QUANTITY ADJUSTMENT IN AN EXCHANGE ECONOMY*

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1. Introduction

As is well known, one of the most crucial assumptions of the Walrasian tatonnement is that every trader believes that he can buy or sell the desired quantities. If this assumption is dropped and transactions at disequilibrium are allowed to take place, then two complications will arise. First, since in a state of disequilibrium realized sales are in general different from planned supplies, realized money income will fall short of the planned one, and hence traders will not be able to finance their planned purchases. If we take this aspect into consideration we get what Leijonhufvud [7, p. 40] calls the cash constrained process. Second, if prices are not flexible enough to clear the markets and disequilibrium prevails, then traders will realize that they cannot complete their desired transactions and hence choose their demands by optimizing subject to quantity constraints. For example, a trader who cannot complete his planned purchase of coffee will increase his demand for tea. This aspect has been referred to as the spillover by Patinkin [9]. We shall call the process which incorporates this intermarket connection the spillover process.

Although the two aspects are obviously interrelated, the distinction between the cash constrained and the spillover process has not been made clear partly because most discussions have dealt with the two-good economy (labor and wage good) and partly because the discussions were verbal. In this paper we study the latter aspect in a monetary exchange economy. In order to avoid complications that will arise from the cash constraint, we shall assume that every trader has a sufficient cash balance so that he can always fill the gap between realized receipts and payments by hoarding or dishoarding of money. This is fairly restrictive, but the interesting implications which result from the cash constraint have been exploited by Howitt [5].

An explicit formulation of the spillover process has been made by Benassy [2]. Veendorp [13] has studied the stability property in a two-person, three-good (including numeraire) model. Benassy formulated a dynamic model where traders

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1) Since we shall study a constant-flow model, the reshuffle of the distribution of initial endowments among traders are not relevant here. For this point, see Negishi [8, pp. 659–65].
2) See Clower [4], and Arrow-Hahn [1, ch. 13].
observe quantity constraints by visiting the markets in turn, while in Veendorp’s model the quantity constraints (which are identical with the realized transactions in his model) are observed at the same time for all markets. Thus we have two versions of the spillover process: the sequential version and the simultaneous one. This paper studies the latter version in a H-person, n-good exchange economy.

A fixprice model of a simultaneous spillover process is constructed in Section 2. In Section 3, the stability of the process is proved under the assumption that all goods are gross substitutes in a certain natural sense. Section 4 deals with the economy in which prices change according to market excess demands, and shows that the process is also stable. Section 5 concludes the paper with a brief remark on the efficiency of equilibrium.

2. The Model

Let there be H traders indexed by h, each endowed with a collection of goods represented by the n-dimensional column vector wh. This bundle of goods flows into the economy at the beginning of each period. Consider period t. Depending on the experience of the past period, each trader has expectation about how much he can transact in the markets. We shall call this the quantity constraint and denote it by qhi(t). Each trader is supposed to optimize his utility function Uh(xh) subject both to the budget constraint and to the quantity constraint. The demand for good i is denoted by xhi(t). We let xh(t) and qh(t) represent the n-dimensional column vector with components xhi(t) and qhi(t), respectively. Thus xh(t)−wh represents the excess demand vector for trader h. We write Z(t) and Q(t) as the matrices with columns zh(t) and qh(t) respectively.

Define

\[ D_h = \{ i; z_h(t) \geq 0 \text{ for all } t \}, \quad S_h = \{ i; z_h(t) \leq 0 \text{ for all } t \}. \]

We make

Assumption 1. \( D_h \cap S_h = \emptyset, \quad D_h \cup S_h = \{1, 2, \ldots, n\}, \text{ all } h. \)

That is, whatever the quantity constraints are, the set of goods which traders demand/supply is predetermined. As is remarked by Benassy [2], this implies an extremely special distribution of resources among traders. But this assumption is not necessary for the proof of fixprice stability. A set of Assumptions 1~7 is no more than a rationalization of the Walras’ Law and Assumption 8 below.

zh(t) is the solution of the following optimizing problem:

\[
\max \ U_h(x_h + w_h) \quad \text{subject to} \begin{cases} 
0 \leq z_h \leq q_h(t), & i \in D_h \\
0 \geq z_h \geq q_h(t), & i \in S_h \\
z_h + w_h \geq 0 \\
pz_h = 0,
\end{cases}
\]

where p denotes the price vector (row vector). The market excess demand for good i, zh(t), is obtained by summing zh(t) over h.

3) The reason we impose the budget constraint will soon become clear. See footnote 4.

— 258 —
Suppose not all markets clear. Since actual transactions must sum to zero, rationing is necessary. Write $T_h(t)$ as the size of ration of good $i$ for trader $h$. The following assumptions may be quite natural.

Assumption 2. $T_h(t)$ is a continuous function of $z_{1}(t)$, $z_{2}(t)$, ..., $z_{i}(t)$, satisfying $\sum_h T_h(t) = 0$, for all $h$, $i$.

Assumption 3. $z_{h}(t) = T_{h}(t)$ for all $h$ if and only if $z_{i}(t) = 0$.

Assumption 4. $0 \leq T_{h}(t) \leq z_{h}(t)$ for $i$ in $D_h$, $0 \geq T_{h}(t) \geq z_{h}(t)$ for $i$ in $S_h$.

Assumption 2 states that rationing is made in each market independently of other markets, and depends continuously on the excess demands expressed in the market. Assumption 3 implies that each trader's plan with respect to good $i$ is brought into realization if and only if the market is in equilibrium. Assumption 4 implies the voluntary exchange. Indeed it is possible that the rationing scheme which satisfies Assumptions 2-4 violates the "financial" constraint $\sum_i p_i T_{h}(t) = 0$. This difficulty can, however, be avoided if we assume that the discrepancy between payments and receipts is filled up with the hoarding or dishoarding of money, and that the resulting unintended change in cash balance has no effect on the flow decisions.4)

Next consider period $t+1$. According to the experience of the preceding period, trader $h$ revises his quantity constraint at the beginning of period $t+1$. He already knows the values of $q_{hi}(t)$ and $z_{hi}(t)$. Taking $T_{hi}(t)$ into account, he determines $q_{hi}(t+1)$. We postulate

Assumption 5. $q_{hi}(t+1)$ depends continuously on $q_{hi}(t)$, $T_{hi}(t)$, and $z_{hi}(t)$.

As for the way $q_{hi}$ is revised, we assume as follows:

Assumption 6.

(a) $z_{hi}(t) > T_{hi}(t) \ (\geq 0) \Rightarrow q_{hi}(t) > q_{hi}(t+1) \ (\geq 0), \ i \in D_h$

(b) $z_{hi}(t) = T_{hi}(t) \ (\geq 0) \Rightarrow q_{hi}(t+1) \leq q_{hi}(t) \ (\geq 0), \ i \in D_h$

(c) $z_{hi}(t) = T_{hi}(t) \ (\leq 0) \Rightarrow q_{hi}(t+1) \geq q_{hi}(t) \ (\leq 0), \ i \in S_h$

(d) $z_{hi}(t) < T_{hi}(t) \ (\leq 0) \Rightarrow q_{hi}(t+1) < q_{hi}(t) \ (\leq 0), \ i \in S_h$.

That is, the expectation of quantity constraints becomes optimistic if the plan is brought into realization and pessimistic if the plan fails to be realized.

Now let us make

Assumption 7. $z_{h}(t)$ is a continuous function of $q_{h}(t)$.

It is readily seen that a sufficient condition for this is that the utility function $U_{h}(.)$ is continuous, strictly increasing, and strictly quasi-concave. Given Assumptions 1-7, we can summarize our spillover process formally as

(1a) $Q(t+1) = F(Q(t))$,

(1b) $Z(t) = G(Q(t))$.

By Assumptions 2, 5, and 7, $F(.)$ and $G(.)$ are continuous functions.

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4) If the cash balance is not an optimal one, there is no reason why the traders follow the budget constraint. In order to incorporate the direct effect of money balances, we must enter the cash balance into the utility function and replace the budget constraint by the new constraint that total planned purchase be less than the amount of cash balance. This is the heart of the cash constrained process, which has already been studied by Howitt [5].
Without loss of generality, we can suppose that the range of $q_{hi}(t)$ is restricted by the conditions

\[ -w_{hi} \leq q_{hi} \leq pw_{hi}/p_i \quad (i=1, 2, \ldots, n). \]

Thus $Q(t)$ belongs to a non-empty, compact, convex set for all $t$. In view of (1a), this implies that there is a fixed point $Q^*$ such that $Q^*=Q(t)=Q(t+1)$. By Assumption 6, the constancy of $q_{hi}$ implies that $z_{hi}(t)=T_{hi}(t)$, and hence by Assumption 3 $z_i(t)=0$ for all $t$. Therefore there exists a market equilibrium in our model.

3. Stability of the Spillover Process

The notion of gross substitutability has played an important role in discussions of the stability of general equilibrium. Its definition is based on price changes, but in principle substitutability depends on preference itself. We shall define $q$-gross substitutes in terms of changes in quantity constraints in the following way.

**Definition 1.** Suppose that the quantity constraint of good $j$ is binding and that of good $i$ is not. Good $i$ and good $j$ are said to be $q$-gross substitutes if a change in the quantity constraint of good $j$ makes the excess demand for good $i$ change in the opposite direction, i.e.,

\[ \frac{\partial z_{hi}}{\partial q_{hj}} < 0 \quad (i \neq j), \]

with other quantity constraints held constant.

**Example.** Consider the Cobb-Douglas utility function: $U_h(x_{h1}, x_{h2}, \ldots, x_{hn})=\sum \alpha_i \log x_{hi}$. Suppose the quantity constraints of goods 1, 2, ..., $m(<n)$ are binding. Then trader $h$ will behave as if his income were $I^*=\sum_{i=1}^n p_iw_{hi}-\sum_{j=1}^m p_jq_{hj}$ and his utility function $U_h^*(x_{h1}, x_{h2}, \ldots, x_{hn})=U_h(q_{h1}, \ldots, q_{hm}, x_{h1}, x_{h2}, \ldots, x_{hn})=\sum \alpha_i \log x_{hi}+\text{constant}$. Thus we have

\[ z_{hi}=\frac{\alpha_iI^*}{p_i}-w_i \quad \text{and} \quad \frac{\partial z_{hi}}{\partial q_{hj}} = -\frac{\alpha_i p_j}{p_i} < 0. \]

To illustrate, consider trader $h$ demanding two goods, coffee and tea. Intuitively, these two goods are substitutes. If the quantity constraint of coffee, $q_{hj}$, is binding and that of tea, $q_{hi}$, is not, then loosening of the coffee's constraint (i.e., $\Delta q_{hj}>0$) will decrease the excess demand for tea. Then, according to the above definition, tea and coffee are $q$-gross substitutes. However, this definition does not always satisfy the symmetric law. In addition, if the quantity constraint of good $i$ is binding or if that of good $j$ is not, then the definition does not make sense. The definition is not complete, but it serves.6)

Now write $\Delta z_{hi}$ as a change in trader $h$'s excess demand for good $i$ caused by a change in $q_{hi}$, other quantity constraints being held constant. Suppose that the

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5) But the converse is not true. Even if the economy is in general equilibrium, the quantity constraints can change.

6) Under a certain restrictive assumption, the present definition is equivalent to the Hicks-Allen definition of substitutes. See Tobin-Houthakker [11]. For a detailed study of the relation between the two types of substitutes, see Pollak [10].

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market for good \(j\) is in a state of excess demand. Some buyers may be able to complete their plan with respect to good \(j\). They will revise their quantity constraints of good \(j\) upwards, and, if goods \(i\) and \(j\) are \(q\)-gross substitutes, they will decrease their excess demand for good \(i\). But most buyers are not able to realize their plan. They will revise their quantity constraints of good \(j\) downwards and increase their excess demand for good \(i\) (i.e., \(\Delta z_{hi}>0\)). Thus if \(q\)-gross substitutability is dominant among the buyers, there will be an increase in their demand for good \(i\) (i.e., \(\sum \Delta z_{ha}>0\), where the summation is over the set of buyers of good \(j\)).

Next consider the sellers of good \(j\). Most sellers can sell all they want, and their \(q_{bj}\) will decrease. If \(q\)-gross substitutability is dominant among the sellers, this will induce them to increase their excess demand for good \(i\) (i.e., \(\sum \Delta z_{ha}>0\), where the summation is over the sellers of good \(j\)). Anyway, we have \(\sum \Delta z_{ha}>0\) when the market for good \(j\) is in excess demand. We can apply the same argument to the case of excess supply of good \(j\): We will have \(\sum \Delta z_{ha}<0\) if \(q\)-gross substitutability is dominant among the traders. Lastly, consider the case in which the market for good \(j\) is in equilibrium (i.e., \(z_j=0\)). Since all traders can complete their desired transactions in good \(j\), the buyers (the sellers) will increase (decrease) their quantity constraints of good \(j\). If \(q\)-gross substitutability is equally dominant among the buyers and sellers, the resulting spillovers by each trader will cancel out each other, so that the net change in the market excess demand for good \(i\) \((\sum \Delta z_{hi})\) will be negligible.

These considerations lead us to make the following

**Definition 2.** Goods \(i\) and \(j\) are said to be \(q\)-gross substitutes for the whole economy if: (a) \(z_j>0 \Rightarrow \Delta z_j>0\), (b) \(z_j<0 \Rightarrow \Delta z_j<0\), (c) \(z_j=0 \Rightarrow \Delta z_j=0\), where \(\Delta z_j=\sum \Delta z_{hi}\) denotes a change in \(z_j\) caused by a change in \(q_{hi}(h=1, 2, \ldots, H)\), other quantity constraints held constant.

Using this definition, we make the following assumptions.

**Assumption 8.** All goods are \(q\)-gross substitutes for the whole economy for all \(Q\).

**Assumption 9.** \(|\Delta z_j|<|z_j(t)|\) for all \(j, t\).

This latter assumption means that the quantity constraints of good \(j\) alone cannot alter the sign of \(z_j(t)\). Of course, the whole change in \(q_{hi}\)'s may alter the sign of \(z_j\). Assumption 9 does not go so far as to eliminate this possibility.

We are now in a position to prove

**Proposition 1.** If Assumptions 8–9 are fulfilled and if the Walras' Law (i.e., \(pz=0\) holds), then the process (1) is quasi-stable. That is, at every limit point of \(Z(t)\), a general equilibrium prevails.

**Proof.** 1) First we show \(V(t)=\sum p_i z_i(t)\) is a Lyapunov function for the process (1a). Let \(P(t)={}^c(P(t) \cap P(t+1))\), \(M=P(t) \cap P(t+1)^c\), and \(N=P(t)^c \cap P(t+1)\), where the subscript \(^c\) represents a complement. By the Walras' Law,

\[7\] The Walras' Law is deducible from the traders' budget constraints.
Note that, by Assumption 8, if \( p_i, z(t+1) - p_i, z(t) > 0 \), it must be due to the spillover from \( P(t) \) to good \( i \).

Two cases may occur: (a) \( \sum_{i \in L} p_i, z(t+1) - \sum_{i \in L} p_i, z(t) \geq 0 \), or (b) \( \sum_{i \in L} p_i, z(t+1) - \sum_{i \in L} p_i, z(t) < 0 \). First consider the case (a). This case can occur only when the spillover from \( M \) occurs. But, since \( p_i, z(t+1) - p_i, z(t) > 0 \) for all \( i \in N \), the spillover from \( M \) to \( N \) must also occur. Then, since the maximum amount of the spillover from \( j \) in \( M \) cannot exceed \( p_j, z(t) \) by Assumption 9, we have

\[
\sum_{i \in L} p_i, (z(t+1) - z(t)) + \sum_{i \in N} p_i, (z(t+1) - z(t)) < \sum_{j \in M} p_j, z(t).
\]

Since \( z(t) < 0 \) for \( i \in N \), this implies

\[
\sum_{i \in L} p_i, z(t+1) - \sum_{i \in L} p_i, z(t) + \sum_{i \in N} p_i, z(t+1) < \sum_{j \in M} p_j, z(t),
\]

and so

\[
\sum_{i \in L} p_i, z(t+1) + \sum_{i \in N} p_i, z(t+1) < \sum_{i \in L} p_i, z(t) + \sum_{i \in M} p_i, z(t).
\]

Hence \( V(t+1) < V(t) \).

Next consider the case (b). Since the spillover from \( L \) to \( N \) must also occur, we get, by the same argument as above,

\[
\sum_{i \in N} p_i, (z(t+1) - z(t)) < \sum_{i \in L} p_i, (z(t) - z(t+1)) + \sum_{i \in M} p_i, z(t).
\]

Since \( z(t) < 0 \) for \( i \in N \), this implies

\[
\sum_{i \in L} p_i, z(t+1) + \sum_{i \in N} p_i, z(t) < \sum_{i \in L} p_i, z(t+1) + \sum_{i \in M} p_i, z(t),
\]

so that \( V(t+1) < V(t) \).

From (a) and (b), we can say that \( V(t) \) is a strictly decreasing function with respect to time if and only if the economy is in disequilibrium. Since \( Z(t) \) is a continuous function of \( B(t) \), \( V(t) \) is a continuous function of \( Q(t) \). Therefore \( V(t) \) is indeed a Lyapunov function for the process (1a).

2) Since \( F(.) \) in (1a) is continuous, the solution of difference equation (1a) is uniquely determined by the initial condition \( Q(0) \) and is continuous in \( Q(0) \). In addition, \( Q(t) \) is bounded as was shown in (2). Then we can apply Uzawa’s stability theorem. \( Q(t) \) approaches to the set \( E=\{Q; \text{ at } Z=G(Q), z_i=0 \text{ for all } i\} \) arbitrarily close as time tends to infinity. Q.E.D.

4. Incorporating Price Changes

It is time we let prices change. The excess demands now depend on prices as well as on quantity constraints. The relevant signal to which prices respond should be our (constrained) excess demands \( z_i(p, Q) \), not the Walrasian (unconstrained) excess demands which are obtained by maximizing utility subject only to the budget constraint. 

Thus the price adjustment equation is:

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8) If the absolute value of quantity constraints is sufficiently large, the constrained excess demand is identical with the unconstrained one.
Assumption 10.

(3a) \[ \dot{p}_i = A_i(z_i(p, Q)), \quad i = 1, 2, \ldots, n, \quad \left( \cdot = \frac{d}{dt} \right) \]

where \( A_i \) is a continuous sign-preserving function which is proportional to \( p_i \) if \( p_i \) is very small.

In order to incorporate price changes, we must lose some generality concerning the way in which quantity constraints are adjusted. We make

Assumption 11. Each trader's expectation of quantity constraints become optimistic (pessimistic) if he is in the short (long) side of the market, and does not change if the market is in equilibrium. More specifically,

(3b) \[ q_{hi} = -B_{hi}(q_{hi}) \cdot z_i(p, Q), \quad i = 1, 2, \ldots, n; \quad h = 1, 2, \ldots, H, \]

where \( B_{hi} \) is a continuous nonnegative function of \( q_{hi} \) such that:

(a) for \( i \) in \( D_i(S_h) \), \( B_{hi} \) is proportional to \( q_{hi}(-q_{hi}) \) if \( q_{hi} \) is close to zero,

(b) \( B_{hi} \) is positive, otherwise.

The restrictions on \( A_i \) and \( B_{hi} \) ensure that \( p_i(t) \) and \( q_{hi}(t) \) do not change their signs if \( p_i(0) > 0 \) and \( q_{hi}(0) \neq 0 \), and if \( z_i \) and \( Q \) are bounded. It should be noted that Assumption 11 requires that every trader must know whether he is on the short side of the market or not.

The assumption of gross substitutability is

Assumption 12. (a) \( \left( \frac{\partial z_i}{\partial q_{hi}} \right) < 0 \), (b) \( \left( \frac{\partial z_i}{\partial p_j} \right) > 0 \), for \( i \neq j \) and \( i, j = 1, 2, \ldots, n; \ h = 1, 2, \ldots, H \), for all \( p, Q \).

An additional technical assumption is

Assumption 13. The solution of the process (3) is bounded, and unique and continuous with respect to initial conditions.

We are now able to prove

Proposition 2. If Assumptions 10 13 are fulfilled and if the Walras' Law holds, then the process (3) is quasi-stable.

Proof. Define

\[ V(t) = \sum_{i} p_i z_i = 2 \sum_{i \in P^+} p_i z_i - 2 \sum_{i \in P^-} p_i z_i, \]

where \( P^+ = \{ i; \ z_i > 0 \} \), \( P^- = \{ i; \ z_i < 0 \} \). Differentiating \( V(t) \) with respect to time and using (3), we obtain

\[ \dot{V} = \sum_h \sum_j \frac{\partial V}{\partial q_{hj}} \dot{q}_{hj} + \sum_j \frac{\partial V}{\partial p_j} \dot{p}_j \]

\[ = - \sum_h \sum_j \frac{\partial V}{\partial q_{hj}} B_{hj} z_j + \sum_j \frac{\partial V}{\partial p_j} A_j. \]

The first term can be rewritten as

\[ - \sum_h \sum_{j \in P^+} \frac{\partial V}{\partial q_{hj}} B_{hj} z_j - \sum_h \sum_{j \in P^-} \frac{\partial V}{\partial q_{hj}} B_{hj} z_j \]

\[ = - \sum_h \sum_{j \in P^+} \frac{\partial}{\partial q_{hj}} (2 \sum_{i \in P^+} p_i z_i) B_{hj} z_j - \sum_h \sum_{j \in P^-} \frac{\partial}{\partial q_{hj}} (2 \sum_{i \in P^-} p_i z_i) B_{hj} z_j. \]

9) Since the excess demand is bounded by the quantity constraints, this implies that \( z_i \) is also bounded.
Since $P^*$ and $P^-$ are non-overlapping, we have $i \neq j$ in the above expression, and by Assumption 12(a), we get
\[ \sum_h \sum_j \frac{\partial V}{\partial q_{ij}} q_{ij} \leq 0 \]
in disequilibrium, since $z_j > 0$ for $j \in P^*$ and $z_j < 0$ for $j \in P^-$. On the other hand, it is well known that, under Assumption 12(b),
\[ \sum_j \frac{\partial V}{\partial p_j} \beta_j < 0. \]
Therefore $\dot{V} < 0$ in disequilibrium. We may use the usual argument to show that $z_i \to 0$ for all $i$. Q.E.D.

5. Concluding Remarks

To conclude this study, let me point out its normative implication. It is obvious that the equilibrium of the process (1) is not Pareto efficient unless the fixed price vector happens to be the Walrasian one. Even if prices are allowed to change, equilibrium is not always Pareto efficient. If prices adjust sufficiently faster than quantities, or if the initial price vector is very close to the Walrasian one, then no quantity constraint will be binding in equilibrium. But otherwise some quantity constraint will be binding in equilibrium and the Pareto efficiency will not result.

Suppose, for example, that trader 1 demands coffee and supplies tea, and trader 2 demands tea and supplies coffee. Suppose also that trader 1 (trader 2) thinks if his quantity constraint of coffee (tea) were more loose he would supply more tea (coffee). In this case, if each trader knew the other trader's quantity constraints, they could both gain by exchanging the two goods. But in our model the markets have no mechanism to transmit such information.

Following Clower [3] and Leijonhufvud [6], [7], let us call our excess demand $z_i(p, Q)$ the effective excess demand and the Walrasian $z_i(p)$ the notional one. In our model, prices are adjusted by the effective excess demands, not by the notional ones, so that prices do not convey the “true” information. The market failure would emerge, and the economy would be trapped in an inefficient state in the case of gross substitutability.

(Notices University)

REFERENCES


