THE PENROSE EFFECT AND OPTIMUM GROWTH*

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The rate of economic growth, generally measured in terms of the annual rate of increase in (per capita) real national income, has recently become one of the basic variables in assessing the economic performance of a country. Thus, in more industrially advanced, as well as less advanced, countries, there is a strong tendency to favor those economic policies such as investment credits, interest subsidies, etc., whose primary effects are to increase the level of investment. In a full-employment economy, investment is increased only through a decrease in the current rate of consumption, but behind these economic measures, there is a widely held view that the increase in future output due to an increase in the stock of capital available will more than compensate the loss in current consumption. The purpose of the present paper is first to formulate a model of capital accumulation in which the long-run as well as short-run effects of investment subsidies are effectively analyzed, and secondly to examine the extent to which the pattern of capital accumulation attained in a market economy diverges from one which is optimum with respect to the welfare criterion based upon individuals' time preference structure. It will be particularly shown that, under fairly general conditions, the rate of investment achieved in a market economy is higher than the optimum rate and an optimum rate of interest subsidies is negative, whenever the aggregate capital-labor ratio is less than a certain critical value—the long-run steady ratio.

The model constructed in the present paper is largely based upon those of the neoclassical growth theory which have been introduced in the past several years, mostly by Tobin [10], Solow [8], Swan [9], Meade [6] and others. It, however, differs from them in two aspects; one relating to the behavior of entrepreneurs concerning investment and the other in terms of the saving decision to be made by households. As for the former, we shall take into account the factors, such as managerial ability, uncertainty, and risk, which are limitation to the process of growth of firms. The

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schedule of marginal efficiency of investment is related to the rate of growth of the
firm, measured in terms of the rate of investment over the existing stock of capital.
The theory of investment on which our model is based, therefore, may be viewed
as a mathematical formulation, in very simple circumstances, of the theory of the
growth of the firm advanced by E. T. Penrose [7]. The behavior of households with
respect to consumption and saving, on the other hand, will be examined in terms of
the theory of time preference as originally discussed by Fisher [2] and recently
formulated by Koopmans [4]. At each moment of time, each household decides to
divide its income between current consumption and saving in the form of an increase
in asset holdings so as to attain the pattern of intertemporal consumption which is
optimum with respect to its time preference schedule. The consumption and saving
functions are determined by the current rate of interest, the level of income, and the
asset holdings. The analysis will be conducted for the case where the time prefer-
ence schedule satisfies a certain number of simplifying assumptions, as in detail
discussed in [12]. Before we proceed with a detailed analysis of the investment and
saving schedules, the structure of the aggregative model will be briefly described.

We are concerned with an economy in which all the economic units are divided
into two classes—business firms and households. Business firms employ labor and
other means of production to engage in productive activities, and the level of employ-
ment and the rate of investment are so determined as to maximize the present value
of future profits, discounted in terms of expected rates of interest, as will be dis-
cussed in detail in a later section. It is assumed that shares are the only means of
external financing for business firms, and the labor market is perfectly competitive.
On the other hand, households receive wages for the labor they provide with busi-
ness firms, and dividends and capital gains for the shares they own. Their income
is divided between current consumption and saving, the latter in the form of an
increase in share holdings, in accordance with their time preference structure.

Output, real capital, and labor are all assumed to be composed of homogeneous
quantities, so that they are measured in unambiguous terms. Let \( K_t \) and \( N_t \) be
respectively the quantities of real capital and labor available to the economy as whole
at a certain time \( t \), and let \( B_t \) be the total number of shares outstanding at time \( t \).
Households’ real income \( Y_t \) is composed of wages \( W_t \), dividends \( D_t \), and expected
capital gains \( G_t^\ast \):

\[
Y_t = W_t + D_t + G_t^\ast.
\]

If \( \pi_t \) stands for the market price (in real terms) of a share, then the value of the
outstanding shares \( V_t \) is given by

\[
V_t = \pi_t B_t,
\]

and the average rate of return to share holdings \( \rho_t \) by

\[
\rho_t = \frac{D_t + G_t^\ast}{V_t}.
\]

The aggregate real income then may be written as

\[
Y_t = W_t + \rho_t V_t.
\]
The desired levels of current consumption \(C_t\) and savings \(S_t\) are in general determined relative to the level of aggregate real income \(Y_t\) and the rate of return \(\rho_t\) to share holdings:

\[
C_t = C(\rho_t, Y_t), \quad S_t = S(\rho_t, Y_t),
\]
where the budgetary equation necessarily holds:

\[
C_t + S_t = Y_t.
\]

The amount of new shares which households plan to purchase (in annual rate) \(\dot{B}_t\) is then given by:

\[
\pi_t \dot{B}_t = S_t - G_t.
\]

The average rate of return to share holdings, \(\rho_t\), has to be equal to the rate of interest prevailing in a bond market, if the latter were available to households and both shares and bonds were held with positive amounts. Hence \(\rho_t\) will be hereafter referred to as the market rate of interest at time \(t\). Now the level of investment which business firms as a whole desire to make is governed first by the current rate of profit through its effects upon the expected rates of future profits and second by the market rate of interest which influences the rates of discount to be used by business firms. Let \(Q_t\) and \(\Phi_t\) be respectively the levels of output and investment. Output \(Q_t\) is divided into wages \(W_t\), dividends \(D_t\), and retained profits \(RP_t\):

\[
Q_t = W_t + D_t + RP_t,
\]
while investment \(\Phi_t\) is partly financed from retained profits \(RP_t\) and the rest from the issue of new shares:

\[
\Phi_t = RP_t + \pi_t \dot{B}_t.
\]

The level of output \(Q_t\) is related to the amount of real capital \(K_t\) and the level of labor employment \(L_t\):

\[
Q_t = F(K_t, L_t),
\]
which is subject to constant returns to scale and a diminishing marginal rate of substitution between capital and labor. The desired level of labor employment is related to the real wage rate \(w_t\), then prevailing in the market.

At each moment of time \(t\), we take as given the stock of real capital \(K_t\), the available quantity of labor \(N_t\), and the number of the outstanding shares \(B_t\), together with the amount of capital gains \(G_t\) which households expect to receive for their share holdings. The real wage rate \(w_t\), the market price of a share \(\pi_t\), and the rate of interest \(\rho_t\) are then determined in such a way that the labor market, the goods-and-services market, and the shares market are all in equilibrium:

\[
L_t = N_t, \quad C_t + \Phi_t = Q_t, \quad \dot{B}_t = \dot{B}_t.
\]

The equilibrium level of investment \(Z_t\) is correspondingly determined in the economy:

\[
\dot{K}_t = Z_t,
\]
while the rate of growth in labor forces is assumed to be constant and exogenously given. For the sake of simplicity, it will be assumed that the size of labor forces remains constant.
To analyze the structure of such an equilibrium system, it is necessary to examine the investment behavior of business firms and the saving behavior of households.

It is generally thought, as in detail described in E. T. Penrose [7], that the managerial and administrative ability required for a firm in the process of growth is basically different from those which are needed in the mere management of the existing administrative structure of the firm. The nature of such a process may be conveniently summarized by a schedule relating the level of investment, as measured in terms of real output, with the rate by which the stock of real capital available to the firm grows. It is assumed that such a schedule is represented by a curve which relates the investment-real capital ratio $\varphi$ with the rate of increase $z$ in real capital, thus indicating that the managerial ability for the growth is endowed with a firm in proportion with its size. Such a schedule is typically illustrated by the heavy curve in Figure 1, where the horizontal axis measures the rate of growth in real capital ratio while the investment-real capital ratio is measured along the vertical axis.

It is assumed that real capital once invested lasts forever and the higher the rate of capital accumulation, the larger is the amount required of output per unit of real capital. Hence, the $\varphi$ curve is convex and passes through the origin, where it may be without loss of generality assumed that the slope is $45^\circ$.

Let us now consider a firm which owns real capital by the amount $K_0$ at time 0 and possesses the $\varphi$-schedule as described in Figure 1. The firm is assumed to have stationary expectations regarding future rates of real wages and interest; i.e. it expects the current wage rate $w$ and interest rate $\rho$ to prevail for the entire future. Then the present value of future profits, discounted at the expected rate of interest, becomes

$$
\int_0^\infty (Q_t - wL_t - \Phi_t)e^{-\rho t}dt,
$$

$$
K_t = Z_t,
$$

where $Q_t = F(K_t, L_t)$: the level of output,
$K_t$: the stock of real capital,
$L_t$: the level of employment,
$\Phi_t$: the level of investment,
$Z_t$: the increase in real capital investment, all at time $t$.

The firm then plans to choose the time-path of investment $\Phi_t$ in such a way that
the present value of future profits (14) is maximized subject to the constraints (15), with initial amount of capital $K_0$. Because of the linear homogeneous concave production function $F(K, L)$ and of the convexity of the $\varphi$-schedule, it is easily shown that, along any optimum path, the employment-capital ratio $L_t/K_t$ is determined as the level at which the marginal product of labor is equal to the real wage rate $w$ and the rate of investment $\varphi_t = \Phi_t/K_t$ is always constant through time. Let the per capita production function be denoted by $f(k)$, $k = K/L$; namely, 

$$f(k) = F(k, 1).$$

Then capital-labor ratio $k$ at which the marginal product of labor is equal to the real wage rate $w$ is uniquely determined and we have 

$$Q_t - wL_t = rK_t,$$

where

$$r = f'(k), \quad w = f(k) - kf'(k).$$

The present value of future profits (14) may now be rewritten as

$$\int_0^\infty (r - \varphi(t))K_te^{-rt}dt = K_0, \quad \int_0^\infty [r - \varphi(z)]e^{-(r - z)t}dt = K_0 \frac{r - \varphi(z)}{\rho - z},$$

when the rate of capital accumulation $z_t = Z_t/K_t$ is kept constant, at the level $z$, over time.

The optimum rate of investment then will be obtained by finding the rate $z$ which maximizes the present value of future profits per unit of initial capital—the demand price of capital—to be given by

$$\frac{r - \varphi(z)}{\rho - z}.$$

The rate of investment $\varphi$ which maximizes the demand price of capital (18) is easily obtained in terms of the diagram in Figure 1. Let $A$ be the point in Figure 1 whose co-ordinates are given by the rate of interest $\rho$ and the marginal product of capital $r$. Then the ratio (18) is nothing but the slope of the line connecting $A$ with the point $B$ on the $\varphi$-schedule corresponding to the planned rate of investment $\varphi$; hence, (18) will be maximized by choosing the rate of investment $z$ such that the line $AB$, where $B = [z, \varphi(z)]$, is tangent to the $\varphi$-curve. The optimum rate of investment $\varphi$ then is determined relative to the rate of interest $\rho$ and the rate of profit $r$; thus we may denote

$$\varphi = \varphi(\rho, r).$$

It is easily seen from Figure 1 that an increase in the rate of interest decreases the rate of investment $\varphi$, while an increase in the marginal product of capital $r$ results in an increase in the rate of investment $\varphi$, and that the rate of investment is positive if and only if the rate of profit exceeds the rate of interest; i.e.,

$$\frac{\partial \varphi}{\partial \rho} < 0, \quad \frac{\partial \varphi}{\partial r} > 0;$$

$$\varphi(\rho, r) > 0 \text{ if, and only if, } r > \rho.$$

1) The present formulation of the investment behavior is, under a very simple circumstance, based upon the Keynesian principle of marginal efficiency of investment (Keynes [3], Chapter 11). For a more detailed analysis, the reader is referred to Treadway [11].
The amount of savings forthcoming at each moment of time, on the other hand, is determined by households in accordance with their time preference schedule. At a certain moment of time \( t \), each consumer possesses a fixed amount of assets, in the form of shares issued by business firms, and expects to receive wages for future periods. The income stream he expects to receive then depends upon the amounts of accumulated savings, in the form of share holdings, and he is primarily concerned with attaining the time-path of consumption which is most preferred, in terms of his time preference structure, among all the feasible consumption paths consistent with his initial asset holdings and the state of expectations regarding future rates of interest and wage. As has been in detail discussed in [12], if the time preference structure satisfies certain properties, it is possible to represent it by a utility functional of the following type:

\[
\int_{0}^{t} u_i e^{-\lambda_i t} dt,
\]

where the instantaneous utility level \( u_i \) is a function of \( c_t \) alone, \( u_i = u(c_t) \), and the accumulated rate of discount \( \lambda_i \) is increased at a rate depending on \( u_i \) alone at each moment of time \( t \):

\[
\dot{\lambda_i} = \delta(u_i),
\]

where \( \delta(u) \) may be referred to as the marginal rate of time preference.

The utility function \( u(c) \) is assumed to have a positive and diminishing marginal utility:

\[
u'(c) > 0, \quad u''(c) < 0,
\]

while the marginal rate of time preference schedule \( \delta(u) \) is assumed to satisfy the following conditions:

\[
\delta(u) > 0, \quad \delta'(u) > 0, \quad \delta''(u) > 0,
\]

\[
\delta(u) - \delta'(u) u > 0, \quad \text{for all} \ u.
\]

Part of conditions (24) and (25) reflects the properties generally attributed to the schedule of time preference that present consumption is preferred to future consumption (of the identical amount), marginal rates of substitution between present and future consumption are diminishing, and each household is better off by a uniform increase in the levels of consumption (or utilities). The last condition in (24), may not be necessarily satisfied for a general time preference schedule. However, as will be easily seen from the analysis presented in [12] and summarized below, the general structure of the consumption and saving functions remains identical when the schedule of the marginal rate of time preference is modified in such a way that the last condition in (24) is satisfied.

We assume that the aggregative behavior of households may be explained in terms of the representative household which is endowed with the unit labor and possesses the Fisherian schedule of the marginal rate of time preference \( \delta(u) \). In accordance with the expectations hypothesis for entrepreneurial decisions concerning investment, households' expectations concerning future real wage and interest rates are both assumed to be stationary, and they expect that capital gains for their share holdings...
will exactly compensate changes in the dividend policy. Let \( v_0 \) be the market value of the shares held by the representative household at a certain time 0, and let \( w \) and \( \rho \) be the expected wage and interest rates. If \( v_t \) and \( y_t \) respectively stand for the market value and real income the representative household plans to have at time \( t \) and if \( c_t \) denotes the level of consumption he desires to make at time \( t \), then the following budgetary equations must be satisfied:

(26) \[ y_t = w + \rho v_t, \]

(27) \[ \dot{v}_t = y_t - c_t, \]

where the real income is to be defined as the sum of wages, dividend payments, and expected capital gains, all in real terms. Given the initial given value of share holdings \( v_0 \), the representative household decides future consumption and pattern of asset accumulation so as to maximize the level of his intertemporal utility (21). The problem is then one of calculus of variations problem: maximize the integral (21) subject to the constraints (22), (26), and (27), with initial condition \( v_0 \). It is fairly easy to obtain the solution to this calculus of variations problem, and the optimum level of consumption is uniquely determined by the level of real income \( y_t \) at each moment of time \( t \) and by the rate of interest \( \rho \):

(28) \[ c_t = c(\rho, y_t) \quad \text{where} \quad y_t = w + \rho v_t, \]

and the level of intertemporal utility will be maximized if the representative household divides his income at each moment between consumption and savings according to (28). It may be noted that the consumption function (29) is in fact obtained as the stable branch of solutions to the following differential equation:

(29) \[ \frac{dc}{dy} = \frac{1}{y - c} \left( 1 - \frac{\rho + \delta' u'(y - c)}{\delta - u' u''(u + u'(y - c))} \right) \]

and it is typically illustrated by the heavy curve in Figure 2, where real income is measured along the horizontal axis and real consumption along the vertical axis. As is seen from Figure 2, the long-run stationary level of real income \( y^* \) is determined as one at which the marginal rate of time preference \( \rho(y^*) \) is equal to the market rate of interest:

(30) \[ \delta(u(c^*)) = \rho, \]

and the optimum level of short-run consumption is less than real income if and only if the marginal rate of time preference corresponding to that level of income is less than the market rate of interest:

(31) \[ c(\rho, y) < y \quad \text{if and only if} \quad \delta(u(y)) < \rho. \]

As has been discussed in [12], the short-run level of real consumption \( c(\rho, y) \) is in-
creased whenever real income $y$ is increased or the rate of interest $\rho$ is decreased:

$$\frac{\partial c}{\partial \rho} < 0, \quad \frac{\partial c}{\partial y} > 0.$$  

We are now in a position to combine the analysis of the investment and saving behaviors of business firms and households to examine the structure of the short-run equilibrium of the whole economy, as expressed in terms of (11). The equilibrium condition for the labor market first entails that the real wage rate $w_t$ is equated to the marginal product of labor when labor is fully employed; namely,

$$w_t = f(k_t) - k_t f'(k_t),$$

where $k_t = K_t/N_t$ is the aggregate capital-labor for the economy as a whole, and $f(k)$ the per capita production function. Hence, the marginal product of capital $r_t$ is also uniquely determined:

$$r_t = f'(k_t).$$

Secondly, the equilibrium condition for the goods-and-service market (namely, the second condition in (11)) may be reduced to the following equation:

$$\varphi(\rho_t, \ r_t) = \frac{f(k_t) - c(\rho_t, y_t)}{k_t},$$

where $\rho_t$ is the market rate of interest and $y_t$ is the per capita real national income:

$$y_t = w_t + \rho_t v_t,$$

and $v_t$ is the market value of the per capita share holdings:

$$v_t = \pi_t h_t,$$

where $\pi_t$ is the market price of a share and $h_t$ is the number of shares held by the representative household.

In view of the hypothesis above concerning expected capital gains and dividends, the per capita real income $y_t$, as defined by (36), in fact becomes identical with the per capita real output $f(k_t)$:

$$y_t = f(k_t).$$

The left hand side of the equilibrium condition (35) represents the rate of investment that the business firms plan to make at the given rate of interest $\rho_t$, while the right hand side stands for the rate of savings forthcoming at the same rate of interest. Hence, the goods-and-services market comes to an equilibrium through an adjustment in the rate of interest, as illustrated in Figure 3.

In Figure 3, the rate of interest $\rho$ is measured along the vertical axis, while the horizontal axis measures either the desired rate of investment or the rate of savings forthcoming. As has been shown in (19), the desired rate of investment is decreased whenever the rate of interest is increased, and it may be represented by the II-curve with a negative slope. It is seen from (29) that the II-curve intersects with the ver-
tical axis when the rate of interest $\rho$ is equal to the marginal product of capital $r_t$. On the other hand, the rate of savings is an increasing function of the rate of interest, as is shown in (32), and the SS-curve describing the savings schedule has a positive slope. The analysis of the saving behavior summarized above implies that the amount of savings will be zero if the rate of interest $\rho$ becomes identical with the marginal rate of time preference $\delta(u(y))$ of the representative household. The II-curve and SS-curve then will have a unique intersection and the market rate of interest $\rho_t$ for which investment equals savings is accordingly uniquely determined. The equilibrium rate of investment and hence the rate of capital accumulation are thus determined.

The shares market now comes to an equilibrium when the market price $\pi_t$ of a share is so adjusted as to satisfy the condition (37) which, in view of (36), may be reduced to:

$$\pi_t = \frac{r_t k_t}{\rho_t d_t}.$$  

The dynamic structure of the economy is then described by the differential equation:

$$\dot{k}_t = z(\rho_t, r_t),$$

where $\rho_t$ is the equilibrium rate of interest and $r_t$ is the marginal product of capital when labor is fully employed.

As is seen from Figure 3, the equilibrium rate of capital accumulation is positive if and only if the marginal product of capital $r_t$ is greater than the marginal rate of time preference $\delta(u(y))$ of the representative household. The marginal product of capital $r = f'(k)$ is decreased as the capital-labor ratio $k$ is increased, as is typically illustrated by $AA$-curve in Figure 4, where the horizontal axis measures the capital-labor ratio. On the other hand, the marginal rate of time preference $\delta(u(y))$ with $y = f(k)$ is an increasing function of the capital-labor ratio $k$, as represented by the $BB$-curve. The $AA$- and $BB$-curves have a unique intersection and the corresponding capital-labor ratio $k^*$ is the long-run stationary ratio for the dynamic system (40). Figure 4 also shows that if the capital-labor ratio $k$ is less (greater) than the long-run stationary ratio $k^*$, the marginal product of capital $r$ is greater (less) than the marginal rate of time preference $\delta(u(y))$, thus having a positive (negative) rate of capital accumulation. Hence, the dynamic system (40) is stable in the sense that the aggregate capital-labor ratio $k_t$ approaches the long-run stationary ratio $k^*$ regardless of the initial capital-labor ratio $k_0$.

We have shown that the model of a two-class economy introduced above has retained most of the properties generally attributed to the standard models in the neo-
classical growth theory. We now proceed to examine how the pattern of capital accumulation in such a model is influenced by an introduction of an investment-subsidy policy. For the sake of simplicity, we assume that the government gives subsidies to business firms proportionally to their investment and levies proportional income taxes on households in such a way that taxes revenues are just equal to investment subsidies. Let \( \sigma \) be the amount of subsidies per unit investment and \( \tau \) be the corresponding rate of income taxes. Then the investment subsidies are given by \( \sigma \cdot \kappa \) and the tax revenues by \( \tau \cdot y \); hence,

\[
\tau = \sigma \cdot \frac{k}{y}.
\]

The effective rate of profit \( \varphi \), based on which business firms determine the desired rate of investment, now becomes

\[
\varphi = \frac{r}{1-\varphi}.
\]

(42)

where \( r \) is the marginal product of capital, and the desired rate of investment is given by \( \varphi(\rho, \varphi) \). On the other hand, consumption and saving are determined by households in relation to the disposable income, \( y = (1-\varphi)y \), and the after-tax rate of interest, \( \rho = (1-\tau) \rho \). Hence, \( c(\rho, \varphi) \) and \( y - c(\rho, \varphi) \) are respectively the amounts of consumption and savings planned by households. The equilibrium condition for the goods-and-services market now becomes:

\[
\varphi(\rho, \varphi) = \frac{y - c(\rho, \varphi)}{k}.
\]

To examine the solution to (43), it may be easier if we take the after-tax rate of interest \( \rho^{*} \) along the vertical axis. Then the II-curve and SS-curve are described as illustrated in Figure 5. The SS-curve intersects with the II-curve at \( (1-\varphi) \rho^{*} = 1-\varphi/1-\rho^{*} \). If the subsidy rate \( \sigma \) is positive, the intersection of the II-curve is higher than the original level and both II- and SS-curves shift to the right. Hence, the equilibrium rate of investment is always increased; namely, a positive investment-subsidy accelerates the rate of capital accumulation. However, the long-run stationary capital-labor ratio \( k^{*} \) remains identical, because the stationary state is characterized as the state where no investment takes place. The level of per capita consumption, on the other hand, is decreased at the beginning but will be increasing at a faster rate than the previous case. In general, the time pattern of consumption, when a positive investment subsidy policy is introduced, is described typically by the BB-curve in Figure 6, where the AA-curve corresponds to the pattern of consumption without any subsidy arrangement.

We are now interested in examining the structure of the investment subsidy policy
which results in the consumption pattern which is most preferred, in terms of the representative household’s time preference schedule, among all feasible patterns. However, before we proceed with the analysis of such an optimum subsidy policy, we consider a related, simpler problem of optimum capital accumulation of the Ramsey type which has recently been discussed by Koopmans [5] and Cass [1]. Suppose that there is a government agency, the central planning bureau, which is empowered to allocate all the scarce resources in the economy in whatever manner it desires and that it is interested in attaining the time path of consumption which is most preferred, in terms of the representative household’s time preference schedule, among all feasible paths starting with the given initial stock of capital. Mathematically, the central planning bureau has to solve the following problem:

Maximize

$$\int_0^\infty u_t e^{-\delta(\omega)\tau} d\tau,$$

subject to the constraints

$$\frac{\dot{k}}{k} = z,$$

with the initial capital-labor ratio $k_0$,

$$u_t = u(c_t),$$

$$c_t + \varphi(z)k_t = f(k_t),$$

where $u(c)$, $\delta(u)$, $\varphi(z)$, and $f(k)$ are respectively the utility function, the marginal rate of time preference, the Penrose function, and the per capita production function.

To solve this calculus of variations problem, let us first represent it in terms of the new independent variable:

$$\Delta = \int_0^\infty \delta(u) ds,$$

which satisfies

$$\dot{\Delta} = \delta(u).$$

The problem then becomes to that of maximizing

$$\int_0^\infty \frac{u}{\delta(u)} e^{-\delta(\omega)\tau} d\tau,$$

subject to the constraints that

$$\frac{\dot{k}}{k} = \frac{dz}{d\tau} = \frac{zk}{\delta(u)},$$

$$u = u(c),$$

$$c + \varphi(z)k = f(k).$$

Let $\lambda$ be the imputed price associated with the differential equation (51) and the
(modified) Hamiltonian form \( H \) be defined by:

\[
H = u + \lambda z k,
\]

with (52-53) satisfied. The Lagrangian form then is written as

\[
\left[ \frac{H}{\partial(u)} - \lambda k \right] e^{-\lambda k}
\]

and the solution to the above calculus of variations problem is obtained by solving the following Euler-Lagrange equations: (45) and

\[
\frac{d}{d\lambda} \left( \frac{d\lambda}{d\lambda} \right) = \frac{\delta(u) - \frac{\partial}{\partial(u)} \lambda}{d\lambda}
\]

where

\[
\frac{\hat{r} - \phi'(z)}{\hat{\beta} - \lambda} = \phi'(z),
\]

\[
\frac{\phi' z^i - \lambda - \phi' \phi' z^i}{\hat{u}} H = 0,
\]

and the transversality condition:

\[
\lim_{A \to \infty} \lambda e^{-\lambda} = 0.
\]

A transformation to the original time variable reduces (51) and (56) to:

\[
\dot{k} = z k,
\]

\[
\dot{\lambda} = \delta(u) - \beta,
\]

where (52), (53), (57), (58) and (59) are satisfied.

It may be first noted that the stationary state \( k^* \) of the differential equation (60-61) is characterized by

\[
r^* = \delta(u(y^*))
\]

where \( r^* = f'(k^*) \), \( y^* = f(k^*) \). Hence the stationary capital-labor ratio \( k^* \) of the Euler-Lagrangian equations coincides with the long-run stationary capital-labor ratio \( \hat{k} \) for the market economy introduced previously. To obtain the solution to the differential equations (60-61) with the transversality condition (59), let us transform (60-61) into the differential equations involving \( k \) and \( c \), instead of \( k \) and \( \lambda \). Differentiating (58) with respect to the time variable \( t \) and eliminating \( \lambda \) entirely, we get

\[
\dot{c} = \beta - \delta - \delta' u' \phi' z k + \frac{\phi''(r - \phi) z}{\delta' u'(u + u' \phi' z k)} - \frac{\phi'' u' + \phi'' k}{\phi' z^2 k}
\]

Combining (63) with (60), we get the differential equation describing the optimum path on the \((c, y)\)-plane:

\[
\frac{dc}{dy} = \frac{\hat{\beta}}{rz k} \left( \frac{\delta + \delta' u' \phi' z k}{\delta' u'(u + u' \phi' z k)} - \frac{\phi'' u' + \phi'' k}{\phi' z^2 k} \right)
\]

where \( y = f(k) \), \( r = f'(k) \).

The optimum path of consumption then is obtained by finding the stable branches of the differential equation (64).

To compare the optimum level of consumption with the level attained in the mar
ket economy, let us assume that the equilibrium rate of interest $\rho$ is decreased whenever there is an increase in the aggregate capital-labor ratio $k$. Then, for the given capital-labor ratio $k$ higher than the long-run stationary ratio $k^*$, the market rate of interest $\rho$ is less than the long-run stationary rate $\rho^* = r^* - \delta^*$. The stationary state for the differential equation (61) lies at a lower level, along the $45^\circ$ line, than that for (29). On the other hand, for the given levels of $y$ and $c$, the magnitude of $dc/dy$ along (61) is lower than that along (29), because

$$\rho(y-c) < rzk, \quad y-c < \varphi^*.$$  

Hence, the stable branch of the solution to (64) has to lie entirely above the path of private consumption path, and, for any given $k$, the optimum level of consumption must be higher than the level achieved by the market mechanism.

We have shown above that the level of consumption along an optimum growth path is always higher than that for a market economy, provided a number of assumptions concerning the expectations and technological structure are satisfied and the aggregate capital-labor ratio $k$ is lower than the long-run stationary level $k^*$. The rate of investment subsidies which would bring the market rate of growth equal to the optimum rate, therefore, has to be negative whenever the capital-labor ratio $k$ is lower than $k^*$. Namely, corporate taxes proportional to the magnitude of investment have to be levied to finance subsidies to households in order to attain the optimum pattern in such a two-class market economy.

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REFERENCES

