THE CUSP CATASTROPHE IN AN EXCHANGE MODEL*

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Recent developments in general equilibrium theory are centered on issues in equilibrium analysis (for example [1], [2], [3]). An economy is identified with a set of parameters and an equilibrium state is a vector of equilibrium prices. The set of pairs of economies and corresponding equilibrium prices form the equilibrium manifold. The projection from the equilibrium manifold to the set of economies, known as the Debreu mapping, describes the dependence of the equilibrium states upon the parameters. Regular and critical economies, corresponding to smooth and discontinuous or non-smooth changes, are identified as the regular and critical values of this mapping. The introduction of concepts from differential topology has led to substantial advances in the theory, such as the result that the set of critical economies is of measure zero in the space of all economies.

The framework of the analysis has an affinity with the models of catastrophe theory. The equilibrium manifold corresponds to the catastrophe manifold, the Debreu mapping to the catastrophe map and critical economies to catastrophe points.

This note presents an explicit model of an exchange economy which yields an equilibrium manifold with a cusp catastrophe. This is the most common type of catastrophe encountered in applications of catastrophe theory [5].

There are two individuals, I1 and I2, and two goods, G1 and G2. The initial endowment vectors are (0, 3) and (3, 0), for I1 and I2 respectively. $x_{ij}$ denotes the quantity of $G_j$ consumed by $I_i$, and throughout $\alpha = \log 7 / \log 2$.

The utility functions of I1 and I2 are, respectively,

$$U_1 = \int_0^{x_{11}} \frac{k^*}{t^2 + 1} dt + x_{12} \quad U_2 = \int_0^{x_{22}} \frac{k}{t^2 + 1} dt + x_{21}$$

where $1 < k^*, k < 5$. The conditions $\alpha = \log 7 / \log 2$ and $1 < k^*, k < 5$ are discussed briefly below.

An economy is identified with an admissible $(k^*, k)$. After normalization, the equilibrium states are identified with equilibrium $p$, the price of $G_1$.

Routine calculations imply that the equilibrium surface is given by $M = \{(k^*, k, p): pk - p^{s-1}(k^* - p) - 1 = 0, \quad 1 < k^*, k < 5\}$ and we are interested in its structure.

It is instructive to consider the relation

$$k = \frac{p^{s-1}(k^* - p) + 1}{p}$$

where $k^*$ is fixed. Taking $1 < k^* < 5$ and admitting only segments with $1 < k < 5$ we obtain

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Irrespective of $k^*$ the function is initially convex and then concave. The value of $p$ at which \( \frac{d^2k}{dp^2} = 0 \) varies inversely with $k^*$. For $k^* < \frac{\alpha}{\alpha - 2}$ the first derivative is always negative. For $k^* = \frac{\alpha}{\alpha - 2}$ the first derivative is negative, except at $p = 1$ where the function has the inflection point. Lastly, for $k^* > \frac{\alpha}{\alpha - 2}$ the function attains a local minimum at some $p < 1$ and a local maximum at some $p > 1$. This means that its graph folds at two points, as in Figure 1 which plots exact calculations for $k^* = 4$. Of course, given the restrictions on $k^*$, it will not be always possible for an admissible iso-$k^*$ curve to exhibit the full behaviour described here.

Returning to $M$, we want to show that it is a smooth manifold. Firstly, this follows from the regular value theorem ([4], p.11; [3], p.44). We see this by considering the manifold $G = \{(k^*, k, p): 1 < k^*, k < 5, 0 < p < \infty\}$ of dimension 3 and the smooth mapping $z = pk - p^{\alpha - 1} \cdot (k^* - p) - 1$ from $G$ to $R$. From the derivative of $z$, it is immediately obvious that 0 is a regular value and hence $M$ is a smooth manifold with $\dim(M) = \dim(G) - \dim(R) = 2$. Alternatively, we can show directly that $M$, locally, looks like $R^2$, or, more formally, that each point of $M$ has a neighbourhood $W \subset M$, where open $W \subset R^3$, diffeomorphic to an open subset $U$ of $R^2$ ([4], p.1). Let $W = \{(k^*, k, p): 1 < k^*, k < 5, 0 < p < \infty\}$ and $f: (k^*, k, p) \mapsto (k^*, p)$. Then it can be shown that $W \cap M = M$ can serve as the neighbourhood of all points in $M$, with $U = f(M)$ as the open subset in $R^2$ and $f$ as the required diffeomorphism. The projection $f$ gives a system of coordinates on $W \cap M$ and the inverse diffeomorphism $f^{-1}$ parametrizes the region $W \cap M$ by pulling back the points of $U$. Finally, in the terminology in [3], p.26, $M$ can be covered by an atlas consisting of a single chart $(M, f)$.

The graph of $M$ around $\left(\frac{\alpha}{\alpha - 2}, \frac{\alpha}{\alpha - 2}, 1\right)$ is given in Figure 2. The manifold is a ruled surface consisting of straight lines obtained by fixing $p[1]$. 
The Debreu mapping is the projection \( \pi: (k^*, k, p) \in M \mapsto (k^*, k) \) and a particular economy is identified as regular or critical depending on whether it is a regular or a critical value of \( \pi \). The catastrophe or bifurcation set, \( B \), consists of all critical values of the Debreu mapping. These are values in the inverse image of which there is at least one point where the derivative is zero (a critical point). Now the definition of the derivative of a function between two manifolds involves proceeding through charts and calculating the Jacobian matrix of a composite function from one Euclidean space to another. Here this is rather simple. In terms of the \((k^*, p)\) parametrization of \( M \) the smooth composite mapping is given by \((k^*, p) \mapsto (k^*, k, p) \mapsto (k^*, k)\) where

\[
k = \frac{p^{\alpha-1}(k^* - p) + 1}{p},
\]

and its critical points are those for which the Jacobian has rank less than 2. It follows that for a critical point it is necessary and sufficient to have \( \frac{\partial k}{\partial p} = 0 \), or, equivalently,

\[
k - (\alpha - 1)p^{\alpha-2}k^* + \alpha p^{\alpha-1} = 0.
\]

Therefore the critical values of \( \pi \) are given by the projection on the \((k^*, k)\) plane of the solution of

\[
kp - k^*p^{\alpha-1} + p^{\alpha-1} - 1 = 0
\]

\[
k - (\alpha - 1)k^*p^{\alpha-2} + \alpha p^{\alpha-1} = 0.
\]

The bifurcation set \( B \) is shown in Figure 2. The argument below justifies its shape.

The two equations imply the relation \( p^{\alpha-1} + (\alpha - 1) p = (\alpha - 2)k^* \). The left-hand-side is a strictly convex function in \( p \). It attains a minimum value, \( \alpha \), at \( p = 1 \). Hence for \( k^* < \frac{\alpha}{\alpha - 2} \) the relation has no solution in \( p \). For \( k^* = \frac{\alpha}{\alpha - 2} \) there is a unique solution \( p = 1 \) and for \( k^* > \frac{\alpha}{\alpha - 2} \) one solution larger and one smaller than 1.

Next from the two equations above we obtain
This system gives $B$ in terms of the parameter $p$.

Clearly $p = 1$ implies $k = k^* = \frac{\alpha}{\alpha - 2}$. It is also straightforward to show that $p > 1$ implies $k > k^*$ and $p < 1$ implies $k < k^*$. Further, through routine calculations we obtain $\frac{dk}{dk^*} > 0$ and $\frac{d}{dp}(\frac{dk}{dk^*}) > 0$ for both $p > 1$ and $p < 1$.

Now as $p$ increases from 1 both $k^*$ and $k$ increase with $k > k^*$. Also $\frac{dk}{dk^*}$ increases and we obtain the convex branch of $B$. On the other hand, as $p$ decreases from 1 both $k^*$ and $k$ increase with $k < k^*$. Also $\frac{dk}{dk^*}$ decreases and we obtain the concave branch of $B$. Hence the bifurcation set consists of two branches with the cusp point at $k^* = k = \frac{\alpha}{\alpha - 2}$.

$B$ is clearly of measure zero in the space of all economies, as was expected from Sard's theorem ([3], p. 42). It is also easy to see that it is the envelope of the iso-$p$ lines on the $(k^*, k)$ space.

The conditions on the parameters are thus sufficient for the equilibrium surface to be a smooth manifold with a cusp catastrophe. It can be shown that there are extensions of the control space for which $M$ does not have the desirable properties. However the restriction $1 < k^*$, $k < 5$ is not unrealistic for it implies that preferences can only vary within certain limits. With respect to $\alpha = \frac{\log 7}{\log 2}$, this has been retained from an initial model which we constructed with the aim of obtaining a smooth manifold with fold catastrophes (Figure 1).

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REFERENCES