ON THE OPTIMAL LEVEL OF RISKY FOREIGN INVESTMENTS*

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Introduction

One of the prominent characteristics of foreign investments is the fact that they involve a great deal of risks and uncertainties. Due to the changes in international political situations, or to the structural changes in the economic system of the country where foreign capital is invested, foreign investments suffer great possibility of default or abrupt changes in their returns.

Needless to say, domestic investments cannot avoid all elements of default or changes in returns. However, to a nation as a whole, the default of an obligation does not necessarily mean the loss of real capital in the economy. Capital loss on one side of the parties is usually accompanied with the decrease of the debt on the other. More or less risks offset each other in a domestic economy. On the other hand, for foreign investments default or capital loss is an actual loss to the investing country as a whole.1)

The purpose of this paper is to examine the optimal level of foreign investments with risks. If one takes account of the fact that foreign investments are being made in the growing world economy, they should be treated in the framework of growth theory.2) Using a von Neumann-Morgenstern type social utility function, this paper focuses attention to the balanced growth path that maximize the expected utility of consumption streams.

After introducing a growth model with foreign investments in Section II, Section III deals with the optimal level of foreign investments given the total saving ratio of an economy. Section IV analyzes the optimal level of foreign investments if a nation can choose the saving ratio by fiscal policy or other measures. This is in a sense a generalization of the so-called Golden Rule3) of capital accumulation to the case where risky foreign investments opportunity is available to a nation.

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Model and Assumptions

Suppose a country where labor is growing exponentially at a constant rate \( n \). The production possibility of the domestic economy is given by a neo-classical production function with constant returns to scale. Let \( P, K \) and \( L \) be domestic product, domestic capital and labor respectively. The production function can be written,

\[
P = F(K, L) = LF(K/L, 1) = Lf(k)
\]

where \( k = K/L \). Assume in addition that the per-capita production function has the following properties:

\[
f'(k) > 0, \quad f''(k) < 0, \quad \lim_{k \to \infty} f'(k) = 0, \quad \lim_{k \to 0} f'(k) = +\infty.
\]

Labor is growing at the rate \( n \),

\[
\dot{L}/L = n.
\]

Capital is assumed to depreciate at the rate \( \delta \).

This economy is located in a world economy, and possesses foreign investment opportunity. Let \( Z \) be the amount of capital invested in foreign countries. Capital invested abroad is also assumed to depreciate at the rate \( \delta \). The return on foreign investments is assumed not to be predetermined but to be stochastic. Here we shall assume the return \( r \) is distributed with expected value \( \mu \) and variance \( \sigma^2 \). On the other hand, the return on domestic capital is assumed to be riskless and equal to its marginal productivity \( f'(k) \). This assumption of riskless domestic investment may not reflect the reality perfectly. However, since most of capital loss is offset by a gain within a country as mentioned in the introduction, we believe that this serves as a first approximation. That is,

\[
E(r) = \mu, \quad \text{Var.}(r) = E[(r - \mu)^2] = \sigma^2.
\]

Suppose the social utility to be maximized is given by the following quadratic function of per-capita consumption \( c \):

\[
U(c) = \alpha c + \beta c^2, \quad \alpha > 0, \quad \beta < 0.
\]

This utility is of von Neumann-Morgenstern type, invariant up to linear transformation. The negative value of \( \beta \) reflects the diminishing marginal utility. To keep the marginal utility of per-capita consumption positive, we shall restrict our discussion to the following range of \( c \):

\[
\alpha + 2\beta c > 0
\]

or

\[
c < -\frac{\alpha}{2\beta}.
\]

What we are interested in is the level of international investments to make the expected utility maximum. In this paper, we shall be concerned only with the comparison of balanced growth paths. By balanced growth path we mean here a growth path where per-capita domestic capital and per-capita foreign investment is kept constant. Of course, the analysis of an unbalanced growth path is as much important.
as that of a balanced path. However, the introduction of uncertainty or risk makes it difficult to analyze the unbalanced growth path. We hope the analysis of balanced growth path in this paper may give some insight into problems of international investment in general.

The Case where Saving Ratio is Constant

Let us consider first the case where national saving ratio is constant. Here the saving ratio is determined by consumer behavior or other mechanisms and not a policy variable. We are interested in the level of foreign investment that maximizes the expected social utility.

Before considering the general case of risky international investment, let us review the case where international investment does not involve any risk. Then $\psi$ always equals $\mu$, and $z$ becomes a nonstochastic variable. Accordingly the maximization of expected utility of per-capita consumption reduces to the maximization of per-capita consumption. The problem is the following maximization problem:

Maximize $(1-s)(f(k)+\mu z)$ with respect to $k$ and $z$
subject to $f(k+z)=s(f(k)+\mu z), 0<s<1$

where $s$ is the overall saving ratio, $z=Z/L$ is per-capita foreign investment, and $\lambda=1+s$.

By a simple calculation, the reader will find the optimal value of $k$ and $z$ can be given as the solution to the following simultaneous equations:

1. $f(k+z)=s(f(k)+\mu z)$
2. $f'(k)=\mu$.

In order that the second order condition for the maximum be satisfied, it is easy to see that

$$\lambda-\mu\mu > 0.$$ 

If $\lambda-\mu\mu < 0$, the economy can maintain the amount of foreign investment as much as it likes. Therefore the maximization problem is not bounded.

Figure 1 illustrates the determination of the optimal level of $k$ and $z$ when $s$ is given and condition (3) is satisfied. $SS$ is the curve of possible combination of $k$ and $z$ by (1) if $s$ is given. That is, $SS$ is the combination of per-capita domestic capital and foreign investment that can maintain itself over time. $TT$ is the optimal condi-
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tion (2). Define \( k^* \) and \( k^{**} \) by the following relationships:

\[
sf(k^*) = \lambda k^*
\]
\[
f'(k^{**}) = \mu.
\]

Then SS intersects with the horizontal axis at \( k^* \). \( k^* \) is the value of per-capita capital of the balanced growth equilibrium in a closed economy with saving ratio \( s \), labor growth rate \( n \) and depreciation rate \( \delta \). Since we exclude the possibility of borrowing from abroad in this paper, \( z \) has to be positive. Therefore when \( \lambda - s\mu > 0 \) and \( k^* > k^{**} \), the problem has an internal solution, namely the solution given by point P. Foreign investments should proceed so much as to equate the domestic marginal productivity of capital to the rate of return on foreign investments. If \( k^* < k^{**} \), the optimal solution becomes a corner solution with \( z = 0 \), and \( k = k^* \). When the rate of return on foreign investments \( \mu \) is so low that \( k^* < k^{**} \), it is not profitable for a nation to invest abroad.

Here let us introduce risks to foreign investments. As explained in Section II \( r \) is assumed to be a random variable, distributed with mean \( \mu \) and variance \( \sigma^2 \). Since we have a social utility function with decreasing marginal utility, or that of a risk-averted, all we have to consider is the case where \( k^* > k^{**} \). For, when \( k^* < k^{**} \), there is no incentive for the country to invest abroad, even if foreign investments are riskless. Why should it care to invest abroad if the mean value of the return from foreign investments remains the same but they involve some risks?

For the same reason as in the case of certainty, we shall also assume \( \lambda - s\mu > 0 \).

Since income itself is a random variable due to the stochastic nature of returns to foreign investment,

\[
s(F(K, L) + rZ)
\]

is not constant. Therefore in order to have a stationary solution, we shall assume

\[
s(F(K, L) + \mu Z), \quad 0 < s < 1
\]

is saved.\(^5\)

The stationary equilibrium can be given by

\[
sf(\hat{k} + \mu Z) = \lambda(\hat{k} + z), \quad 0 < s < 1.
\]

Per-capita consumption, which is also a random variable, can be given by

\[
c = f(\hat{k}) + rz - \lambda(\hat{k} + z).
\]

Since

\[
E(c) = f(\hat{k}) + \mu z - \lambda(\hat{k} + z)
\]

and

\[
\text{Var. } (c) = \sigma^2 z^2,
\]

expected utility is

\[
E(U(c)) = \alpha E(c) + \beta (\sigma^2 z^2 + (E(c))^2).
\]


\(^5\) This is not necessarily consistent with the permanent income hypothesis. Instead of consumption, saving is related to the permanent component of income here. This procedure, however, can be justified from the fact that the motive of saving from international investment income would normally be for accumulation for its own sake rather than for consumption.
In order to maximize $E(U(c))$ subject to constraint (4), we introduce a Lagrangean expression,

$$
\theta = \alpha (f(k) + \mu - \lambda (k + x)) + \beta [\sigma^2 + \gamma (f(k) + \mu - \lambda (k + x))^2] + \delta (s(f(k) + \mu) - \lambda (k + x)),
$$

where $\theta$ is a Lagrangean multiplier.

The optimal condition for internal maximum is, writing $E(c)$ as $\xi$, and $f'(k)$ as $f'$,

$$
\begin{align*}
\phi_1 &= \alpha (f' - \lambda) + 2\beta \delta (f' - \lambda) + \xi (f' - \lambda) - 0 \\
\phi_2 &= \alpha (\mu - \lambda) + 2\beta \delta (\mu) + \xi (\mu - \lambda) - 0.
\end{align*}
$$

Eliminating $\theta$,

$$
\frac{\mu - \lambda}{\xi f' - \lambda} = \frac{(\mu - \lambda)(\alpha + 2\beta \delta) + 2\beta \delta^2}{(f' - \lambda)(\alpha + 2\beta \delta)}.
$$

Solving for $x$,

$$
(5)
$$

If (5) and (6) jointly give positive solutions of $k$ and $x$, they are the optimal stationary value of per-capita capital and foreign investments. Figure 2 illustrates how these values of $k$ and $z$ are determined. As in Figure 1, $SS$ shows relation (5). $TT$ plots the optimal condition (6) under the assumption $f'-\lambda > 0$ and $k^* > k^{**}$. $TT$ intersects with the horizontal axis where $\mu - f'(k)$, namely at $k^{**}$, and it is asymptotic to the vertical line $k = k^{***}$ where $k^{***}$ is the value of $k$ such that $sf'(k^{***}) = \lambda$. Incidentally, $k^{***}$ corresponds to the maximal value of $x$ along $SS$. It is also clear that $k^{**} > k^{***}$ if $\lambda - s\mu > 0$.

Figure 2 shows that under the assumption $\lambda - s\mu > 0$ and $k^{**} < k^*$, there exist $k$ and $x$ which satisfy both equations (5) and (6). The optimal value of $k$, $k^*$, is given as

$$
k^{**} < k^*.
$$

That is, capital deepening in domestic economy proceeds more than the case of certainty where $k^* - k^{**}$.

**The Case where Saving Ratio can be chosen**

So far has it been assumed that the saving ratio is given to the economy as a

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6) Since $f'(k^{***}) = \lambda/s > \mu = f'(k^{**})$ and $f'(k) < 0$, $k^{***}$ is smaller than $k^{**}$.

7) Naturally, the optimal value of $x$ and $s/k$ are smaller than those of the certainty case.
datum. The adjustment for the optimum is made mainly by the amount of foreign investments. However, by fiscal policy or other measures, a nation may be able to choose the desirable combination of saving and consumption. In this section we shall assume that the decision of choice between saving and consumption can be made by a central authority. Suppose an economy can choose per-capita capital and per-capita foreign investment. Here also we are interested in the comparison of balanced growth paths, where per-capita capital and per-capita foreign investment is kept constant. Our question is what is the optimal values of $k$ and $z$ for maintaining the maximal level of expected utility.\(^8\)

Expected social utility can be written

$$E(U(c)) = \alpha (f(k) + \mu z - \lambda (k + z)) + \beta \left[ e^\alpha + (f(k) + \mu z - \lambda (k + z))^\alpha \right].$$

We have to choose the values of $k$ and $z$ that maximize $E(U(c))$. For internal maximum, differentiating by $k$ and $z$,

\begin{align*}
\alpha (f' - \lambda) + 2\beta \alpha (f' - \lambda) &= 0, \\
\alpha (\mu - \lambda) + 2\beta \alpha (\mu - \lambda) + 2\beta \gamma z &= 0.
\end{align*}

From (7)

$$f' = \lambda.$$

From (8)

$$(\alpha + 2\beta \gamma) (\mu - \lambda) = -2\beta \gamma z$$

or

$$z = \frac{\alpha + 2\beta \gamma}{(-2\beta \gamma)} \frac{\mu - \lambda}{\sigma^2}.$$

Because $(\alpha + 2\beta \gamma)/(-2\beta \gamma)$ is positive by assumptions, there is internal optimum with positive $k$ and $z$ so long as $\mu > \lambda$. If $\mu < \lambda$, $z = 0$ and $f' = \lambda$ is the optimum solution.\(^9\)

Equation (9) shows that the Golden Rule of accumulation is also valid for an open economy which allows the possibility of investing abroad. (10) means that if the average return on foreign investments is larger than the sum of labor growth rate and depreciation rate, it is profitable to invest abroad. The level of optimum investment is related inversely to the variance of the return to foreign investments.

The amount of per-capita consumption is

$$c = f(k) + \mu z - \lambda (k + z),$$

and its expected value can be written using (9),

$$\bar{c} = f(k) + \mu z - \lambda (k + z)$$

$$= f(k) - f'(k) k + (\mu - \lambda) z$$

$$\geq f(k) - f'(k) k.$$

This is greater than the earning of wage earners which is the maximal level of balanced growth consumption for a closed economy.\(^10\) Thanks to the opportunity for

\(^8\) If there is no risk for $\sigma$, and $\sigma$ is constant, the problem becomes unbounded so long as $\mu > \lambda$. Cf. Hamada, op. cit., Section II.

\(^9\) Negative solution of $z$ would be meaningful, if borrowing were allowed and interest payments for international debt had the same distribution as the return from abroad. But this assumption would be somewhat artificial.

\(^10\) See, Phelps, op. cit.
foreign investments, the expected value of the sustained consumption level is higher by the amount \((\mu - \lambda z)\).\(^{11}\) The larger the risks, the smaller the value of optimal \(z\) and the smaller the level of expected consumption. The less dangerous foreign investment opportunity becomes, the more can a country enjoy the fruit of foreign investments.

To conclude this short analysis, let us consider how the rational behavior of individual investors in a competitive market can divert from the level of foreign investments desirable to a nation as a whole. Suppose an investor tries to allocate his capital between domestic and foreign investment. If he is risk-neutral, he will choose the investment with higher return. But if he is a risk-avertor, he will choose a mix of domestic and foreign investment to maximise the expected utility.

Since the law of the large number does not work for an individual investor, risks for foreign investment is much higher, objectively as well as subjectively, than those for the national economy as a whole. Thus individual rational behavior does not necessarily achieve the optimal foreign investment for a nation. For an individual investor, of course, not only foreign but also domestic investment involves some risk. However, in some case where the individual risk of domestic investment is very small but that of foreign investment is much higher than the risk for a nation, rational behavior of individual investors might lead to smaller foreign investments than the national optimum. The argument that foreign investments tend to be excessive, from the standpoint of default and risks,\(^{12}\) could allow some exception if individual investors are also risk-averters. In such a case the reduction of individual risks by foreign investment insurance will work towards improving the social welfare of a country.\(^{13}\)

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11) If there is no risk, optimal \(z\) is not finite. But the existence of the risk checks the unlimited expansion of foreign investment.

12) Kemp, op. cit.