Semi-dynamic Markovian Path Flow Estimator Considering the Inconsistencies of Traffic Counts

Hiroshi SHIMAMOTO a*, Atsushi KONDO b

a Department of Civil and Environmental Engineering, University of Miyazaki, 889-2192 Miyazaki, Japan
b Domestic Consulting Operations, NIPPON KOEI CO., LTD, Japan

E-mail: shimamoto@cc.miyazaki-u.ac.jp
E-mail: a8370@n-koei.co.jp

Abstract: This paper proposes a semi-dynamic path flow estimator (PFE) that does not require enumeration of the paths set. The proposed model takes traffic assignments derived by the absorbing Markov chain approach and incorporates them into the nonlinear PFE model while considering the inconsistency of traffic counts described by Chen et al. (2009). The proposed model was applied to two types of toy networks. As a result, we confirmed that the proposed model can estimate the link volume with high accuracy if the observed link volumes are error-free. We further confirmed that the total origin-destination volumes as well as link volumes are underestimated if the observation interval of the traffic counts is longer than the length of the time period.

Keywords: Absorbing Markov chain approach, Origin-destination matrix, Path flow estimation, Time-dependent

1. INTRODUCTION

The origin-destination (OD) matrix contains fundamental information for urban and transportation planning. The OD matrix can be derived in part by the four-step transportation demand model. However, because the input data of the four-step model are not updated regularly due to the costs related to surveying, it is difficult to update the OD matrix frequently. An alternative approach is to estimate the OD matrix using link traffic counts, which is interpreted as the inverse of the traffic assignment problem. Because this approach can estimate the OD matrix of any day using the daily observed link traffic counts, many researchers have proposed the OD matrix estimation model using the link traffic counts data. There are two types of approaches for estimating the OD matrix using link counts data: statistical estimation and mathematical programming approaches. Statistical estimation approaches estimate the OD matrix based on statistical models, such as the Bayesian inference technique (Pitombeira-neto et al. 2016), Gibbs sampling and Kalman filtering (Cho et al. 2009), and so on. However, it is difficult to utilise these approaches to forecast how the OD demands (or path flows) are affected by the change of network structure (e.g. a link closure or opening a new road) because they do not explicitly consider the topological information. The latter approaches can further be classified as maximum entropy, generalised least squares, bi-level programming, and path flow estimation (Bell and Iida 1997).

Van Zuylen and Willumsen (1980) first proposed the maximum entropy approach. They assumed that a trip table that maximised entropy subject to constraints imposed on it by link

* Corresponding author.

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measurements would be the most likely trip table (Bell and Iida 1997). Although their model adopted a proportional-assignment approach (i.e. users’ route choice probabilities are given and independent from the congestion level), Fisk (1988) imposed the UE routing principle to a similar matrix estimation problem, resulting in a maximum entropy model subject to a variational inequality constraint. Tsekeris et al. (2003) proposed the time-dependent OD matrix estimation problem with a simulation-based model. Because their model iteratively updates the link use proportion, their model can consider the effect of congestion onto the link use proportion. Xie et al. (2010, 2011) proposed the subnetwork OD matrix estimation model based on the maximum entropy method.

One of the disadvantages of the maximum entropy approach is that it cannot consider the measurement error of link flows. In contrast with this disadvantage, the generalised least squares (GLS) approach allows for errors from various sources. Most initial GLS approach models assume the link choice proportions of the traffic are independent from congestion levels in the network, but the bi-level programming approach incorporating a traffic assignment in the lower problem was further proposed, which facilitates describing the effect of congestion. Yang (1995) proposed a bi-level model whose upper model is based on the GLS approach, and a lower model serves the user equilibrium (UE) condition formulated as a variational inequality problem. Two heuristic solution algorithms for the bi-level model were also proposed in the paper. Lundgren et al. (2008) also formulated the OD estimation model as a bi-level optimisation problem; they further reformulated it as a single-level mathematical problem and proposed a heuristic solution algorithm. Nie et al. (2010) also relaxed the UE condition in bi-level problems based the OD estimation model while taking users’ route choice behaviour into account. They also developed an efficient solution algorithm for the relaxed model.

The path flow estimator (PFE) approach is an alternative approach for estimating path flows (hence OD flows as well) with a single-level formulation while taking users’ route choice behaviour into account. Sherali et al. (1994) first proposed a linear PFE model, where users were assumed to choose their route in a UE manner. Although their model assumes that all link volumes are counted, Sherali et al. (2003) expanded their model to consider the cases where only a partial set of link volumes are counted. Bell et al. (1997) proposed a log-linear PFE model based on the stochastic user equilibrium (SUE) assignment. The proposed model was solved by combining the iterative balancing scheme and the column generation method. Although Bell et al. (1997)’s PFE also admits the cases where only a partial set of link volumes are counted, their model does not consider the measurement errors in traffic counts. Chen et al. (2005) examined the capability of the PFE model in capturing the total demand of the study network as well as individual OD demands, and considered the allowed measurement error for each observed traffic count as the constraint condition of the model. However, the allowed measurement error for each traffic count is required to set a priori, which may be difficult to apply to real networks. Chen et al. (2009) further proposed a norm approximation method to internally handle the inconsistent traffic counts. As described later, the absolute error of link volume is carried via a virtual path, and the virtual path flows are included in the objective function of their model. Nie et al. (2005) proposed another approach for considering measurement errors by incorporating the PFE into the GLS model framework. Recently, efficient algorithms for solving the PFE have also been proposed. For example, Tang et al. (2013) replaced the iterative balancing scheme by the primal-dual heuristic. Abareshi et al. (2017) reduced the constraints by applying an L-shaped algorithm. Note that most of the PFE models so far implicitly limit the path set due to column generation, K-shortest path ranking procedure, or other heuristic methods. However, it is important to consider the entire possible path set, especially when the PFE is applied to evaluate the
fluctuation of path flows due to road closures.

Thus far, only the static OD estimation models have been summarised, but many dynamic OD estimation models have also been proposed by extending the static OD estimation models mentioned above. For example, Lu et al. (2013) proposed a dynamic PFE model, where a dynamic network loading model based on Newell’s simplified kinematic wave theory is employed in a DUE assignment process. Their model can consider the flow propagation accurately in continuous time. However, because their model, as well as other full dynamic approach based models, requires more detailed input data than that of statistic models, a semi-dynamic approach (considering discrete time) may be reasonable for practical applications. A space-time expanded network (STEN) is one such method for dealing with a semi-dynamic approach. Nie et al. (2008) extended the aforementioned relaxed-based formulation (Nie et al. (2010) to the dynamic regime using STEN. Wong et al. (2013) also utilised STEN for the estimation of the time-dependent OD matrix. However, STEN requires many more links than a static model for the calculation, which may prevent real network application. Alternatively, semi-dynamic assignment models have been developed. The semi-dynamic assignment assumes a static network equilibrium in each time period, and the residual flow, which cannot reach its destination within the time period, is carried to the next time period. The semi-dynamic assignment model can be classified into three categories based on how residual flow is handled: i) the demand modification approach (Fujita et al. 1988, Ujii et al. 2003, Nakayama et al. 2012); ii) the link flow modification approach (Fujita et al. 1989); and iii) the queue approach (Akamatsu et al. 1998). Of those three approaches, the demand modification approach requires less computational cost and the uniqueness of the solution is guaranteed. Recently, Shimamoto et al. (2017) evaluated tsunami evacuation scenarios in a real-size network using the semi-dynamic, multi-class assignment model.

Based on this background, this paper proposes a path flow estimator without path enumeration. The estimator takes traffic assignments derived by the absorbing the Markov chain approach and incorporates them into the nonlinear PFE model considering the inconsistency of traffic counts proposed by Chen et al. (2009). The proposed model estimates time-dependent OD flows as well as time-dependent path flows by applying the idea of semi-dynamic assignment. Furthermore, although the PFE model by Chen et al. (2009) was formulated as path-based problem, our model reformulates it as a link-based problem to avoid enumerating all of the possible paths. Therefore, our proposed PFE model has distinctive features that can explicitly consider all possible paths without path enumeration as well as consider inconsistency of traffic counts.

The remainder of this paper is organised as follows: Section 2 describes the proposed model formulation. Section 3 describes the solution procedure of the proposed model. Section 4 provides numerical examples by applying the proposed model to hypothetical networks. Finally, Section 5 summarises the conclusions and proposes future works.

2. MODEL FORMULATION

2.1. Static Model Formulation

This section presents a static model formulation. As Chen et al. (2009) noted, the best approximation of a path flow pattern is the solution that keeps the deviation between the estimated and observed link-flows as small as possible. The estimation error for each link with traffic counts is given as follows:

$$\psi_a = |v_a - \sum_{r \in \Omega} \sum_{k \in K} f_{rs}^k \delta_{rs}^a|, \forall a \in M$$  \hspace{1cm} (1)
where \( M \) is the set of links with traffic counts, \( v_a \) is the observed flow on link \( a \), \( f^{k}_{rs} \) is the estimated flow on path \( k \) between \( rs \), \( \delta^{k}_{rs} \) is the path-link indicator that uses 1 if link \( a \) is on path \( k \) between \( rs \) and 0 otherwise, and \( \psi_a \) is the absolute error between the observed and estimated link flow. However, the definition of the estimation error for all the links with traffic counts is not uniquely determined. Therefore, for the definition of the estimation error, Chen et al. (2009) utilised the norm approximation method as follows:

\[
\|\Psi\|_p = \left( \sum_{a \in M} \psi_a^p \right)^{1/p}
\]

(2)

Three different norms with different \( p \) values are considered for evaluating the approximate solutions: (i) minimise the maximum absolute error for \( p \Rightarrow \infty \); (ii) minimise the average absolute error for \( p = 1 \); and (iii) minimise the average squared error for \( p = 2 \). Then, the virtual paths that carry the flow equivalent to the absolute error were defined in the model proposed by Chen et al. (2009), and both the physical and virtual path flows were assigned in the network. (Note that only virtual path that carries a flow equivalent to the maximum absolute error among all the links with traffic counts for \( p \Rightarrow \infty \), but there are as many virtual paths as the number of observed links for \( p = 1 \) and \( p = 2 \); each virtual path carry flow equivalent to \( \psi_a \).)

With this idea, three different models can be formulated with the three different norm approximations. Because Chen et al. (2009) numerically confirmed the problem with \( p = 2 \) may fall into the ill-conditioned problem with some penalty cost, we adopt \( p = 1 \) (i.e. minimising the average absolute error between estimated and observed link flows) in this study. Then, the PFE model can be formulated as a non-linear optimisation problem with regard to both physical and virtual paths as following. (We call this PFE model the [PFE/Norm-path] because the path flows are unknown variables.)

\[
\begin{align*}
\text{min} & \quad Z_{path} = \sum_{a \in A} \int_0^{x_a} t_a(w)dw + \frac{1}{\theta} \sum_{rs \in \Omega} \sum_{k \in K_{rs}} f^{k}_{rs}(\ln f^{k}_{rs} - 1) + \frac{1}{\theta} \sum_{a \in M} \psi_a \ln(\psi_a - 1) \\
\text{subject to} & \quad x_a = \sum_{rs \in \Omega} \sum_{k \in K_{rs}} f^{k}_{rs} \delta^{a,k}_{rs}, \forall a \in A \\
& \quad q_{rs} = \sum_{k \in K_{rs}} f^{k}_{rs}, \forall rs \in \Omega \\
& \quad x_a \geq v_a - \psi_a, \forall a \in M \\
& \quad x_a \leq v_a + \psi_a, \forall a \in M \\
& \quad x_a \leq C_a, \forall a \in U \\
& \quad \psi_a \geq 0, \forall a \in M \\
& \quad f^{k}_{rs} \geq 0, \forall rs \in \Omega, \forall k \in K_{rs}
\end{align*}
\]

(3)

where \( U \) is the set of links without traffic count, \( A \) is the set of links in the network (i.e. \( A = M \cup U \)), \( \Omega \) is the set of OD pairs, \( K_{rs} \) is the set of paths connecting OD pair \( rs \), \( \theta \) is the dispersion parameter, \( \rho_a \) is the penalty cost on link \( a \), \( q_{rs} \) is the estimated flow of OD pair \( rs \), \( x_a \) is the estimated flow on link \( a \), \( C_a \) is the capacity of link \( a \), and \( t_a(\cdot) \) is the travel time function of link \( a \).

The first two terms of the objective function corresponds to the SUE assignment.
third term tries to reduce the entropy of the virtual path flows, and the fourth term tries to reduce the penalty for the virtual paths. The penalty cost for each observed link, \( \rho_a \), should be determined based on the level of confidence of each observation. Similar to the SUE assignment model, Eqs. (4) and (5) respectively represent relationship between path flows and link flows, and the path flows and the OD flows. Equations (6) and (7) respectively represent the lower boundary and upper boundary of the estimated link flows. Note that it is not necessary to determine these two boundaries of the estimated link flows, which are optimised as the virtual flow, \( \psi_a \). Equation (8) represents the capacity constraints for the unobserved links, and Eqs. (9) and (10) represent the non-negative constraints for physical path flows and virtual path flows, respectively.

Because the path flow, \( f_{rs}^\prime \), is included in the objective function of [PFE/Norm-path], it is impossible to calculate this function in the framework of the model proposed in this study, which tries to explicitly consider all the possible paths without employing path enumeration for the estimation. (Note that the model of Chen et al. (2009) can calculate this objective function because their model enumerates candidate paths by a column generation procedure.) To avoid the path enumeration for calculating the entropy function of the path flow, Akamatsu (1997) showed the decomposition of the entropy function, and derived a link-based SUE model. Similarly, we can derive a link-based PEF model (called [PFE/Norm-link]), which is equivalent to [PFE/Norm-path] as follows:

\[
\begin{align*}
\text{[PFE/Norm-link]} \\
\min_{(x,\psi)} Z_{\text{link}} = & \sum_{a \in A} \int \limits_0^{x_a} t_a(w)dw + \frac{1}{\theta} \sum_{r \in R} \left( \sum_{a \in IN(j)} \sum_{a \in M} (x_a^r) \ln \left( \sum_{a \in A} x_a^r \right) - \sum_{a \in A} x_a^r \ln x_a^r \right) \\
& + \frac{1}{\theta} \sum_{a \in M} \psi_a \ln(\psi_a - 1) + \sum_{a \in M} \rho_a \psi_a 
\end{align*}
\]

Subject to

\[
\begin{align*}
\sum_{a \in IN(k)} x_a^r - \sum_{a \in OUT(i)} x_a^r + q_{rs} \delta_{rk} - \sum_{s \in S} q_{rs} \delta_{sk} &= 0, \ \forall k \in N, \forall r \in R \\
x_a^r &= \sum_{r \in R} x_a^r, \ \forall a \in A \\
x_a^r &\geq 0, \ \forall a \in A, \ \forall r \in R \\
\end{align*}
\]

Eqs. (6) – (9),

where \( R \) is the set of origins, \( S \) is the set of destinations, \( N \) is the set of nodes, \( IN(i) \) is the set of links that leads into node \( i \), \( OUT(i) \) is the set of links that leads out of node \( i \), \( x_a^r \) is the estimated flow of link \( a \) starting from origin \( r \), and \( \delta_{ab} \) is the Kronecker’s delta, which uses 1 if \( a = b \) and 0 otherwise.

The Lagrangian function of [PFE/Norm-link] can be expressed as follows:

\[
\begin{align*}
(f(q), \Psi, l, u, d) &= Z_{\text{link}} + \sum_{a \in M} l_a \left( v_a - \sum_{r \in \Omega} \sum_{k \in K_{rs}} f_{rs,k} \delta_{rk}^a - \psi_a \right) \\
& + \sum_{a \in M} u_a \left( v_a - \sum_{r \in \Omega} \sum_{k \in K_{rs}} f_{rs,k} \delta_{rk}^a + \psi_a \right) + \sum_{a \in \Omega} d_a \left( c_a - \sum_{r \in \Omega} \sum_{k \in K_{rs}} f_{rs,k} \delta_{rk}^a \right)
\end{align*}
\]

The first optimal conditions with respect to the path flow variables analytically lead to
flows of both physical paths and virtual paths as follows:

\[ f_{rs,k} = \frac{\exp\left(-\theta\sum_{a \in A} t_a(x_a) \delta_{rs}^a - \sum_{a \in M} (u_a + l_a) \delta_{rs}^a - \sum_{a \in U} d_a \delta_{rs}^a\right)}{\sum_{a' \in K_{rs}} \exp\left(-\theta\sum_{a \in A} t_a(x_a) \delta_{rs}^a - \sum_{a \in M} (u_a + l_a) \delta_{rs}^a - \sum_{a \in U} d_a \delta_{rs}^a\right)} q_{rs}, \quad \forall rs \in \Omega, \forall k \in K_{rs} \]  

(16)

\[ \psi_a = \exp(\theta(-\rho_a + l_a - u_a)), \quad \forall a \in M \]  

(17)

2.2. Expansion to Semi-dynamic Path Flows Estimation

In this section, we expand the proposed model for the multiple time period estimation. By applying the idea of the demand modification approach based on the semi-dynamic assignment model (Fujita et al. 1988, Ujii et al. 2003, Nakayama et al. 2012), the modified OD volume at time period \( \tau \), \( \tilde{q}_{rs}^{\tau} \), is given as follows:

\[ \tilde{q}_{rs}^{\tau} = g_{rs}^{\tau-1} + q_{rs}^{\tau} - g_{rs}^{\tau} \]  

(18)

where \( g_{rs}^{\tau} \) is the residual flow from time period \( \tau \) to time period \( (\tau + 1) \) in OD pair \( rs \) and can be given as follows:

\[ g_{rs}^{\tau} = \frac{C_{rs}^{\tau}}{2T_w} q_{rs}^{\tau} \]  

(19)

where \( T_w \) is the length of each time interval and \( C_{rs}^{\tau} \) is the minimum travel cost between \( rs \). Note that \( q_{rs}^{\tau} \) is the OD volume at time period \( \tau \), which is estimated in this model. As shown in Eq. (19), the residual flow to time period \( (\tau + 1) \), \( g_{rs}^{\tau} \), cannot be determined until the OD volume at time period \( \tau \), \( q_{rs}^{\tau} \), is estimated. Therefore, the OD volumes are estimated successively in each time interval in the proposed model.

3. SOLUTION PROCEDURE

Because the objective function of [PFE/Norm-link] is convex with respect to the unknown variables, this problem can be solved with any commercial solver if the objective function can be evaluated with given OD volumes. (Note that the objective function of [PFE/Norm-link] is minimised by the interior point method using the ‘fmincon’ function in MATLAB in this study.) Therefore, this section presents the procedure to compute the objective function with given OD volumes.

3.1. Computing the Objective Function

Similar to Chen et al. (2009) and other transportation assignment models, the [PFE/Norm-link] is solved by the partial linearisation method, in which the linearised objective function with regard to the link cost function is solved iteratively until the convergence. The detailed procedure is summarised as follows.

Step 1: Set up the problem and initialise.

Set \( \tau = 1, n = 0, x_a^{r,n} = l_a^{r,n} = u_a^{r,n} = d_a^{r,n} = 0, \quad \forall a \in A \) (\( n \) is the count of the outer loop) and set the residual flow from the previous time period as \( g_{rs}^{0} = 0, \forall rs \in \Omega \). Set link costs as \( t_a^{r,n} = t_a(0) \) and \( n = n + 1 \).

Step 2: Add the residual flow from the previous time period.

The residual flow from the previous time period is added to the OD volumes as following;
\[
\tilde{q}_{rs}^{\tau} = q_{rs}^{\tau} + g_{rs}^{\tau-1}
\]  

(20)

Note that \( q_{rs}^{\tau} \) is assumed to be given when calculating the objective function.

**Step 3:** Solve the sub-problem.

Solve the following partially linearised problem [Sub-PFE/Norm-link], in which the first term of [PEF/Norm-link] is linearised with regard to \((x^{r,n}, \Psi^{r,n})\).

[Sub-PFE/Norm-link]

\[
\begin{aligned}
\min_{(x^{r,n}, \Psi^{r,n})} & Z_{\text{sub}}^{\text{link}} \\
= & \sum_{a \in A} t_a^{x^{r,n}} x_a^{x^{r,n}} + \frac{1}{\theta} \sum_{r \in R} \left\{ \sum_{j \in J} \sum_{a \in \text{IN}(j)} (x_a^{r,r,n}) \ln \left( \sum_{a \in \text{IN}(j)} x_a^{r,r,n} \right) - \sum_{a \in A} x_a^{r,r,n} \ln x_a^{x^{r,n}} \right\} \\
+ & \frac{1}{\theta} \sum_{a \in M} \psi_a^{x^{r,n}} \ln(\psi_a^{x^{r,n}} - 1) + \sum_{a \in M} \rho_a \psi_a^{x^{r,n}}
\end{aligned}
\]

(21)

Subject to

Equations (6)–(9) and (12)–(14).

The [Sub-PFE/Norm-link] is solved using the iterative balancing scheme described in Section 3.2.

**Step 4:** Conduct the convergence test.

If the difference between \((x^{r,n}, \Psi^{r,n})\) and \((x^{r,n-1}, \Psi^{r,n-1})\) is not sufficiently small, then set \(n = n + 1\), update link travel times as shown in Eq. (22) and return to Step 3. Otherwise, go to Step 5.

\[
\tilde{t}_a^{x^{r,n}} = t_a^{x^{r,n-1}} - u_a^{r,n-1} - d_a^{r,n-1}, \forall a \in A
\]

(22)

**Step 5:** Calculate residual flow to the next time period.

Calculate the residual flow \( g_{rs}^{r} \) using Eq. (19).

**Step 6:** Update time period.

If \( \tau = T \) (the end of the time period), then terminate. Otherwise, \( \tau = \tau + 1 \) and return to Step 2.

To avoid the complexity of notation, we omit the index representing time period, i.e. \( \tau \), hereafter in this chapter.

### 3.2. Iterative Balancing Scheme

The [Sub-PFE/Norm-link] is a non-linear program with linear inequality constraints, which can be solved by the iterative balancing scheme (Bell et al. 1997, Chen et al. 2009). With a fixed OD demand and link travel time, the iterative balancing scheme can be summarised as follows:

**Step 3-1:** Set up the sub-problem and initialise.
Set \( m = 0, u_a^m = d_a^m = 0, l_a^m = \eta_0 \) for all links, where \( \eta_0 \) is the threshold for termination (e.g. \( 10^{-6} \)). Note that \( m \) is the counter of the inner loop (iterative balancing scheme). Set \( m = m + 1 \).

**Step 3-2:** Compute flows.
Compute link flows, \( x^m \), by an absorbing Markov Chain approach as described in Section 3.3.
Compute flows on the virtual paths as follows:

\[
\psi_a^m = |v_a - x_a^m|, \forall a \in M
\]  

(23)

**Step 3-3:** Update dual variables.
Update the dual variables for each link as follows:

\[
d_a^m = \min\{0, d_a^{m-1} + \lambda_a\}, \forall a \in U
\]  

(24)

\[
l_a^m = \max\{0, l_a^{m-1} + \beta_a\}, \forall a \in M
\]  

(25)

\[
u_a^m = \min\{0, u_a^{m-1} + \pi_a\}, \forall a \in M
\]  

(26)

Note that \( \lambda_a, \beta_a \) and \( \pi_a \) are the adjustment factors, which are given as follows (Chen et al. 2009):

\[
\lambda_a = \frac{1}{\theta} \ln \left( \frac{v_a x_a^m}{\psi_a^m} \right), \forall a \in U
\]  

(27)

\[
\beta_a = \frac{1}{\theta} \ln \left( \frac{v_a}{\psi_a^m + \psi_a^m} \right), \forall a \in M
\]  

(28)

\[
\pi_a = \frac{1}{\theta} \ln \left( \frac{v_a + \sqrt{v_a^2 + 4x_a^m \psi_a^m}}{2x_a^m} \right), \forall a \in M
\]  

(29)

**Step 3-4:** Conduct the convergence test.
Set the maximum difference between the dual variables of current and previous steps as follows:

\[
\xi = \max_a \{|l_a^m - l_a^{m-1}|, |u_a^m - u_a^{m-1}|, |d_a^m - d_a^{m-1}|\}
\]  

(30)

If \( \eta_0 < \xi < \eta \), set \( m = m + 1 \) and return to Step 3-2. Otherwise, terminate, set \( l_a^n = l_a^m, u_a^n = u_a^m, \) and \( d_a^n = d_a^m \) and go to Step 4. Note that \( \eta \) is the upper limit of the difference allowed (e.g. \( 10^6 \)).

Note that the auxiliary flows given in Eq. (23) follow the definition of the absolute error shown in Eq. (1), although Chen et al. (2009) calculated the auxiliary virtual flows using the first optimal condition of [PFE/Norm-link], as shown in Eq. (17). The values of both definitions of the virtual flows at the equilibrium point should be theoretically identical. However, in our application to a larger size network, the virtual flows defined in Eq. (17) became too large to obtain a reasonable link flow if the dual variable \( l_a^m \) was more than some threshold or the dual variable \( u_a^m \) was less than some threshold. (Note that from the definition of the Lagrange function shown in Eq. (15), \( u_a^{m-1} \) should be a negative value.)

3.3. Traffic Assignment by an Absorbing Markov Chain Approach
To explicitly consider all possible paths without path enumeration, we applied the absorbing Markov chain approach as the assignment procedure. With regard to this approach, Sasaki (1965) demonstrated an assignment model based on the absorbing Markov chain. Although Sasaki (1965) assumed that the link transition probability is pre-determined, Akamatsu (1996) demonstrated that an absorbing Markov chain based assignment is equivalent to the Logit type assignment model under a certain link transition probability.

Suppose the number of nodes in the network is $n$, and the number of origins and destinations out of them are respectively $g$ and $a$. The transition matrix, $P$, which represents for the probabilities of vehicle movement in the Markov process, is assumed to be arranged as follows:

$$
P = \begin{bmatrix}
I & 0 \\
R & Q
\end{bmatrix}
\begin{bmatrix}
a \\
q - a \\
a & q - a
\end{bmatrix}
$$

where $R$ is a matrix of the transition probabilities from a node except for destinations to a destination node, $Q$ is a matrix of the transition probabilities between a node pair except for destinations, and $I$ is a unit matrix.

Furthermore, because no vehicle is absorbed in origin nodes, the matrix $Q$ is assumed to be arranged as follows:

$$
Q = \begin{bmatrix}
0 & Q_1 \\
0 & Q_2 \\
a & q - a
\end{bmatrix}
\begin{bmatrix}
g \\
q - g - a \\
a & q - a
\end{bmatrix}
$$

When the transition is repeated infinitely, all vehicles should be absorbed into the destination nodes, which means $\lim_{k \to \infty} Q^k = 0$. Therefore, the probability that vehicles traverse from intermediate nodes $i$ to $j$ can be calculated as follows:

$$
I + Q^1 + Q^2 + \cdots = [I - Q]^{-1} = \begin{bmatrix}
I & Q_1 [I - Q_2]^{-1} \\
0 & [I - Q_2]^{-1}
\end{bmatrix}
$$

Furthermore, if the element of $P$ is given as below, the Markov process is equivalent to the logit type assignment (Akamatsu 1996) as follows:

$$
p(j|i) = \exp(-\theta t_{ij}) \frac{V_{js}}{\sum_{s} V_{js}}
$$

where

$$
V_{is} \equiv \sum_{k=1}^{\infty} \exp(-\theta c_{is}^k) \quad \text{and}
$$

where $c_{is}^k$ is the cost of path $k$ from intermediate node $i$ to destination $s$ and $ij$ is a link leading out from node $i$ and leading in to node $j$. Although the above definition of $V$ requires path enumeration, Akamatsu (1996) further showed that if the matrix $W$ satisfies the Hawkins-Simon condition, $V$ can be obtained by following the below matrix operation:

$$
V = W + W^2 + W^3 + \cdots = [I - W]^{-1} - I
$$
where the element of $W$ is given as follows:

$$w_{ij} = \begin{cases} \exp(-\theta t_{ij}), & \forall \alpha \in A \\ 0, & \text{otherwise} \end{cases} \quad (37)$$

Because the path flows are given by link travel time and dual variables as shown in Eq. (16) in this study, the element of $W$ should be modified as follows (see Appendix A for detail):

$$w_{ij} = \begin{cases} \exp(-\theta(t_{ij} - u_{ij} - l_{ij})), & \forall (ij) \in M \\ \exp(-\theta(t_{ij} - d_{ij})), & \forall (ij) \in U \\ 0, & \text{otherwise} \end{cases} \quad (38)$$

4. NUMERICAL EXAMPLE

The proposed model was applied to two types of networks, a grid network and the Sioux Falls network, as shown in Figures 1 and 2, to demonstrate the model characteristics. The grid network, which is the same network as applied by Chen et al. (2009), consists of 9 nodes, 4 links, and 9 OD pairs. The shaded nodes in Figure 1 represent the centroids; nodes 1, 2, and 4 are origin nodes and nodes 6, 8, and 9 are destination nodes. The Sioux Falls network consists of 20 nodes and 76 links. Only the shaded nodes in Figure 2 are assumed to be the centroids in this study.

Figure 1. Grid network.
In this study, RMSEP (root mean squared error of percentage) is utilised to evaluate the accuracy of the estimates. The RMSEP is defined as follows:

\[
RMSEP = 100 \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left( \frac{x_i - x_i^*}{x_i^*} \right)^2}
\]  

(39)

where \(x_i\) and \(x_i^*\) respectively represent the estimate and true values. Note that the RMSEP is independent from the network size because it is normalised by the true value. The dispersion parameter \(\theta\) and the penalty parameter \(\rho_a\) in the PFE model are set as 1.50 and 11.27, respectively, for the estimation in this chapter.

4.1. Grid Network

4.1.1 Case setting
Three time periods are each considered to be 10 minutes in duration, and the residual flows from the previous time period are not considered at the beginning of time period 1. The length of each time period, which is assumed to be 10 minutes, may be too small considering the actual situation. However, the size of the grid network used in this chapter is also small, and free-flow travel time from 1 to 9, which is the farthest OD pair, is 6 minutes. The true link volumes are obtained by the logit-based SUE assignment model with dispersion parameter $\theta$ as 1.50. It is assumed that traffic volumes are available in only links 3, 5, 6, 7, 8, 10, 11, and 13 with traffic counts. Note that the setting conditions so far are equivalent with Chen et al. (2009). Then, following three scenarios are considered with different observed link flow patterns:

Tables 1 and 2 summarise the characteristics of the grid network as well as the assumed true OD volume (see also Chen et al. 2009). Three time periods are each considered to be 10 minutes in duration, and the residual flows from the previous time period are not considered at the beginning of time period 1. The length of each time period, which is assumed to be 10 minutes, may be too small considering the actual situation. However, the size of the grid network used in this chapter is also small, and free-flow travel time from 1 to 9, which is the farthest OD pair, is 6 minutes. The true link volumes are obtained by the logit-based SUE assignment model with dispersion parameter $\theta$ as 1.50. It is assumed that traffic volumes are available in only links 3, 5, 6, 7, 8, 10, 11, and 13 with traffic counts. Note that the setting conditions so far are equivalent with Chen et al. (2009). Then, following three scenarios are considered with different observed link flow patterns:

### Table 1. Link characteristics of grid network.

<table>
<thead>
<tr>
<th>Link</th>
<th>From</th>
<th>To</th>
<th>Capacity</th>
<th>Free-flow travel time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>280.0</td>
<td>2.00</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>290.0</td>
<td>1.50</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>5</td>
<td>280.0</td>
<td>2.00</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>4</td>
<td>280.0</td>
<td>1.00</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>5</td>
<td>600.0</td>
<td>1.00</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>6</td>
<td>300.0</td>
<td>2.00</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>5</td>
<td>500.0</td>
<td>2.00</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>7</td>
<td>400.0</td>
<td>1.00</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td>6</td>
<td>500.0</td>
<td>1.50</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>8</td>
<td>700.0</td>
<td>1.00</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>9</td>
<td>250.0</td>
<td>2.00</td>
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<tr>
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<td>6</td>
<td>9</td>
<td>300.0</td>
<td>1.00</td>
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<tr>
<td>13</td>
<td>7</td>
<td>8</td>
<td>350.0</td>
<td>1.00</td>
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<tr>
<td>14</td>
<td>8</td>
<td>9</td>
<td>220.0</td>
<td>1.00</td>
</tr>
</tbody>
</table>

### Table 2. Assumed true OD volume of grid network.

<table>
<thead>
<tr>
<th>From/To</th>
<th>6</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>120</td>
<td>150</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>130</td>
<td>200</td>
<td>90</td>
</tr>
<tr>
<td>4</td>
<td>80</td>
<td>180</td>
<td>110</td>
</tr>
</tbody>
</table>

### Table 3. True link volumes and observed link volumes for each case.

<table>
<thead>
<tr>
<th>Link</th>
<th>True value</th>
<th>Scenario 1</th>
<th>Scenario 2</th>
<th>Scenario 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>TP1</td>
<td>TP2</td>
<td>TP3</td>
<td>TP1</td>
</tr>
<tr>
<td>1</td>
<td>124</td>
<td>150</td>
<td>150</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>137</td>
<td>162</td>
<td>162</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>109</td>
<td>136</td>
<td>137</td>
<td>109</td>
</tr>
<tr>
<td>4</td>
<td>77</td>
<td>95</td>
<td>95</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>467</td>
<td>526</td>
<td>527</td>
<td>467</td>
</tr>
<tr>
<td>6</td>
<td>77</td>
<td>95</td>
<td>95</td>
<td>77</td>
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<tr>
<td>7</td>
<td>212</td>
<td>254</td>
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<td>212</td>
</tr>
<tr>
<td>8</td>
<td>295</td>
<td>328</td>
<td>329</td>
<td>-</td>
</tr>
<tr>
<td>9</td>
<td>303</td>
<td>354</td>
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<td>303</td>
</tr>
<tr>
<td>10</td>
<td>400</td>
<td>457</td>
<td>459</td>
<td>400</td>
</tr>
<tr>
<td>11</td>
<td>85</td>
<td>105</td>
<td>106</td>
<td>85</td>
</tr>
<tr>
<td>12</td>
<td>50</td>
<td>61</td>
<td>61</td>
<td>-</td>
</tr>
<tr>
<td>13</td>
<td>295</td>
<td>328</td>
<td>329</td>
<td>295</td>
</tr>
<tr>
<td>14</td>
<td>165</td>
<td>189</td>
<td>189</td>
<td>-</td>
</tr>
</tbody>
</table>

Tables 1 and 2 summarise the characteristics of the grid network as well as the assumed true OD volume (see also Chen et al. 2009). Three time periods are each considered to be 10 minutes in duration, and the residual flows from the previous time period are not considered at the beginning of time period 1. The length of each time period, which is assumed to be 10 minutes, may be too small considering the actual situation. However, the size of the grid network used in this chapter is also small, and free-flow travel time from 1 to 9, which is the farthest OD pair, is 6 minutes. The true link volumes are obtained by the logit-based SUE assignment model with dispersion parameter $\theta$ as 1.50. It is assumed that traffic volumes are available in only links 3, 5, 6, 7, 8, 10, 11, and 13 with traffic counts. Note that the setting conditions so far are equivalent with Chen et al. (2009). Then, following three scenarios are considered with different observed link flow patterns:
Scenario 1: Link flows are observed without any errors for all of the time periods.
Scenario 2: Link flows are observed without any errors at time period 1, and the same values as time period 1 are observed at time periods 2 and 3.
Scenario 3: Link flows are observed with errors at time period 1, and the same values as time period 1 are observed at time periods 2 and 3.

Note that observed link flows at time period 1 in scenario 3 are generated by independent Poisson variates with mean and variances equal to the true link volumes (see Chen et al. 2009). Table 3 summarises the observed link volumes in each scenario as well as the true link volumes.

4.1.2 Discussion
Figure 3 compares true and estimated link volumes as well as the RMSEP values for each scenario and time period. In scenario 1, where the link flows are observed without any errors

Figure 3. Comparison between true and estimated link volumes with RMSEP values.
for all of the time periods, the links with traffic counts can be estimated with high accuracy for all time periods. The links without traffic counts can also be estimated with high accuracy in general for all of the time periods, although the RMSEP values of links without traffic counts are larger than that of links with traffic counts. In scenario 2, where the link flows are observed without any errors at time period 1 but the same values as time period 1 are observed at time periods 2 and 3, both of the links with and without traffic counts can be estimated with the same as those in scenario 1 at time period 1. However, we can confirm that the volumes of links with traffic counts tend to be underestimated at both time periods 2 and 3; the observed link flows are smaller than true ones as shown in Table 3, because residual flows from the previous time period are assumed not to be included in the observation data. Therefore, the RMSEP values of links with traffic counts in scenario 2 are larger than those in scenario 1 at both time periods 2 and 3. This finding suggests that if the length of the time period in the PFE model does not correspond to the length of time period of traffic counts data, the estimation accuracy may be reduced. In scenario 3, where the link flows are observed with errors at time period 1 and the same value as time period 1 are observed at time periods 2 and 3, although the RMSEP values of links with traffic counts at time period 1 become worse than that in scenario 1 due to the measurement errors of link flows, the estimated link volumes are still close to the actual link volumes. Furthermore, similar to the description of the comparison of the estimated link volumes between scenarios 1 and 2, the volumes of links with traffic counts tend to be underestimated at both time periods 2 and 3.

Thus far, only the estimation accuracy of link flows has been discussed. Here, we evaluate the estimation results of OD volumes. Figure 4 shows the comparison between true and estimated OD volumes for each scenario and time period. Table 4 lists RMSEP values and
the coefficient of determination of each scenario and time period. Figure 4 shows that the estimation accuracy of OD volumes is lower than that of link volumes for all the scenarios. Therefore, as shown in Table 4, RMSEP values of OD volumes are in general higher than those of link volumes at all scenarios and time periods because the proposed model uses only the volume data at (several) observed links and does not use any other data, such as prior information of OD trip pattern and the total volume. It should be mentioned that this is a common tendency with other PFE models using only observed link volumes data (e.g. Bell et. al. 1997, Chen et al. 2009). There is room to improve our proposed model by using not only the observed link volume data, but also other data, to improve the estimation accuracy of OD volumes as well as that of link volumes.

Furthermore, Table 4 shows that RMSEP values in scenario 1 are not the smallest among all of the scenarios. This result is contrary to the estimation result of link volumes. A possible reason for this is that the volumes of OD pairs that are overestimated in scenario 1 are estimated with a lower value in scenarios 2 and 3 in time period 1, because the residual flows in the observed link volumes are not considered, which consequently lead the estimated OD volumes to closer to the true values. However, Table 5 shows that although the estimated total OD volumes do not change greatly along all time periods in scenario 1, the estimated total OD volumes are clearly reduced at time periods 2 and 3 in both scenarios 2 and 3. Therefore, we can confirm that the total OD volumes might be underestimated if the observation interval of the traffic counts is longer than the length of the time period.

4.1.3 Comparison of estimated OD and link volumes with the existing model

The observed link volume used in scenario 3 is identical to that used by Chen. Furthermore, it is assumed that the residual flow is zero at the beginning of period 1. Therefore, it is possible to compare the estimated results in period 1 of scenario 3 with the result of Chen et. al. (2009).

Figure 5 compares the estimated OD and link volumes between the proposed model and the model of Chen et al. (2009). Note that the RMSEP values for OD volume of the proposed model and the model of Chen et. al. (2009) respectively are 31.77 and 50.84, and the RMSEP values for link volume of the proposed model and the model of Chen et. al. (2009) respectively are 17.34 and 6.14. Though the RMSEP value for link volume of the proposed model is a little larger than that of the model of Chen et. al. (2009), the estimation accuracy for both OD volumes and link volumes of the two models is comparable.
4.2. Sioux Falls Network

This section investigates the behaviour of the proposed model in the Sioux Falls network (Transportation Networks for Research Core Team) as shown in Figure 2, which is often utilised as one of the benchmarks of a transportation network problem. The characteristics of the Sioux Falls network are published online (Transportation Networks for Research Core Team). Three time periods are 60 minutes each, and the residual flows from the previous time period are not considered at the beginning of time period 1. The true OD demands in time periods 1 and 3 are assumed to be identical to the published one, and those in time period 2 are assumed to be twice those in time periods 1 and 3. Then, the true link volumes are obtained by the logit-based SUE assignment model with dispersion parameter $\theta$ as 1.50.

Figure 6 compares true and estimated link volumes when true volumes are observed in all links. This is because the estimated link volume at point A in Figure 6 is far from the true link volume. (Note that from the definition in Eq. (39) the RMSEP values tend to become large if the estimation error is large with a link of small true value.) Therefore, the proposed PFE model can reproduce link volumes with high accuracy when the true volumes are observed in all links.

However, not all link volumes are observed in the real world, and there may be observation errors in the traffic counts. Figure 7 compares the RMSEP values with different coverage ratios of traffic counts and observation errors. The coverage ratios of traffic counts range from 0.5 to 0.7, and the average observation errors range from 0.00 to 0.05. For each combination of the coverage ratio and the average observation error, the observation links are
randomly selected and the observation link volumes are assumed as the following:

$$x_{i}^{obs} = x_{i}^* + E \cdot r$$

(40)

where $x_{i}^*$ and $x_{i}^{obs}$ respectively are the true and observed link volumes, $E$ is the average observation error, and $r$ is a random variable following normal distribution. To avoid the effect of the random variable, link volumes are estimated five times with different random variables for each combination of the coverage ratio and the average observation error. Figure 7 shows that the estimation accuracy of link volumes is improved as the coverage ratio of traffic counts increases with the same observation errors. Furthermore, if the observation ratio of traffic counts reaches 90%, the estimation accuracy of the link volumes is almost equal to the case when the true volumes are observed in all links. On the other hand, estimation accuracy with and without observation error is close for the same coverage ratio of traffic counts. Therefore, observation error does not affect the estimation accuracy significantly in this case study.

5. CONCLUSION

This paper proposed a path flow estimator that does not require enumeration of paths set. The proposed model took traffic assignments derived by the absorbing Markov chain approach and incorporated them into the nonlinear PFE model considering the inconsistency of traffic counts proposed by Chen et al. (2009). The proposed model was formulated based on links. We further expanded the model semi-dynamically, and described the solution algorithm of the proposed model. Then, the proposed model was applied to two types of toy networks. As a result, we confirmed that the proposed model can estimate the link volume with high accuracy if the observed link volumes are error-free. We further compared the relationship between the observation error and the coverage ratio of the traffic counter and confirmed that the coverage ratio of the traffic counter has more influence on the estimation accuracy than the observation error. Therefore, when the observation error ratio of the traffic counter is not high, the estimation accuracy can be improved by collecting a large number of link volume data even if the observation error is included. We further confirmed that the total OD volumes as well as link volumes are underestimated if the observation interval of the traffic counts is longer than the length of the time period.

Although we fixed the dispersion parameter $\theta$ as 1.50 in the numerical example, this value will also affect the estimation result. Therefore, there is a room to examine the sensitivity of the dispersion parameter. Because we need to calculate the inverse matrix whose elements are Eq. (38), the computability needs to be considered when examining the sensitivity of $\theta$ in a large size network. Furthermore, as the proposed model, as well as most of the PFE model, gives the dispersion parameter $\theta$ exogenously, there is a room to expand the model to estimate the dispersion parameter endogenously. In addition, we confirmed that the estimation accuracy of the OD volumes is not high. One of the reasons for this is that prior information is not utilised as input data in the proposed model. Therefore, there is a room to expand the model in future work to utilise prior information for the estimation.

ACKNOWLEDGEMENTS

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estimation in the Sioux Falls network.

REFERENCE


**Appendix A**

In this appendix, we show that the element of \( W \) in Eq. (38) gives link flows that approach the observed link flows via given dual variables. Let us introduce the following link cost:
\[ \tilde{c}_a = \begin{cases} 
(t_a - u_a - l_a) , & \forall a \in M \\
(t_a - d_a) , & \forall a \in U \\
0 , & \text{otherwise} 
\end{cases} \]  
(A-1)

Then, because
\[ \sum_{a \in A} t_a \delta_{ka}^{rs} - \sum_{a \in M} (u_a + l_a) \delta_{ka}^{rs} - \sum_{a \in U} d_a \delta_{ka}^{rs} = \sum_{a \in M} (t_a - u_a - l_a) \delta_{ka}^{rs} + \sum_{a \in U} (t_a - d_a) \delta_{ka}^{rs} \equiv \sum_{a \in A} \tilde{c}_a \delta_{ka}^{rs}, \]  
the physical path flows shown in Eq. (16) can be simplified as follows:
\[ f_{rs,k} = \frac{\exp(-\vartheta \tilde{c}_k)}{\sum_{k \in K_{rs}} \exp(-\vartheta \tilde{c}_k)} q_{rs}, \forall rs \in \Omega, \forall k \in K_{rs}, \]  
(A-2)

where \( \tilde{c}_k = \sum_{a \in A} \tilde{c}_a \delta_{ka}^{rs} \). Note that the physical path flow shown in (A-2) is consistent with the logit type assignment model with the link cost \( \tilde{c}_a \). Furthermore, because the element of \( W \) defined by Eq. (38) can be represented as
\[ w_{ij} = \exp(-\vartheta \tilde{c}_{ij}), \forall (ij) \in A, \]  
the \( n \)-th power of \( w_{ij} \), \( w_{ij}^n \), is given as
\[ w_{ij}^n = \sum_{k \in K_{ij}} \exp[-\vartheta \tilde{c}_k], \]  
similar to the discussion above. Therefore, the element of \( W \) defined by Eq. (38) is also consistent with the definition provided by Akamatsu (1996).