CONSUMERS' CHOICE AND THE FUNDAMENTAL DUALITY*

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1. Introduction

In a von Neumann type model, Burmeister-Kuga [2] showed that a generalization of the factor-price frontier, which they call the minimum real wage frontier, is mathematically identical with the optimal transformation frontier. This duality relation is an extension of the results obtained by Hicks [4] and by Bruno [1]. Fujimoto [3] also presents the duality in a von Neumann model, in which a commodity can serve both as a capital good and as a consumption good.

In these models, however, there exists only one consumption good or a basket of consumption goods is fixed, implying the prohibition of consumers' choice. Morishima investigates von Neumann models in which workers can correspond to price changes in [7, 8]. He is also concerned with the uniqueness of von Neumann growth equilibrium, allowing for consumers' choice in [9, 10]. The purpose of this note is to extend the duality relation to a von Neumann model, taking into account workers' freedom to choose consumption goods.

2. Model and Assumptions

Our model is a von Neumann growth model. There exist m goods and n processes. The period of production is uniform among processes and so taken as the unit of time, say one year. We use the following notation.

\[ A: \text{ input matrix } (m \times n) \text{ of material (each process as a column)}, \]
\[ B: \text{ output matrix } (m \times n), \]
\[ C_0: \text{ given standard basket of consumption goods, column } m\text{-vector}, \]
\[ L: \text{ row } n\text{-vector of labour input, the unit of which is man-year}, \]
\[ x: \text{ column } n\text{-vector of activity intensity of processes}, \]
\[ y: \text{ row } m\text{-vector of prices (labour as numeraire), } y=(y_1, y_2, \ldots, y_m), \]
\[ r: \text{ rate of profit, } 1+r>0, \]
\[ g: \text{ rate of balanced growth, } 1+g>0, \]
\[ S_m: \text{ } m\text{-simplex, i.e., } S_m=\{y|\sum_{i=1}^{m}y_i=1, y\geq 0\}. \]

A is non-negative, while B, L and C_0 are semi-positive. The inequality sign \( \geq \) for vectors is used to mean \( \geq \) but not \( = \). The sign \( > \) means a strict inequality holds in the comparison of each element.

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One more symbol, \( C(y) \), is defined like this. Given a price vector \( y \), \( C(y) \) is a vector of consumption goods which minimize \( yC(y) \), size of expenditures to buy \( C(y) \), subject to \( u(C(y)) \geq u(C_0) \). In this definition, \( u \) is a utility function which is assumed to be common among workers and homothetic (See the next paragraph). It is also assumed that \( C(y) \) is uniquely determined for any \( y \geq 0 \) and each entry is continuous and finite over \( y \geq 0 \). \( C(y) \) thus defined is homogeneous of degree zero in \( y \). For, suppose \( C(y) \neq C(\lambda y) \) for \( \lambda > 0 \) and \( \lambda \neq 1 \), implying that \( \lambda yC(\lambda y) < \lambda yC(y) \) while \( u(\lambda yC(\lambda y)) \geq u(C_0) \). This reveals a contradiction because a worker could buy \( C(\lambda y) \) when a price vector \( y \) is prevailing, which keeps him at the utility level not lower than with \( C_0 \), making a less expenditure than to buy \( C(y) \). Thus \( C(y) \) is homogeneous of degree zero. Note also that by definition \( yC(y) \geq yC(y) \) for any \( y \geq 0 \) and any \( y \geq 0 \).

In this note, homotheticity is defined as follows: if \( u(C_1) \geq u(C_2) \), then \( u(\lambda C_1) \geq u(\lambda C_2) \) for any pair of consumption baskets \( C_1 \) and \( C_2 \) and for any non-negative scalar \( \lambda \). Then, homotheticity insures the following proposition holds. That is, when the size of expenditures is unity and a price vector \( y \) obtains, then, a worker prefers the consumption basket \( vC(y) \) to other kinds of baskets. Here, \( v \) is defined as \( v = 1/(yC(y)) \). In order to prove the above proposition, we have to show \( u(vC(y)) \geq u(D) \) for any consumption basket \( D \) such that \( yD \leq 1 \) and \( D \neq vC(y) \). Suppose the contrary. Then by the homotheticity of \( u \), we have, for some \( E \),

\[
u C(y) \leq u(E) \quad \text{for } E \text{ such that } yE \leq 1/v = yC(y) \text{ and } E \neq vC(y),\]

putting \( E = D/v \). Thus, we have a contradiction to the minimality of \( yC(y) \) or the uniqueness of \( C(y) \).

We also make the following assumptions:

Assumption A1: \( Lx > 0 \) for \( x \) which satisfies the inequality (1) below with an arbitrarily given \( y \geq 0 \).

Assumption A2: There is at least one positive entry in each row of \( B \).

A1 says that labour is indispensable to produce a consumption basket and implies that \( C(y) \) cannot be 0-vector for any \( y \). A2 means that every commodity can be produced by at least one process. This assumption was introduced into a von Neumann model by Kemeny-Morgenstern-Thompson [6].

3. Duality

First, let us consider the following two problems:

Problem I: Minimize \( Lx \) subject to \( x \geq 0 \), \( y \geq 0 \) and

\[
Bx \geq (1+g)Ax + C(y) \quad (1)
\]

Problem II: Maximize \( yC(y) \) subject to \( y \geq 0 \) and

\[
yB \leq (1+r)yA + L. \quad (2)
\]

Supposing our economy is in a long-run steady balanced growth, the above problems may be interpreted as follows. Problem I requires that the employment should be the smallest for producing the supply which is not less than the demand, and this demand consists of input requirements in the next period and one consumption basket which is preferred when a price vector \( y \) is prevailing. Problem II requires
that the value of a consumption basket, $C(y)$, should be the largest while insureing that no process can earn more than normal profits. Note that in Problem I, $x$ and $y$ are variable vectors, while in Problem II $y$ is a variable vector and that here wages are paid at the end of each production period.

Now denote by $G$ the supremum value of the set of $g$'s for which the set of feasible vectors $x$ in Problem I is not empty for any $y \geq 0$. This $G$ is larger than $-1$, i.e., $1+G>0$, owing to the assumption A2 ($G$ may be infinite). For, by A2, there exists a number $g_0$ and a vector $x_0 \geq 0$ such that $1+g_0>0$ and $Bx_0>(1+g_0)Ax_0$. Thus, given $g=g_0$, the set of feasible $x$ in Problem I is not empty for any $y$, implying $1+G>0$. In the following, we confine ourselves to the case in which $g=r<G$.

Let us first suppose there exist solutions to the above problems. Write a solution of Problem I as $x^*$ and similarly $y^*$ for Problem II. Then, define $h(g)=1/(Lx^*)$, and $k(r)=1/(y^*C(y^*))$. These functions are well defined so far as $g$ and $r$ are smaller than $G$ and as $y^*C(y^*)$ is positive. By virtue of our choice of the unit for labour power, $Lx^*$ stands for the minimum number of workers necessary to produce input requirements and one consumption basket which has the same utility level as the standard one. Thus, $h(g)$ stand for the maximum number of consumption baskets available to a worker per year. Moreover, there exists a price vector under which those baskets are really preferred to other baskets.

On the other hand, $k(r)$ stands for the minimum number of baskets which a worker could buy using all his wages. This is because in our model the money wage is unity. Thus, $h(g)$ may be called the optimal transformation frontier (OTF) and $k(r)$ the minimum real wage frontier (MWF) in accordance with [2].

Here, we make one more assumption that the set of feasible vectors in Problem II for $r<G$ is bounded. This assumption is not so restrictive as would be expected at first sight. Now, we can state the fundamental duality theorem, whose proof will follow in the next section.

**Theorem:** $h(g)=k(r)$ for $g=r<G$.

4. **Proof**

Take any $y_0$ in $S_m$, and solve the Problem I and II where the constraint (1) is replaced by $Bx \geq (1+g)Ax + C(y_0)$ and the maximand $yC(y)$ replaced by $yC(y_0)$. Since these are now ordinary dual linear programming problems, actually we can solve them and obtain a solution pair $x_1$ and $y_1$. Then, normalize $y_1$ into $y_2$ so that $y_2$ is in $S_m$. Similarly normalize all the solutions to Problem II and write these normalized vectors as $\phi(y_0)$. Thus we have a mapping $\phi$ from $S_m$ into $S_m$.

$\phi: S_m \ni y_0 \rightarrow \phi(y_0) \in S_m$.

First let us show that $\phi(y_0)$ is closed and convex. It is well known that the set of solution vectors for a linear programming problem is closed and convex. Thus, the set of normalized solution vectors is also closed and convex, since the solution set is bounded by the assumption made in the preceding section. Next, $\phi$ is upper semi-continuous (See Appendix). Therefore, by the Kakutani fixed-point theorem
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[5], \( \phi \) has a fixed-point \( y_{00} \). Denote by \( y^* \) an actual solution (before normalization) to Problem II corresponding to \( y_{00} \). In the same way, write \( x^* \) as an actual solution to Problem I corresponding to \( y_{00} \). Then, by virtue of duality in linear programming, we have \( Lx^* = y^*C(y_{00}) \). Since \( y_{00} \) is a normalized vector of \( y^* \), we obtain \( Lx^* = y^*C(y^*) \) due to homogeneity of \( C(y) \), and their common value is positive by the assumption A1.

Now we have to show that \( x^* \) and \( y^* \) are really solutions to Problem I and II without predetermining the argument \( y \) in \( C(y) \). Take an arbitrary pair \( (x, y_0) \) which satisfies (1) and also an arbitrary \( y \) which satisfies (2). Postmultiply (2) by \( x \) and we get

\[
yBx - (1+r)yAx \leq Lx. \tag{3}
\]

Similarly, premultiply (1) by \( y \) and we have

\[
yBx - (1+g)yAx \geq yC(y_0) \geq yC(y). \tag{4}
\]

The last inequality in (4) is due to the property of \( C(y) \) noted in Section 2. Thus, by the inequalities (3) and (4), \( Lx \geq yC(y) \) since \( g=r \). This means that if \( Lx^* = y^*C(y^*) \), \( x^* \) and \( y^* \) are solutions to Problem I and II respectively. This completes the proof.

5. Some Remarks

As a straightforward corollary, we have

Corollary: If \( g_1 > g_2 \), then \( h(g_1) < h(g_2) \). Similarly, if \( r_1 > r_2 \), then \( k(r_1) < k(r_2) \). This says that if there are two balanced growth paths, one with a higher rate of growth than the other, then the former cannot enjoy a higher sustainable consumption level than the latter, and that between two balanced growth paths, one with a higher rate of profit than the other, the former cannot allow a higher real wage level than the latter. A reasonable consequence.

As for von Neumann balanced growth, existence and uniqueness, we can argue in a way similar to those in [3, 10].

Appendix

In this Appendix, we give a proof that the mapping \( \phi \) defined in Section 4 is upper semi-continuous. First let us define two symbols. \( Y \) is the set of feasible vectors \( y \) for Problem II and \( e \) is a row \( m \)-vector with each entry unity, viz., \( e = (1, 1, \ldots, 1) \). Now, suppose that \( \phi \) is not upper semi-continuous. That is, there is a sequence \( \{y_n\} \) in \( S_m \), \( y_n \rightarrow y_0 \), and a sequence \( \{\eta_n\} \), \( \eta_n \in \phi(y_n) \), \( \eta_n \rightarrow \eta_0 \), and \( y_0 \in \phi(y_0) \). Denote by \( \eta^*_n \) the actual solution for Problem II corresponding to \( \eta_n \). Since \( Y \) is closed and bounded, we have a converging subsequence of \( \{\eta^*_n\} \). Let us think that \( \{\eta^*_n\} \) is already such a converging subsequence and write the limit of this sequence as \( \eta^*_0 \). Clearly, the normalized vector of \( \eta^*_0 \) is \( \eta_0 \). Since \( \eta_0 \in \phi(y_0) \), \( \eta^*_0 \) cannot be a solution for Problem II with the parametric value \( y_0 \). On the other hand, \( \eta^*_0 \) satisfies the constraint (2) in the text because of the closedness of the set \( Y \). It follows that there exists a vector \( y \) in \( Y \) such that \( yC(y_0) > \eta^*_0C(y_0) > 0 \). Define \( \delta = yC(y_0) - \eta^*_0C(y_0) > 0 \). Since \( y \rightarrow y_0 \) and \( \eta^*_0 \rightarrow \eta^*_0 \) and each entry of \( C(y) \) is continuous,
we can find, for any positive value \( \varepsilon \), a suffix number \( k \) such that \( |C_i(y_0) - C_i(y_k)| < \varepsilon \) for each \( i \) and \( |\eta_i^* - \eta_{0i}^*| < \varepsilon \) for each \( i \). Here, the suffix \( i \) stands for the \( i \)-th element of respective vectors. Let us take \( \varepsilon \) smaller than unity. Then, we have

\[ |yC(y_k) - yC(y_0)| < \varepsilon e', \quad \text{and} \quad |\eta_0^*C(y_0) - \eta_k^*C(y_k)| < \varepsilon (\eta_0^* e' + eC(y_0) + m). \]

Therefore, if we take \( \varepsilon \) sufficiently small so that \( \varepsilon (\eta_0^* e' + eC(y_0) + m) < \delta \), we obtain \( yC(y_k) > \eta_k^* C(y_k) \). This is a contradiction to the fact that \( \eta_k^* \) is a solution for Problem II when \( C(y) \) is fixed at \( C(y_k) \).

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REFERENCES


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