrix of \( \mu(\mathbf{C}) \) with respect to \( \mathbf{C} \), evaluated at the equilibrium \( \mathbf{C}^* \). Then \( \mathbf{C}^* \) is stable with respect to (11) if and only if all the eigenvalues of \( \mathbf{A} \) are within the unit circle (i.e. with norm less than one).

\textbf{Example 2—Stability and instability.} Consider a two-route one-OD network with cost functions as

\[
c_1 = 4 \left[ 1 + 0.15 \left( \frac{x_1}{1000} \right)^4 \right] \quad \text{and} \quad c_2 = 3 \left[ 1 + 0.15 \left( \frac{x_2}{900} \right)^4 \right].
\]

Demand is elastic and depends on the lesser of the perceived costs:

\[
d = x_1 + x_2 = 2000 - 50 \cdot \min[C_1, C_2].
\]

Parameter in the logit choice model is set at \( \theta = 1.2 \) and the unique equilibrium is then given as

\[
(C_1^*, C_2^*) = (4.181, 3.880) \quad \text{with} \quad (x_1^*, x_2^*) = (742, 1064).
\]

We have already shown that the equilibrium solution is independent of the updating parameter \( \beta \). Stability is analyzed here for two different values of \( \beta \):

Case 1: \( \beta = 0.5 \); Case 2: \( \beta = 0.7 \).

Starting from the same initial point \((C_1^{(0)}, C_2^{(0)}) = (4.2, 3.9)\), the system evolution for the two cases is shown in Figure 3. Equilibrium in case 1 is stable and therefore the system evolution is convergent. However, the same equilibrium solution is unstable when \( \beta \) takes larger values. In case 2, even by starting very close to equilibrium, the system evolution quickly diverges.

Trip-makers’ behavior is influential on the system stability, as shown in Example 2. System stability is more likely to be maintained when trip-makers’ cost updating behavior is more conservative (i.e. with a smaller \( \beta \)). However, as trip-makers’ behavior is an intrinsic feature in the traffic dynamics, there can be no firm supposition regarding the value of \( \beta \). The
parameter $\beta$ can take on any value between 0 and 1 while still be considered as “rational.” Therefore, no guarantee can be made to ensure a small $\beta$ for system stability.

![Figure 3 Stable and unstable equilibria](image)

3.2. Stability fails to uphold

When stability fails to uphold, by appropriately modifying the system parameters, stability can be regained. The equilibrium solution, as a result, may be shifted. The only exception is that the equilibrium remains unchanged with varying $\beta$. However, the parameter $\beta$ is behavioral in nature and inherently related to the trip-makers. External measures intending to change $\beta$ are usually futile efforts. Traffic engineers should therefore focus on other features in the dynamical system. Travel cost functions can be easily modified by means of road pricing, traffic signal resetting, and/or capacity modification such as lane adding/shutting. The details of how these modifications can be systematically made exceed the scope of the present paper. An example is shown below to demonstrate the possibility of regaining stability.

**Example 3—Stability regained.** Consider case 2 in Example 2. The current equilibrium is unstable and a divergent evolution (Figure 3) is observed from initial state $(C_1^{(0)}, C_2^{(0)}) = (4.2, 3.9)$. Stability, however, can be regained through capacity improvement. Consider the case where capacity of route $2$ is enlarged from day 20 on and the cost function is then changed to

$$c_2 = 3 \left[ 1 + 0.15 \left( \frac{x_2}{1200} \right)^4 \right].$$

The system evolution under the modified cost structure is shown in Figure 4, which converges to the new equilibrium

$$(C_1^*, C_2^*) = (4.079, 3.487) \text{ with } (x_1^*, x_2^*) = (602, 1224).$$
4. ATTAINABILITY AND ATTRACTION BASIN

Stability governs the system evolution only in the neighborhood of equilibrium. It remains uncertain whether evolution with an arbitrary initial state will converge to equilibrium or not. A stable equilibrium may as well be unattainable from some given initial state. The attraction basin is the property that associates convergence to the equilibrium from an initial state. Evolution starting from an initial state within the attraction basin will converge to equilibrium for sure.

4.1. Attraction basin

Attraction basin is a useful concept in depicting the convergence behavior from a global perspective of the state space. Consider day-to-day dynamics $x^{(n+1)} = f(x^{(n)})$, the stable equilibrium $x^*$ ($x^* = f(x^*)$) is called attainable from a point $x$ if

$$
x^{(0)} = x \Rightarrow \lim_{n \to +\infty} x^{(n)} = x^*.
$$

Therefore, if equilibrium is attainable from a point, the dynamic evolution starting at that point converges to equilibrium. The equilibrium’s attraction basin (or domain of attraction), denoted as $B(x^*)$, consists of all points that it is attainable from:

$$
B(x^*) = \{x^{(0)} : \lim_{n \to +\infty} x^{(n)} = x^*\}.
$$

Any points inside the attraction basin are therefore attracted to the equilibrium while points outside are not. The equilibrium is said to be globally attainable if its attraction basin covers the whole state space $S$, i.e. $\lim_{n \to +\infty} x^{(n)} = x^*$, $\forall x^{(0)} \in S$. In the case of global attainability, evolution with an arbitrary initial state converges to equilibrium.
The region of attraction basins can be estimated by “tracing back” the orbits of system evolution. We can determine the inverse mapping (if existing)

\[ x^{(n-1)} = f^{-1}(x^{(n)}) \]

and start the trace-back from around equilibrium. If the inverse function does not exist, we may need to identify multiple solutions and map out all pre-images. Repeat this procedure and the estimate is gradually enlarged.

The utility of an attraction basin is its sufficiency and necessity in telling whether an initial point is attracted to the equilibrium or not. An initial point is attracted to the equilibrium if and only if the initial point lies inside the equilibrium’s attraction basin. Attraction basin can then perform as a divisor of the global state space. All states inside an attraction basin evolves towards the same equilibrium while any states outside is not attracted to that equilibrium.

**Example 4—Attraction basin.** Consider the three equilibria in Example 1 with additional information of \( \beta = 0.2 \). If we trace back the orbits, the phase portrait in Figure 5 is generated. Equilibrium \( (g_1, g_2)_I = (-2.45, 2.89) \) on the left side of the figure and equilibrium \( (g_1, g_2)_II = (1.95, 0.19) \) on the right are stable while equilibrium \( (g_1, g_2)_III = (0.30, -1.34) \) in the middle is unstable. Points on the dash-dot line represent the states that will evolve towards the unstable equilibrium. All points on the left of the dash-dot line are attracted to \( (g_1, g_2)_I \), while all points on the right are attracted to \( (g_1, g_2)_III \). Therefore \( B[(g_1, g_2)_I] \) is given as the whole region left of the dash-dot line and \( B[(g_1, g_2)_III] \) as the whole region right.
of the dash-dot line. The global state space is then divided into three parts: two areas and a curve. Once an initial point is given, its eventual image in the day-to-day dynamics can be immediately told by the part that it lies within. For instance, an evolution starts right of the dash-dot line for sure converges to equilibrium $(g_1, g_2)_{III}$.

4.2. Outside attraction basin: transitional attainability

It is important to identify the equilibrium’s corresponding attraction basin because points outside are not attracted. When the initial state falls out of the attraction basin, it may be desirable to change the network dynamics in order to make the equilibrium attainable. This can be accomplished by temporary network alteration which for a period of time directs the dynamics in a different way. If this temporary alteration is properly done, the traffic dynamics will move the outside point, which otherwise is not attracted, into the attraction basin. Since the alteration is not intended to be permanent, when the alteration is removed and the original network restored, the dynamic evolution having been moved to a different point from which the desired equilibrium is attainable.

Equilibrium is *transitionally attainable* to a point outside the attraction basin if convergence from this point can be achieved by the above method of temporary network alteration. The point is therefore in the equilibrium’s *transitional attraction basin*, which forms an addition to the attraction basin. When global transitional attainability is ensured, the equilibrium can be obtained from an arbitrary initial state.

![Figure 6 Transitional attainability](image)
Example 5—Transitional attainability. Consider Example 4 with initial point at \((g_1, g_2)^{(0)} = (1, 2)\). Because the initial point is in the attraction basin of \((g_1, g_2)^*_{III} = (1.95, -0.19)\), the system evolution soon converges to \((g_1, g_2)^*_{III}\). However, if we examine the system cost then we will found \((g_1, g_2)^*_{I}\) is more desirable. The total actual travel cost is \(TC(x^*_I) = 7.06\) under equilibrium \((g_1, g_2)^*_{I}\) and \(TC(x^*_II) = 8.92\) under equilibrium \((g_1, g_2)^*_{II}\). Therefore we may want to direct the network flow started at \((g_1, g_2)^{(0)} = (1, 2)\) to equilibrium \((g_1, g_2)^*_{I} = (-2.45, -2.89)\), which is better off in terms of total travel cost.

For the temporary network alteration, we consider modifying the cost function on route 2 by adding an additional cost \(\tau = 0.2\). This can be easily achieved by charging route 2 users an equivalent price. While \((g_1, g_2)^{(0)} = (1, 2)\) is attracted to \((g_1, g_2)^*_{I}\) in the original network, \((g_1, g_2)^*_{III}\) is no longer an equilibrium solution in the new network with \(c_2(x) = 2x_1 + x_2 + 2 + \tau\). Instead, it is attracted to the new equilibrium located at \((g_1, g_2)^*_{III} = (-2.77, -2.96)\), which lies in the attraction basin of \((g_1, g_2)^*_{I}\). When the system evolution has moved close enough to the new equilibrium, we can then restore the original system by setting \(\tau = 0\). The restored dynamics, starting from a new initial point, will evolve towards \((g_1, g_2)^*_{I}\), the desired equilibrium. This process of transitional attraction by temporary network alteration is shown in Figure 6.

5. BEYOND EQUILIBRIUM

The dynamic approach to traffic assignment equilibrium puts another challenge on the robustness of equilibrium. We have shown in the previous sections that only points within the attraction basin are attracted to the equilibrium. Points outside the attraction are not attracted to the equilibrium. Instead, they may be attracted to other types of attractors such as cycles and chaos.

5.1. Periodic attractor

Consider the dynamical system \(x^{(n+1)} = f(x^{(n)})\), periodic orbits are characterized by the following stationary point of \(f^{(k)} (k = 1, 2, \ldots)\):

\[
x^* = f^{(k)}(x^*).
\]  

We call \(x^*\) a period-\(k\) point if \(k\) is the smallest number such that (17) is held. The series \(x^*, f(x^*), f^2(x^*), \ldots, f^{(k-1)}(x^*)\) forms a cycle of period \(k\). A \(k\)-cycle is stable if, intuitively speaking, points in the neighborhood of anyone of the \(k\) periodic points are attracted to the cycle. Similar with fixed point, a cycle attractor also has its attraction basin, where all inside points are attracted to the cycle.

Example 6—Periodic attractor. Consider Example 2 with \(\beta = 0.44\) and the capacity of route 2 reduced:

\[
c_2 = \left[1 + 0.15 \left(\frac{x_2}{600}\right)^4\right].
\]
The unique equilibrium is located at \((C_1^*, C_2^*) = (4.490, 4.608)\) with \((x_1^*, x_2^*) = (950, 825)\) but is unstable. Instead, a 4-cycle attractor is observed. The attraction process of initial point \((C_1^{(0)}, C_2^{(0)}) = (4.2, 3.9)\) is shown in Figure 7. Cycles of other periods are also observed with different values of \(\beta\). For instance, a 2-cycle is observed with \(\beta = 0.4\) and a 32-cycle with \(\beta = 0.457\).

![Figure 7 Periodic attractor: a 4-cycle](image)

5.2. Chaotic attractor
Chaotic attractor is one that follows no periodicity. A chaos is literally a condition of great disorder. Mathematical quantification of chaos may be difficult but some observations can be readily made by a numerical example.

**Example 7—Chaotic attractor.** Consider Example 6 with \(\beta = 0.5\). The dynamic evolution from any given initial state (except the unstable equilibrium) appears to converge to a specific shape, which is shown in Figure 8 by plotting \((C_1^{(n)}, C_2^{(n)})\) from day 200~2000. However, these orbits follow no periodicity.

6. CONCLUSIONS
This paper investigated the day-to-day traffic dynamics in the pursuit of equilibrium. Traffic dynamics was modeled by a daily updating process which simulates trip-makers’ learning behavior on route travel costs. The dynamic approach toward equilibrium is valuable because it allows studies on how equilibrium is achieved. We pointed out that existence and uniqueness of equilibrium does not guarantee its attractiveness. The property of stability, which reflects the equilibrium’s attractiveness in the neighborhood and its tolerance toward small perturbations, is important in that only stable equilibrium is maintainable and therefore
observable. Unstable equilibrium is not likely to last because even very small fluctuations lead to divergence. It is essential that traffic engineers should ensure the designed equilibrium to be stable. Where instability occurs, redesigning the network appears necessary to direct the traffic dynamics into a stable equilibrium.

Stability of equilibrium, though ensuring convergence from its neighborhood, does not guarantee attraction from the global state space. The attraction basin specifies initial states that eventually will converge to equilibrium. In other words, equilibrium is only attainable from points within the attraction basin. From an initial state outside the attraction basin, the desired equilibrium may be achieved through temporary network alteration. This transitional attainability acts as an expansion of the attraction basin. When global attainability is confirmed, the equilibrium can be established from an arbitrary initial state.

The dynamic approach of traffic dynamics brings another question: why is equilibrium considered as the only eventuality in the first place? There are also other attractors, including cycles and chaos. Their existence implies that traffic dynamics in the real world can follow more complicated patterns than the idealistic equilibrium.
Although based on a very simplistic and stringent model of day-to-day learning process, the shown attraction patterns in this paper do represent a wide range of possibilities in traffic dynamics. We hope that this paper will widen the understanding of day-to-day traffic dynamics and inspire more advanced models in simulating such dynamics.

ACKNOWLEDGEMENTS

This study is partially supported by the Competitive Earmarked Research Grant from the Research Grants Council of the Hong Kong Special Administrative Region (Fund number: HKUST6283/04E).

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