TRANSACTION COSTS AND THE PRECAUTIONARY DEMAND FOR MONEY

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IN THE RECENT literature of monetary theory, the emphasis has focused on both the role of transactions costs and the effect of uncertainty. The purpose of this paper is to examine the effect of three types of transaction costs on the precautionary demand for money. In the theory of demand for money we have two theories which have not so far been connected each other, although it is because they deal with different motives of money holding. One is the cube root formula, Whalen [5] and Miller and Orr [2], and the other is the square root formula, Baumol [1] and Tobin [4]. We will construct a simple two period model under uncertainty and derive both the cube root formula and the square root formula from the same model under different assumptions of transactions costs. It will be shown that the cube root formula is obtained from the assumption of constant transaction cost and the square root formula from that of proportional transaction cost. Finally we will proceed to the analysis of linear cost model and derive its several properties.

In applying a rational behavior to the precautionary demand for money, Whalen [5] used a function which conservatively relates the probability of insufficient cash to precautionary cash balances, being constructed from Tchebycheff’s inequality. The questions may arise whether such a function exists, is uniquely determined and differentiable, but we assume these conditions to be satisfied. In this paper we note that by his approach we can easily investigate the effects of transaction costs (in his terms, the cost of illiquidity) on the precautionary demand for money. Although we use his notations with a few additional notations, we will formulate a maximization problem instead of his minimization problem in order to clarify the difference between his measure of risk aversion and that of Portfolio Selection Theory. There is no difference between these two optimization problems, but it is only for convenience. The point is to incorporate the transaction costs and examine their effects.

The Model

Let an individual have initial wealth $W_0$ and money balances $M$. The fraction $(W_0 - M)$ of his wealth is assumed to be invested in bonds which earn the interest

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rate $r$ (the opportunity cost of holding the precautionary money balances). If net disbursements $D$ which is a random variable with the probability density function $f(D)$, are less than money holding, i.e., $D \leq M$, his final wealth is

$$W_1 = (M - D) + (1+r)(W_0 - M). \quad (1)$$

But if holding money balances are insufficient, (i.e., $D > M$); that is, he fails to provide adequate money funds to meet required disbursements, he incurs the transaction costs because money is the only means of payment by definition. We assume the following linear transaction cost function

$$c + v(D - M) \quad (2)$$

where $c$ represents the constant cost per occurrence and $v$ the cost proportional to the size of a money deficiency $(D - M)$. Hence in the case of insufficient money holding his final wealth is

$$W_2 = (M - D) + (1+r)(W_0 - M) - c - v(D - M). \quad (3)$$

For simplicity we will rule out the complication that in the case of bankruptcy (i.e., $D > W_0$), another additional costs are charged.

Now let us turn to the behavioral hypotheses. An individual is assumed to determine the optimal money holding by maximizing his expected final wealth $1)

$$E(W) = \int_{-\infty}^{\infty} W_1 f(D)dD + \int_{M}^{\infty} W_2 f(D)dD \quad (4)$$

where $E$ denotes the expectations operator and $W$ his final wealth. Next let us formulate a kind of risk averse behavior in order to obtain as simpler solutions as possible. In fact this may be a vital limitation of our analysis, but we are able to get the cube and square root formula in the cost of analytical generality. Let us assume, first, that the probability density function of net disbursements $f(D)$ has a mean of zero and a standard deviation $S$ and that an individual estimates conservatively the probability of adverse situations that his holding money is insufficient. A measure of this risk averse behavior may be formalized by the Tchebycheff's inequality $2)$. According to this theorem, the probability that a variable will deviate from its mean by more than $k$ times its standard deviation is equal to or less than $(1/k^2)$. Letting $k = M/S$, his subjective probability function must satisfy $3)$

$$\int_{-\infty}^{\infty} f(D)dD = \int_{-\infty}^{\infty} f(D)dD = 1 - \frac{1}{2} - \frac{S^2}{2M^2}. \quad (5)$$

Substituting (1) and (3) into (4), and rearranging it using (5), we get

$$E(W) = \int_{-\infty}^{\infty} W_1 f(D)dD - c - \frac{1}{2} - \frac{S^2}{M^2} - v \int_{M}^{\infty} (D - M)f(D)dD. \quad (6)$$

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1) According to Portfolio Selection Theory, we must formulate this problem as that of the expected utility maximization, using the utility function $U(\ )$ which has well-known properties. The maximand will be

$$EU(W) = \int_{-\infty}^{\infty} U(W_1)f(D)dD + \int_{M}^{\infty} U(W_2)f(D)dD.$$

It seems impossible to derive an explicit solution such as the cube or square root formula even if we make the shape of transaction costs simpler.

2) For the detail explanation, see Whalen [5].

3) The difference of coefficient between Whalen's [5] and ours comes from the fact that we consider $M$ to be positive and $f(D)$ to be symmetric. This is the reason why we multiply 1/2 by $1/k^2$. 

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Taking the first derivative with respect to $M$ and setting it equal to zero, we get
\[
\frac{\partial E(W)}{\partial M} = -r + c \frac{S^2}{M^2} + v \int_{M}^{\infty} f(D)dD
\]
\[
= -r + c \frac{S^2}{M^2} + v \frac{1}{2} \frac{S^2}{M^2} = 0
\]
where we used (5) in the last term. This first order condition for the maximum will play the important role in our analysis. Since $c$ and $v$ are positive by assumption and we don’t permit negative money holding, the second order condition is always satisfied.

**Constant Cost Model and Proportional Cost Model**

Now let us consider, to begin with, the effect of the constant cost per occurrence on the equilibrium money holding. This will be done by letting $v=0$ in equation (7) and solving for $M$.

\[
M = \sqrt[3]{\frac{S^2 c}{r}}. \tag{8}
\]

This is the case Whalen [5] analysed. The only difference is that of coefficient which has already explained. We are now familiar with the cube root formula in the theory of cash management and it can also be derived from another rationalization, Miller and Orr [2]. Their derivation is based on the analysis of stochastic process. Although we can not compare their analysis with ours, it should be noted that their model is also a constant transaction cost model (in their terms, fixed transfer cost) under uncertainty.

Next let us consider the effect of proportional cost by letting $c=0$ in equation (7) and solving for $M$.

\[
M = \sqrt{\frac{S^2 v}{2r}}. \tag{9}
\]

The expression (9) is very akin to the square root formula by Baumol [1] and Tobin [4]. Thus it may be able to conclude that the cube root formula is obtained under the assumption of constant transaction cost and the square root formula is under the assumption of proportional transaction cost. Since, if $(D-M)$ is positive, it means the individual made a decision of inadequate money holding, it incurs the penalty. The constant cost model (8) deals with the case where the penalty does not depend on the amount of money he failed to provide. Moreover even if the individual did make a correct decision, he incurs the constant cost $c$.

On the other hand, in the proportional cost model, the penalty increases as the degree of failure to provide a correct amount of money increases, but if he made a correct decision, he does not incur any cost. It will be claimed intuitively that in an usual situation the individual has a tendency to hold more money when the constant cost is applied than when the proportional cost is applied. The condition that this is the case is that $8c^3 > (S^2/r)v^3$.

A natural extention we can manage mathematically may be the analysis of the linear transaction cost which will be treated in the following section.
Properties of Linear Cost Model

Finally let us consider the linear cost model. In this case, we can not get an explicit solution unfortunately. So that the analysis will be confined to the elasticity analysis. Let $\gamma(S^2)$ and $\gamma(r)$ be demand elasticities of the square of the standard deviation $S^2$ and the interest rates $r$, respectively; that is, $\gamma(S^2)=S^2/M\cdot\partial M/\partial S^2$ and $\gamma(r)=-r/M\cdot\partial M/\partial r$. Calculating from (7), we get

$$\gamma(S^2)=\gamma(r)=\frac{2M\gamma}{6M^2r-S^2v}.$$  

From (10) we can know the interesting properties of the demand function for money.

(a) Two elasticities are equivalent in absolute value, just the same as the cube or square root formula, (8) or (9). It is natural because the analysis is a mere extension of above two models. (b) From equation (7) we can obtain the comparative static results. By partially differentiating $M$ with respect to $c$, $v$, $S^2$ and $r$, we get

$$\partial M/\partial c=2S^2/6M^2r-S^2v>0, \quad \partial M/\partial v=MS^2/6M^2r-S^2v>0, \quad \partial M/\partial S^2=2c+Mv/6M^2r-S^2v>0, \quad \partial M/\partial r=-2M^3/6M^2r-S^2v<0.$$  

(c) Because the proportional transaction cost $v$ is positive, elasticities are greater than $1/3$. Of course, zero proportional transaction cost results in elasticities of $1/3$, as shown in (8). The value of elasticity depends on the definition of money. Our definition that money is the means of payment is broader than that which consists of currency only. So that the results does not seem to be inconsistent with the empirical evidence, Teigen [3] (p. 98). (d) Because $S^2$ measures the lack of synchronization between the pattern of receipts and disbursements, $S^2$ may represent the degree of uncertainty attached to these patterns. Differentiating (10) with respect to $M$, we get

$$\frac{\partial \gamma(S^2)}{\partial M}=-S^2v\frac{4Mr}{(6M^2r-S^2v)^2}<0.$$  

Hence we can conclude that the more he holds money, the more insensitive he becomes to uncertainty. This relationship between the optimal money holding and elasticities is described in Figure. (e) Initial wealth has entirely no effect on the optimal money holding. This property comes from our formulation of the problem, especially the hypothesis of maximizing the expected wealth, not from the transaction cost function. Suppose the general transaction cost function $K(D-M)$, $K'>0$, $K(0)=0$, where the prime denotes the first derivative. Then the first order condition (7) should be replaced by

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4) Positivity of elasticities is shown to be satisfied when we consider equation (7). That is, $6M^2r-S^2v=6M^2(S^2+Sv[2M^2]-S^2v=6cS^2|M+2Sv>0$.  

5) Combined with equation (11), we can get the comparative static results of elasticities, $\partial \gamma/\partial c<0$, $\partial \gamma/\partial v<0$, $\partial \gamma/\partial S^2<0$, $\partial \gamma/\partial r>0$.  

6) "Most of those studies reported interest elasticities of demand of $-0.6$ to $-0.9\ldots$. However, it should be noted that several of the earlier studies used the short rate and obtained interest elasticities of the demand for money of from $-0.6$ to as high as $-1.16$. While part of this difference could be due to differences in the period covered, restriction of the income elasticity to be unity, etc., much of it must be due to disregard of the supply relationship."
which does not depend on initial wealth $W_0$.

Some of these properties may depend on the analytical method taken. Indeed our analysis has used a very restrictive measure of risk aversion. But the results are interesting in itself and it seems to be worth noting.

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REFERENCES


