THE NEO-CLASSICAL THEOREM
AND THE TWO-SECTOR MODEL OF ECONOMIC GROWTH

BY MASAO FUKUOKA AND KUNIO KAWAMATA

Introduction

Recent issues of the Review of Economic Studies have produced a sizable literature on a two-sector model of economic growth, the main incentive for which is in finding sufficient conditions to ensure the relative stability of a "Golden-age" growing equilibrium. At the same time, the special issue of the Review on Production Functions and Economic Growth contains a number of papers concerning the "Neo-classical Theorem," which states the conditions for the highest consumption path obtainable in a system growing at its "natural" rate. Since the latter works have mostly been posed in terms of the aggregate production function, it may be of some interest to study the theorem in a somewhat broader setting of a two-sectoral economy, where investment-goods and consumption-goods are produced by different production functions. Thus the purpose of the present paper is to investigate the validity of the theorem in the context of the two-sectoral growth model as has been developed in the literature earlier mentioned. A similar attempt was undertaken by Mordecai Kurz, but his analysis was confined exclusively to the case of Cobb-Douglas production functions. We shall treat the problem in a more general model as formulated by Uzawa and many others including Shirai, Takayama, Drandakis, Amano, Inada and Kawamata.4

The next section will describe briefly the structure of the basic model just as the necessary preliminaries for the following analysis. The main results will be given in the third section, where it will be shown that all propositions of the Neo-classical Theorem remain invariant for the two-sector model under consideration. To this we shall append the final section which provides a few remarks on the case of Kaldorian saving functions.

Preliminaries

The basic model, which is now well-known, may be summarized as follows. There are two-sectors in the economy; sector 1 produces investment-goods and sector 2

1) Uzawa [19], [20], Solow [15], Takayama [18], Drandakis [4], Inada [5], [6], and Amano [1].
2) Robinson [12], Meade [10], Champernowne [3], and Black [2]. As for earlier formulations, see Swan [17] and Phelps [11].
3) Kurz [8].
4) However, our result might be regarded as a special case of Professor Solow's even more general treatment. See, Solow [16]. We are indebted to Professor Samuelson for this remark.
consumption-goods. Let $F_1(K_1, L_1)$ and $F_2(K_2, L_2)$ be the quantity of the investment-goods $Y_1$ and that of the consumption-goods $Y_2$ respectively, where $K_i$ and $L_i$ are the quantities of capital and labor allocated to sector $i$. Both capital and labor are fully employed so that the sum of $K_1$ and $K_2$ equals the total available amount of capital $K$, and similarly that of $L_1$ and $L_2$ the total amount of labor $L$. As the allocation of the factors is competitively determined, the marginal-productivity equalities are always satisfied. Consumption forms a certain fraction $1-s$ of total gross income, and savings as residuals are all invested. Investment-goods depreciate ("evaporate") at a fixed rate of depreciation $\mu$, and labor expands at an exogenously-determined rate $n$.

Denoting the price of the $i$-th goods by $p_i$, the rental rate of capital by $r$, and the wage rate of labor by $w$, we can write down all the relations described above as

(1) $Y_1=F_1(K_1, L_1), \ Y_2=F_2(K_2, L_2)$
(2) $K_1+K_2=K$
(3) $L_1+L_2=L$
(4) $\frac{\partial F_1}{\partial K_1}=p_1, \ \frac{\partial F_2}{\partial K_2}=p_2, \ \frac{\partial F_1}{\partial L_1}=p_1, \ \frac{\partial F_2}{\partial L_2}=w$
(5) $p_1Y_1=(1-s)(rK+wL)$
(6) $\dot{K}=Y_1-\mu K$
(7) $\dot{L}=nL$

Under the assumptions of (a) constant returns to scale, and (b) positive marginal productivities, (1)~(7) are further restated as

(8) $y_1=f_1(k_1)l_1, \ y_2=f_2(k_2)l_2$
(9) $k_1l_1+k_2l_2=k$
(10) $l_1+l_2=1$
(11) $\omega+k_1=\frac{f_1(k_1)}{f_1''(k_1)}, \ \omega+k_2=\frac{f_2(k_2)}{f_2''(k_2)}$
(12) $p=\frac{f_2'(k_2)}{f_1'(k_1)}$
(13) $y_1=(1-s)(\omega+k)f_2'(k_2)$
(14) $\frac{\dot{k}}{k}=\frac{s}{k}f_1(k_1)+s\left(1-\frac{k_1}{k}\right)f_1'(k_1)-(n+\mu)$, where

\[ k_1=\frac{K_i}{L_i}, \ k=\frac{K}{L}, \ l_i=\frac{L_i}{L}, \ y_i=\frac{Y_i}{L}, \ \omega=\frac{w}{r}, \ \dot{p}=\frac{p_1}{p_2}, \]

and

\[ f_i(k_i)=F_i(k_i, 1). \]

It has been proved that, with additional assumptions of (c) $f_1''(k_1)<0$, and (d) $f_1'(0)=-\infty, f_1'(\infty)=0$, either if the investment-goods sector is always less capital intensive than the consumption-goods sector, or if the elasticity of substitution between total capital and total labor is not less than unity, a path of balanced growth is uniquely determined, and any growth process approaches the balanced path in its relative configuration.
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**Main Results**

Suppose now that either of the above conditions for relative stability is satisfied. Then, from the long-run viewpoint, it may be considered that a balanced path is always realized in accordance with each possible set of parameters including the fraction of income saved and invested $s$. The Neo-classical Theorem is concerned with the problem: What is the best value of the saving ratio $s$ in the sense that it maximizes the consumption level per head along each path of "Golden-age" steady configuration?

In view of (8), (9), (10), (11) and (13), we obtain

$$k = \frac{(1-s)(\omega + k_2)f_1(k_2)}{(1-s)k_1 + sk_2 + \omega}.$$  \hfill (15)

Accordingly, if we substitute (15) into (13), and use (11) again, we can easily verify that the consumption level to be maximized is expressed in the form

$$y_2 = \frac{(1-s)(\omega + k_2)f_1(k_2)}{(1-s)k_1 + sk_2 + \omega}.$$  \hfill (16)

On the other hand, by (15), equation (14) may be rewritten as

$$\frac{\dot{k}}{k} = \frac{s(\omega + k_2)f_1(k_2)}{k_1k_2 + sow_1 + (1-s)\omega k_2} - (n+\mu),$$  \hfill (17)

so that, by putting $\dot{k}/k=0$, the "Golden-age" steady path is characterized by the condition

$$s(\omega + k_2)f_1(k_1) - (n+\mu)(k_1k_2 + sow_1 + (1-s)\omega k_2) = 0.$$  \hfill (18)

With all these considerations, our problem is now framed in terms of a familiar constrained maximum problem maximizing (16) subject to constraint (18). Following the conventional procedure, fix a Lagrangean expression as

$$\Phi(s, \omega, \lambda) = \frac{(1-s)(\omega + k_1)f_1(k_1)}{(1-s)k_1 + sk_2 + \omega} + \lambda\{s(\omega + k_2)f_1(k_1) - (n+\mu)(k_1k_2 + sow_1 + (1-s)\omega k_2)\},$$

where $\lambda$ is a Lagrangean multiplier, and maximize $\Phi$ by varying $s$, $\omega$ and $\lambda$. A straightforward calculation shows that the following first-order conditions have to be satisfied.

$$\frac{\partial \Phi}{\partial s} = \frac{(\omega + k_1)(\omega + k_2)f_2(k_2)}{[(1-s)k_1 + sk_2 + \omega]^2} + \lambda[(\omega + k_2)f_1(k_1) + (n+\mu)(k_2 - k_1)\omega] = 0,$$  \hfill (20)

$$\frac{\partial \Phi}{\partial \omega} = \frac{(1-s)[s(\omega + k_1)^2 \frac{d}{d\omega} + (1-s)(\omega + k_1)^2 \frac{d}{d\omega} + s(k_2 - k_1)(\omega + k_2)f_2(k_2)]}{(\omega + k_1)(\omega + k_2)}.$$  \hfill (21)

$$= \lambda(n+\mu),$$

and

$$\frac{\partial \Phi}{\partial \lambda} = s(\omega + k_2)f_1(k_1) - (n+\mu)(k_1k_2 + sow_1 + (1-s)\omega k_2) = 0.$$  \hfill (22)

From (20) and (22), one gets

$$\lambda = \frac{s}{n+\mu} \frac{(\omega + k_2)f_1(k_2)}{[(1-s)k_1 + sk_2 + \omega]^2k_2}.$$  \hfill (23)

Substituting (23) into (21) and rearranging terms, it can be shown that
(24) \[\frac{(1-s)(\omega+k_1)k_2-s(\omega+k_2)\omega}{(1-s)(\omega+k_1)k_2-s(\omega+k_2)\omega} \cdot \frac{s(\omega+k_2)^2}{\omega} + \frac{(1-s)(\omega+k_1)^2}{\omega} \cdot \frac{dk_2}{\omega} = 0.\]

Hence, in view of \( \frac{dk_1}{\omega} = 0, \frac{dk_2}{\omega} = 0, \) it is obvious that

(25) \((1-s)(\omega+k_1)k_2-s(\omega+k_2)\omega\)

must hold along any "Golden-age" path with the highest consumption.

Now, using (11), we can verify that (25) is equivalent to

(26) \[s(\omega+k_2)\frac{f_i'(k_1)}{k_1k_2 + s\omega k_1 + (1-s)\omega k_2} = f_i'(k_1).\]

With the condition of balanced growth (18), the left side of (26) equals \( n + \mu \), which is the sum of the "Golden-age" rate of capital growth plus the rate of depreciation. On the other hand, by (4), the right side of (26) equals the gross rate of profit \( r/p_1 \). Accordingly it follows that

(27) \[\frac{K}{K} = n = \frac{r}{p_1} - \mu,\]

i.e., the rate of capital accumulation at its natural rate equals the net rate of profit. Then from (4), (6) and (27), it immediately follows that

(28) \[p_1Y_1 = rK.\]

Investments equals profits. Finally, from the homogeneity postulate, (2), (5) and (28), it is easy to see

(29) \[s = \frac{rK}{rK + wL},\]

which says that the fraction of income invested equals the relative share imputed to profits.

Thus we have seen that the decomposition into two sectors does not affect the Neo-classical Theorem at all.

**Additional Remarks**

In developing the two-sector models of economic growth, some writers assume that there are certain differences between saving rates from profits and wages. In this section we shall show that this decomposition also leaves the validity of the theorem unaffected.

Let the fraction of profits saved be \( s_p \), and that of wages be \( s_w \). Then the equation (5) becomes

(30) \[p_2Y_2 = (1-s_p)rK + (1-s_w)WL,\]

and accordingly (16) and (18) become

(31) \[y_2 = \frac{[(1-s_w)\omega + (1-s_p)k_2]f_i(k_2)}{(1-s_p)k_1 + s_p k_2 + \omega}\]

and

(32) \[(s_w\omega + s_p k_2)f_i(k_2) - (n + \mu)(k_1k_2 + s_w\omega k_1 + (1-s_w)\omega k_2) = 0\]

respectively. Similar but a little more laborious calculation gives us

(33) \[(1-s_1)(\omega+k_1)k_2 = s_w\omega(\omega+k_2)\]

in the place of (25). Hence, to see if the theorem is still valid, we only need to rewrite (26) as
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\[
\frac{(s_o \omega + s_I k_2) f_2(k_1)}{k_1 k_2 + s_o \omega k_1 + (1 - s_o) w k_2} = f'_1(k_1)
\]

and (29) as

\[
\frac{r K}{s_r r K+w L + s_w r K+w L} = \frac{r K}{r K+w L}.
\]

In concluding our paper, it is interesting to note that (33) implies

\[
\frac{r K_2}{r K_1+w L_2} = \frac{s_w}{1-s_p}.
\]

I.e., the ratio of capital’s relative share in the consumption-goods sector to labor’s relative share in the investment-goods sector equals the ratio of laborers’ propensity to save to capitalists’ propensity to consume.

Keio University

REFERENCES