COMPETITIVE EQUILIBRIUM AND FIXED-POINT THEOREM II

WALRAS' EXISTENCE THEOREM AND
BROUWER'S FIXED-POINT THEOREM*

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THE PURPOSE of this note is to show the equivalence of two fundamental theorems—Walras' Existence Theorem on the one hand and Brouwer's Fixed-Point Theorem on the other.

Walras' theorem*3 is concerned with the existence of an equilibrium in the Walrasian system of general equilibrium and has been a problem of some importance in formal economic analysis since his work [12] appeared in 1874–7. It was, however, not until Wald's contributions, [10] and [11], that the existence problem was rigorously treated. Recent contributions, in particular those of Arrow and Debreu [2], McKenzie [6], Nikaidô [7], and Gale [4], have shown that Walras' theorem is essentially a necessary consequence of Brouwer's Fixed-Point Theorem. The latter theorem,² first proved by Brouwer [3] in 1911, also bears a fundamental importance in mathematics. It may be hence of some interest to see that Brouwer's theorem is in fact implied by Walras' theorem. It would indicate the reason that the general treatment of the existence problem in the Walrasian system had to wait for the development of the twentieth century mathematics.

1. According to Gale [4] and Nikaidô [7], Walras' theorem may be formulated as follows.

Let there be \( n \) commodities, labeled \( 1, \ldots, n \), \( p = (p_1, \ldots, p_n) \) and \( x = (x_1, \ldots, x_n) \) be a price vector and a commodity bundle, respectively.

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1) The reader is referred to Schumpeter [8], Part IV, especially Chapter 7, for an appraisal of Walras' theory of general equilibrium.

2) See, for example, Alexandroff and Hopf [1], pp. 376–8 and p. 480, and Lefschetz [5], p. 117.
Price vectors are assumed to be nonzero and nonnegative; commodity bundles are arbitrary \(n\)-vectors. Let \(P\) and \(X\) be the sets of all price vectors and of all commodity bundles:

\[ P = \{ p = (p_1, \ldots, p_n) : p_i \geq 0, i=1, \ldots, n, \text{ but } p \neq 0 \} \]

\[ X = \{ x = (x_1, \ldots, x_n) \} . \]

The excess demand function \(x(p)=[x_1(p), \ldots, x_n(p)]\) is a mapping from \(P\) into \(X\).

A price vector \(\bar{p}\) is called an equilibrium if

\[ x_i(\bar{p}) \leq 0, \quad (i=1, \ldots, n) \]

with equality unless \(\bar{p}_i = 0, (i=1, \ldots, n)\).

Walras' Existence Theorem. Let an excess demand function \(x(p)\) satisfy the following conditions:

(A) \(x(p)\) is a continuous mapping from \(P\) into \(X\).

(B) \(x(p)\) is homogeneous of order 0; i.e., \(x(tp) = x(p)\), for all \(t > 0\) and \(p \in P\).

(C) Walras' law holds:

\[ \sum_{i=1}^{n} p_i x_i(p) = 0, \text{ for all } p \in P. \]

Then there exists at least an equilibrium price vector \(\bar{p}\) for \(x(p)\).

2. Brouwer's theorem, on the other hand, is concerned with a continuous mapping on the simplex.

The fundamental \((n-1)\)-simplex \(\Pi\) is the set of all nonnegative \(n\)-vectors whose component sums are one:

\[ \Pi = \{ \pi = (\pi_1, \ldots, \pi_n) : \pi \geq 0, \sum_{i=1}^{n} \pi_i = 1 \} \]

Brouwer's Fixed-Point Theorem. Let \(\varphi(\pi)\) be a continuous mapping from \(\Pi\) into itself. Then there is at least a fixed-point \(\bar{\pi}\) in \(\Pi\):

\[ \bar{\pi} = \varphi(\bar{\pi}). \]

3. Equivalence Theorem. Walras' Existence Theorem and Brouwer's Fixed-Point Theorem are equivalent.

Proof: It is well established\(^3\) that Brouwer's theorem implies Walras'

\(^3\) See Nikaidô [7], Gale [4], and Uzawa [9], Appendix.
theorem of the form above. We shall therefore prove that Walras' theorem implies Brouwer's theorem.

Let \( \varphi(\pi) \) be any continuous mapping from \( \Pi \) into itself. We construct an excess demand function \( x(p) = [x_1(p), \ldots, x_n(p)] \) by

\[
x_i(p) = \varphi_i \left( \frac{p}{\lambda(p)} \right) - p_i \mu(p), \quad (i=1, \ldots, n, p \in P),
\]

where

\[
\lambda(p) = \sum_{i=1}^{n} p_i
\]
\[
\mu(p) = \frac{\sum_{i=1}^{n} p_i \varphi_i \left( \frac{p}{\lambda(p)} \right)}{\sum_{i=1}^{n} p_i^2}
\]

It may be noted that \( \varphi_i(p/\lambda(p)) \) and \( p_i \mu(p) \) are both positively homogeneous of order 0.

It is evident that the excess demand function thus defined satisfies conditions (A), (B), and (C). Hence, applying Walras' theorem, there is an equilibrium \( \bar{p} \). Then, by (1), we have

\[
\varphi_i \left( \frac{\bar{p}}{\lambda(\bar{p})} \right) \leq \bar{p}_i \mu(\bar{p}), \quad (i=1, \ldots, n)
\]

with equality unless \( \bar{p}_i = 0 \).

Letting

\[
\pi = \frac{\bar{p}}{\lambda(\bar{p})}, \quad \beta = \lambda(\bar{p}) \mu(\bar{p})
\]

the relation (2) may be written

\[
\varphi_i (\pi) \leq \beta \pi_i,
\]

with equality unless \( \pi_i = 0 \).

Summing (3) over \( i=1, \ldots, n \), and noticing that \( \pi, \varphi(\pi) \epsilon \Pi \), we have \( \beta = 1 \); hence,

\[
\varphi_i (\pi) \leq \pi_i
\]

with equality unless \( \pi_i = 0 \).

The relation (4), again together with \( \pi, \varphi(\pi) \epsilon \Pi \), implies that

\[
\varphi_i (\pi) = \pi_i, \quad (i=1, \ldots, n);
\]

i.e., \( \pi \) is a fixed-point for the mapping \( \varphi(\pi) \). Q.E.D.
REFERENCES