THE SAMUELSON RECIPROCITY RELATION
IN THE JOINT PRODUCTION CASE*

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I. Introduction

Basic theories of international trade as the Samuelson Reciprocity Relation, the Stolper-Samuelson Theorem and the Rybczynski Theorem are based on the assumption of non-joint production. Several authors have explicitly investigated the applicability of these theories to the joint production case; Woodland [9] and, Jones and Scheinkman [5] denied the applicability of the Stolper-Samuelson and the Rybczynski Theorems. Chang, Ethier and Kemp [1] showed these two theories can be generalized to fit the joint production case if a composite commodity is introduced. However, they failed to show that for the strictly concave production possibility frontier (p.p.f.), the Samuelson Reciprocity Relation still holds true under joint production.

In this paper, assuming neo-classical technologies and perfect competition, we will first show the dependence of the input coefficient (matrix) and output coefficient (matrix) on commodity prices and factor endowments, where the number of industries (K) and that of primary factors (m) do not have to be equal. Second, the Samuelson Reciprocity Relation is shown to remain valid. Third, for K = m, the effects of the changes in commodity prices on the factor prices (and the dual effects of the changes in the factor endowments on the commodity outputs) are considered while it will be shown how the Stolper-Samuelson and the Rybczynski Theorems are modified. Lastly, the same effects are investigated for K < m.

In the next section, notations and assumptions are introduced.

II. Notations and Assumptions

There exist n commodities, m primary factors, and K industries.

- \( p_i > 0, i = 1, \ldots, n \): unit price of commodity i
- \( p = (p_1, \ldots, p_n) \): commodity price vector
- \( w_j \geq 0, j = 1, \ldots, m \): factor price of input j
- \( y_{ki} \geq 0, i = 1, \ldots, n \): the amount of commodity i produced by industry k
- \( y^k = (y_{k1}, \ldots, y_{kn}) \geq 0 \): output vector of industry k
- \( x^k = (x_{k1}, \ldots, x_{km}) \geq 0 \): input vector of industry k
- \( T_k \subset R^{n+m}, k = 1, \ldots, K \): the production set of industry k. It is a closed, convex cone with free disposal. \((y^k, -x^k) \in T_k\) is an element of \(T_k\) where \(y^k \geq 0\) is an output vector and \(-x^k\) is an input vector. It is assumed that \(y^k\) is finite for any finite \(x^k\).

* I am grateful for the careful reading and the invaluable comments of an anonymous referee on the earlier draft of this paper though I alone am responsible for any possible remaining errors.
\( L_j > 0, j = 1, \ldots, m \): the amount of factor endowments \( j \)
\( L = (L_1, \ldots, L_m) > 0 \): vector of factor endowments.

**Assumption 1** Perfect competition prevails in both commodity markets and primary factor markets.

Let

\[ (1) \quad y(L) = \{ y \in \mathbb{R}^n \mid y = \sum_{k=1}^{K} y^k, (y^k, -x^k) \in T_k, k = 1, \ldots, K, \sum_{k=1}^{K} x^k \leq L \} \]

be the production possibility set (p.p.s.), and

\[ (2) \quad \pi(p, L) = \sup \{ p \cdot y \mid y \in y(L) \} \]

be the value of output function.

Furthermore let

\[ (3) \quad r^k(p, x^k) = \sup \{ p \cdot y^k \mid (y^k, -x^k) \in T_k \} \]

be the value added function of industry \( k, k = 1, \ldots, K \).

Then \( r^k \) is a non-negative function which is linearly homogeneous, non-decreasing, convex and continuous in \( p \), and linearly homogeneous, non-decreasing, concave and continuous in \( x^k \). Now let

\[ C^k(w, y^k) = \inf \{ w \cdot x^k \mid (y^k, -x^k) \in T_k \} \]

be the cost function of industry \( k, k = 1, \ldots, K \).

It is a non-negative function which is linearly homogeneous, non-decreasing, concave, continuous and differentiable in \( w \), and linearly homogeneous, non-decreasing, convex and continuous in \( y^k \). (For these properties of the value added function and cost function, see Shephard [8] and Sakai [7].) Next following Woodland [9] and Diewert [3], the unit value added function of industry \( k \) \( G^k(p, w) \) is introduced:

\[
G^k(p, w) = \begin{cases} 
\min \{ w \cdot x^k \mid r^k(p, x^k) > 1 \} & \text{if } p \in Q_k \\
+\infty & \text{if } p \notin Q_k
\end{cases}
\]

where

\[ Q_k = \{ p \in \mathbb{R}_+^n \mid x^k > 0, \exists r^k(p, x^k) > 0 \}. \]

But since \( r^k(p, x^k) \) is non-decreasing and linearly homogeneous in \( x^k \), it is rewritten as

\[ (6) \quad Q_k = \{ p \in \mathbb{R}_+^n \mid r^k(p, x^k) > 0, x^k > 0 \} \]

Then \( G^k \) is

(i) a positive extended real valued function defined for all \( w > 0 \) and \( p \geq 0 \),

(ii) continuous from above and quasi-concave in \((p, w)\),

(iii) homogeneous of degree \(-1\) in \( p \), and

(iv) non-decreasing and linearly homogeneous in \( w \).

If \( p \notin Q_k \), then industry \( k \) suffers losses if it produces at all. Hence it produces nothing.

**Assumption 2** The commodity price vector \( p > 0 \) is contained in

\[ (7) \quad Q = \bigcap_{k=1}^{K} Q_k \neq \emptyset. \]

That is, the price vector \( p > 0 \) is constrained so that all industries operate at a positive level.

**Assumption 3** \( G^k(p, w) \) is continuously differentiable in \((p, w)\), and \( r^k(p, x^k) \) is continuously differentiable in \((p, x^k)\), \( k = 1, \ldots, K. \)

Let
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(8) \( z_k = r^k(p, x^k) \)
and denote \( z_k \) to be the activity level of industry \( k \), which is equal to the value added of its industry.

Since \( T_k \) is a cone, we obtain

(9) \( r^k(p, x^k) = C_k(w, y^k) \).

Let \( a_k = (a_{k1}, \ldots, a_{kn}) \), \( b_k = (b_{k1}, \ldots, b_{km}) \) with

(10) \( a_{ki} = y_{ki}/z_k, b_{kj} = x_{kj}/z_k, i = 1, \ldots, n, j = 1, \ldots, m, k = 1, \ldots, K. \)

Denote \( a_k, b_k \) to be respectively the output coefficient vector and input coefficient vector of industry \( k \). Further, denote \( A = \begin{bmatrix} a_1 \\ \vdots \\ a_K \end{bmatrix} \) and \( B = \begin{bmatrix} b_1 \\ \vdots \\ b_K \end{bmatrix} \) as the output coefficient matrix and input coefficient matrix respectively. Then by definition it follows that

(11) \( 1 = r^k(p, b_k) = G^k(p, w) \) for \( p \in Q \)
and

(12) \( \partial G^k(p, w)/\partial p_i = -a_{ki}, \partial G^k(p, w)/\partial w_j = b_{kj}, \)
(13) \( y^k = z_k \cdot a_k, x^k = z_k \cdot b_k, i = 1, \ldots, n, j = 1, \ldots, m, k = 1, \ldots, K. \)

III. Results
First we show in Theorem 1 the dependency of the input and output coefficients on the commodity prices \( p \) and factor endowments \( L \).

Theorem 1
\[ \{x \in \mathbb{R}^n | \pi(p, x) \geq 1\} = H[\bigcup_{k=1}^{K} \{x^k | r^k(p, x^k) \geq 1\}] \]
where \( H(A) \) is the convex hull of the set \( A \).

Proof of Theorem 1 is shown in Appendix I.

For \( r^k \)'s and \( C^k \)'s, it is known that

Lemma 1
(14) \( \partial r^k(p, x^k)/\partial x^k_j = w_j, \)
(15) \( \partial r^k(p, x^k)/\partial p_i = y^k_i, \)
(16) \( \partial C^k(w, y^k)/\partial y^k_i = p_i, \)
(17) \( \partial C^k(w, y^k)/\partial w_j = x^k_j, i = 1, \ldots, n, j = 1, \ldots, m, k = 1, \ldots, K. \)

Here the p.p.f. \( \mathcal{Y}(L) \) is defined to be the set of efficient output vectors \( y = (y_1, \ldots, y_n) \in \mathcal{Y}(L) \) — i.e., the set of output vectors \( y \ni \) : there exists no \( \tilde{y} \) with \( \tilde{y} \geq y \) (\( \tilde{y}_i \geq y_i, \forall i \), but \( \tilde{y} \neq y \) and \( \tilde{y} \in \mathcal{Y}(L) \). —

Theorem 1 shows how \( a_k \) and \( b_k \) are determined. To gain an intuitive idea of Theorem 1, refer to Fig. 1, which illustrates the case of \( K = m = 2 \). The shaded area shows the convex hull \( H[\bigcup_{k=1}^{2} \{x^k | r^k(p, x^k) \geq 1\}] \). As long as the factor endowments vector \( \lambda L \) (\( \lambda > 0 \) is an arbitrary scalar) lies in \( AOB \) (the diversification cone) the tangent points \( A, B \) of the plane \( \pi(p, x) = 1 \) with the convex hull determine \( b_1, b_2 \) respectively. And the normal to the plane \( \pi(p, x) = 1 \) determines the vector of factor prices \( \mu w' = w \) (\( \mu > 0 \) is an arbitrary scalar) from

1) and 2) For these results, see Diewert and Woodland [4].
Eq. (14), and $1 = G^k(p, w)$ determines the value of the scalar $\mu$. Hence in this case, the $w_j$'s, and $b_k$'s, $a_k$'s (determined from Eq. (12)) depend only on $p$. But if the factor endowments vector $\lambda L$ lies outside $AOB$, then $b_k$'s and $a_k$'s depend not only on $p$ but also on $L$. Hence in general, the $w_j$'s, $b_k$'s, and $a_k$'s depend on $p, L$. Here it is easy to see that the above arguments follow with arbitrary numbers of $K$ and $m$. Denote the dependency of the output coefficient matrix and input coefficient matrix $A, B$ on $p, L$ as

$$(18) \quad A = A(p, L), \quad B = B(p, L).$$

When the p.p.f. is strictly concave to the origin, $y$ and hence $z$ are determined by $p, L$. Therefore in this case we can denote $y = \bar{y}(p, L)$ and $z = \bar{z}(p, L)$. Now we consider the Samuelson Reciprocity Relation.

**Theorem 2**

Let $\tilde{y}(L)$ be strictly concave toward the origin. Then

(i) $\partial y_i / \partial p_i = y_i, \partial y_i (p, L) / \partial L_j = w_j, i = 1, \ldots, n, j = 1, \ldots, m.$

(ii) Further if $y(p, L)$ is twice continuously differentiable in $(p, L)$, then $\partial y_i / \partial p_j = \partial y_j / \partial p_i, \partial w_j / \partial L_i = \partial w_i / \partial L_j, \partial y_i / \partial L_j = \partial w_j / \partial p_i, (\partial y_i / \partial p_j)$ is positive semi-definite and $(\partial w_j / \partial L_i)$ is negative semi-definite (hence in particular $\partial y_i / \partial p_i \geq 0$ and $\partial w_j / \partial L_j \leq 0$). \(^3\)

**Proof**

Since $\pi(p, L) = \sum_{k=1}^{K} \tilde{r}^k(p, \tilde{x}(p, L))$ where $x^k = \tilde{x}^k(p, L)$ is defined from $x^k = z_k b_k$ and $z = \tilde{z}(p, L)$, it follows that from Lemma 1 $\partial \pi / \partial p_i = \sum_{k=1}^{K} (\partial \tilde{r}^k / \partial p_i + \sum_{j=1}^{m} \partial \tilde{x}^j_i / \partial x^j_i / \partial p_i) = \sum_{k=1}^{K} \tilde{r}^k_i + \sum_{j=1}^{m} w_j \cdot \partial x^j_i / \partial p_i$. But since $\sum_{k=1}^{K} \tilde{x}^k_j (p, L) = L_j, j = 1, \ldots, m, \sum_{k=1}^{K} \partial x^j_i / \partial p_i = 0$.

\(^3\) $(\partial y_i / \partial p_j)$ is a matrix of order $(n \times n)$ whose $(i, j)$ element is $\partial y_i / \partial p_j$.

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Hence $\partial \pi / \partial p_i = y_i$. Similarly $\partial \pi / \partial L_j = \sum_{k=1}^{K} \sum_{m=1}^{M} \partial \pi / \partial x^k_i \cdot \partial x^k_i / \partial L_j = \sum_{k=1}^{K} \sum_{m=1}^{M} w_{ij} \cdot \partial x^k_i / \partial L_j = \sum_{j=1}^{J} w_{ij} \cdot \delta_{ij}$ since $\sum_{i=1}^{I} \partial x^k_i / \partial L_j = \delta_{ij}$ (Kronecker’s delta).

For (ii), we obtain from (i) and the concavity of support function in $p$ and the convexity in $L$ (for these properties, see Appendix II)

$\partial^2 \pi / \partial p_i^2 = \partial y_i / \partial p_i \geq 0$ from the convexity of support function in $p$,

$\partial^2 \pi / \partial L_j^2 = \partial w_j / \partial L_j \leq 0$ from the concavity of support function in $L$,

$\partial^2 \pi / \partial p_i \partial p_i = \partial^2 \pi / \partial p_i \partial p_i$ implies $\partial y_i / \partial p_j = \partial y_j / \partial p_i$,

$\partial^2 \pi / \partial L_j \partial L_j = \partial^2 \pi / \partial L_j \partial L_j$ implies $\partial w_i / \partial L_j = \partial w_j / \partial L_i$,

$\partial^2 \pi / \partial L_j \partial p_i = \partial^2 \pi / \partial p_i \partial L_j$ implies $\partial y_i / \partial L_j = \partial w_j / \partial p_i$.

Further $(\partial y_i / \partial p_j)$ is positive semi-definite from the convexity of support function in $p$, and $(\partial w_i / \partial L_j)$ is negative semi-definite from the concavity of support function in $L$.

Theorem 2 states that the Reciprocity Relation also holds in the case of joint production as long as the p.p.f. is strictly concave toward the origin. The next corollary further investigates how the Stolper-Samuelson Theorem (or its dual, the Rybczynski Theorem) is modified in case of joint production.

**Corollary 1**

Assume $K = m$, and

\[
\begin{vmatrix}
11 & \cdots & 1m \\
\vdots & \ddots & \vdots \\
1k & \cdots & km
\end{vmatrix}
\neq 0. \text{ Then,}
\]

\[
\frac{\partial w_k}{\partial p_i} = \sum_{j=1}^{J} b_{kj} a_{ji} = \begin{vmatrix}
11 & \cdots & 1k & a_{11} b_{1k+1} & \cdots & 1m \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
11 & \cdots & km & a_{1m} b_{mk+1} & \cdots & km
\end{vmatrix}
\]

where $(b_{kj}) = (b_{kj})^{-1}$, and $a_{ji} = y_{ij} / z_j$.

**Proof**

From Eq. s (11) and (12) it follows that $A(p, w) dp = B(p, w) dw$.4)

Corollary 1 was obtained also by Woodland [9] and Chang et al. [1].

As an example consider the simplest case where $K = m = 2$. Then

\[
\frac{\partial w_1}{\partial p_1} = \begin{vmatrix}
a_{11} b_{12} & b_{12} & b_{11} & b_{12} \\
a_{21} b_{22} & b_{22} & b_{12} & b_{22}
\end{vmatrix}, \quad \frac{\partial w_2}{\partial p_1} = \begin{vmatrix}
b_{11} a_{11} b_{12} & b_{11} & b_{12} \\
b_{21} a_{21} b_{22} & b_{21} & b_{22}
\end{vmatrix},
\]

\[
\frac{\partial w_1}{\partial p_2} = \begin{vmatrix}
a_{11} b_{12} & b_{12} & b_{11} & b_{12} \\
a_{21} b_{22} & b_{22} & b_{12} & b_{22}
\end{vmatrix}, \quad \frac{\partial w_2}{\partial p_2} = \begin{vmatrix}
b_{11} a_{12} b_{12} & b_{11} & b_{12} \\
b_{21} a_{22} b_{22} & b_{21} & b_{22}
\end{vmatrix}.
\]

Hence if industry $i$ uses input $i$ more intensively (i.e., $b_{1i} / b_{12} > b_{2i} / b_{22}$), then e.g., $\partial w_1 / \partial p_1 > (>) 0$ iff $a_{1i} b_{12} > (>) a_{2i} b_{22}$. That is, $\partial w_1 / \partial p_1$ is positive (negative) iff industry 1 produces output 1 relatively more (less) intensively with respect to input 2 (i.e., it produces more (less) output 1 per unit of input 2) than industry 2 does. Therefore the results obtained here are radically different from the non-joint production case; not only input coefficients but also output coefficients have to be considered. Here in Corollary 1 the Reciprocity Relation holds so that

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4) For $K = m$, $w$ depends only on $p$. --- 35 ---
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Next we consider the case $K < m$.

**Theorem 3** Assume that $\tilde{y}(L)$ is strictly concave toward the origin, $K < m$, $G^k$ is twice continuously differentiable in $(p, L)$, and the matrix $C = \begin{pmatrix} (\xi_{jt})_{K \times 1} & (b_k)_{1 \times 1} \\ (b_k)_{1 \times 1} & 0 \end{pmatrix}$ is non-singular$^5)$ where

$$\xi_{jt} = \sum_{k=1}^{K} z_k b_{kjt}, \quad b_{kjt} = \frac{\partial^2 G^k}{\partial w_j \partial w_t}, \quad j, t = 1, \ldots, m.$$

Then

$$\left(\frac{\partial w_j}{\partial p_i}\right) = N' A - M \rho'$$

where

$$\begin{bmatrix} -I_m & (b_k) \end{bmatrix}^{-1} = \begin{bmatrix} M & N' \\ N & R \end{bmatrix},$$

$A = (a_{ik})$, and $\rho = (\rho_{ij})$, $\rho_{ij} = \sum_{k=1}^{K} z_k c_{kuj}$, $c_{kuj} = \frac{\partial^2 G^k}{\partial p_i \partial w_j}$.

**Proof of Theorem 3** is shown in Appendix III.

In the case of non-joint production if $n < m$, the weak Stolper-Samuelson holds, i.e., a rise in the price of one commodity raises the price of at least one of the factors and lowers the price of at least one of the other factors as is shown by Jones and Scheinkman $^5$. But in the case of joint production if $K < m$, even this weak Stolper-Samuelson does not necessarily hold because $M = 0$ does not always hold.$^6$)

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**Appendix I**

**Proof of Theorem 1**

(i) $\{x \in R^n_+ | \pi(p, x) \geq 1\} \subseteq H[\bigcup_{k=1}^{K} \{x^k | r_k(p, x^k) \geq 1\}]$. Let $x \in H[\bigcup_{k=1}^{K} \{x^k | r_k(p, x^k) \geq 1\}]$. Then $x = \sum_{k \in K} \lambda_k x^k$ where $\lambda_k \geq 0$, $\sum_{k \in K} \lambda_k = 1$, $R \subseteq N = \{1, \ldots, K\}$. Now let $p \cdot y^k = r_k(p, x^k) \geq 1$, with $\lambda_k (y^k, -x^k) \in T_k, \quad k \in R$. Since $\left( \sum_{k \in R} \lambda_k y^k, -x \right) \in \tilde{y}(x)$, it follows that $\pi(p, x) \geq p \cdot \sum_{k \in R} \lambda_k y^k = \sum_{k \in R} \lambda_k r_k(p, x^k) \geq \sum_{k \in R} \lambda_k = 1$.

(ii) $\{x \in R^n_+ | \pi(p, x) \geq 1\} \subseteq H[\bigcup_{k=1}^{K} \{x^k | r_k(p, x^k) \geq 1\}]$. Let $\tilde{x} \in \{x \in R^n_+ | \pi(p, x) \geq 1\}$, and $\pi(p, \tilde{x}) = \hat{\pi} \geq 1$. Then $\hat{x} \subseteq \tilde{x}$ where $\hat{x} = \sum_{k=1}^{K} \lambda_k x^k$ and $\hat{\pi} = p \cdot \sum_{k=1}^{K} y^k$ with $p \cdot y^k = r_k(p, \tilde{x}^k), \quad k = 1, \ldots, K$. Then $p \cdot \sum_{k=1}^{K} y^k = \hat{\pi}$. Define $\lambda_k = p \cdot y^k / \hat{\pi} \geq 1$ and $x^k = \frac{\hat{\pi} \tilde{x}^k}{p \cdot y^k}, \quad k = 1, \ldots, K$. Hence $\sum_{k=1}^{K} \lambda_k x^k = \sum_{k=1}^{K} \tilde{x}^k = \tilde{x}$, and $r_k(p, x^k) = r_k(p, \tilde{x}^k / \lambda_k) = \frac{1}{\lambda_k} r_k(p, \tilde{x}^k) = \frac{\hat{\pi}}{p \cdot y^k} r_k(p, x^k) = \hat{\pi} \geq 1$. Hence $r_k(p, x^k) \geq 1, \quad k = 1, \ldots, K$.

**Appendix II**

First we show that the support function $\pi$ is convex in $p$. Let $\pi(p, L) = p \cdot y^0, \pi(p', L) = p' \cdot y'$ and $\pi(p + p', L) = (p + p') \cdot y'$. Then by definition

$$\begin{align*}
(A.2.1) \quad \pi(p + p', L) &= p \cdot y' + p' \cdot y' \leq p \cdot y^0 + p' \cdot y' = \pi(p, L) + \pi(p', L).
\end{align*}$$

$^5$) The matrix $C$ is non-singular almost everywhere since Eq. s (A.3.1) and (A.3.2) together determine $z_k$'s and $w_i$'s as functions of $p, L$ (which is made possible by the strict concavity of the p.p.f.). For this, see Katzner [6], Lemma B. 5-4 (p. 202).

$^6$) I owe the anonymous referee for pointing this out.
Further, it is easy to see that
\[ \lambda \pi(p, L) = \pi(\lambda p, L) \quad \forall \lambda \in R_+^1. \]
Eq. s (A.2.1) and (A.2.2) together imply the convexity of \( \pi \) in \( p \). Next we show that \( \pi \) is concave in \( L \). Let \( \pi(p, L) = p \cdot y^0, \pi(p, L') = p \cdot y' \) and \( \pi(p, L + L') = p \cdot y'' \). Next let \( y^0 = \sum_{k=1}^{K} y^k \) with \( (y^k, -x^k) \in T_k, k = 1, \ldots, K, \) and \( y' = \sum_{k=1}^{K} y'^k \) with \( (y'^k, -x'^k) \in T_k, k = 1, \ldots, K \). Then \( \sum_{k=1}^{K} x^k = L, \sum_{k=1}^{K} x'^k = L' \) and \( \sum_{k=1}^{K} x''^k = L + L' \). Since \( (y^k + x^k, -x^k) \in T_k \) as \( T_k \) is a convex cone, \( k = 1, \ldots, K, \) \( y^0 + y' \in \mathcal{Y}(L + L') \). Hence
\[ p(y^0 + y') \leq p \cdot y'' \] or
\[ (A.2.3) \quad \lambda \pi(p, L + L') \leq \pi(p, L) + \pi(p, L'). \]
Next we show \( \pi(p, \lambda L) = \lambda \pi(p, L) \forall \lambda \in R_+^1 \). In fact let \( \pi(p, \lambda L) = p \cdot y^0 \) with \( y^0 = \sum_{k=1}^{K} y^k \), \( (y^k, -x^k) \in T_k, k = 1, \ldots, K \), \( y' = \sum_{k=1}^{K} y'^k \) with \( (y'^k, -x'^k) \in T_k, k = 1, \ldots, K \), \( \lambda \sum_{k=1}^{K} x^k = \lambda L \). Then since \( \lambda(y^k, -x^k) \in T_k, k = 1, \ldots, K \), and \( \lambda \sum_{k=1}^{K} x^k = \lambda L \), \( \lambda \sum_{k=1}^{K} y^k \in \mathcal{Y}(\lambda L) \). Hence \( \lambda \pi(p, \lambda L) \leq p \cdot y^0 \) or \( \lambda \pi(p, \lambda L) \leq \pi(p, \lambda L) \). On the other hand, since \( (y'^k/\lambda, -x'^k/\lambda) \in T_k, k = 1, \ldots, K \) and \( \sum_{k=1}^{K} x^k/\lambda = L, \sum_{k=1}^{K} y^k/\lambda \in \mathcal{Y}(L) \). Hence \( p \cdot (y/\lambda) \leq p \cdot y \) or \( \pi(p, \lambda L) \leq p \cdot \lambda \pi(p, L) \). In short
\[ (A.2.4) \quad \lambda \pi(p, L) = \pi(p, \lambda L) \quad \forall \lambda \in R_+^1. \]
Eq. s (A.2.3) and (A.2.4) together imply the concavity of \( \pi \) in \( L \). 

Appendix III

By totally differentiating the following equation
\[ \sum_{k=1}^{K} z_k b_{kj} = L_j, \quad j = 1, \ldots, m \]
we obtain
\[ (A.3.1) \quad \sum_{k=1}^{K} b_{kj} dz_k + \sum_{i=1}^{m} \xi_i dw = dL_j - \sum_{i=1}^{m} \rho_{ij} dp_i, \quad j = 1, \ldots, m. \]
Next by totally differentiating Eq. (11), we observe
\[ (A.3.2) \quad (a_{ki}) dp = (b_{kj}) dw. \]
From Eq. s (A.3.1) and (A.3.2)
\[ (A.3.3) \quad \left[ \begin{array}{c} (\xi_i) \\ (b_{kj}) \end{array} \right] \left[ \begin{array}{c} dw \\ dz \end{array} \right] = \left[ \begin{array}{c} dL - (\rho_{ij}) dp_i \\ (a_{ki}) dp \end{array} \right] \]
where \( (b_{kj}) \) is of order \( (K \times m) \), and \( (\xi_i) \) of order \( (m \times m) \), hence \( C \) is of order \( ((m + K) \times (m + K)) \).

Now by totally differentiating the equation \( y^k_i = a_{ki} z_k \), we obtain \( dy^k_i = a_{ki} dz_k - z_k \sum_{j=1}^{m} a_{kij} dp_i \). \( dp_i - z_k \sum_{j=1}^{m} c_{kij} dw_j \), which implies \( dy_i = \sum_{k=1}^{K} dy^k_i = \sum_{k=1}^{K} a_{ki} dz_k - \sum_{j=1}^{m} \eta_{ij} dp_i \). \( - \sum_{j=1}^{m} \rho_{ij} dw_j \), where
\[ \eta_{ii} = \sum_{k=1}^{K} z_k a_{kin}, \quad a_{kin} = \frac{\partial^2 G^k}{\partial p_i \partial p_j}. \]
(A.3.4)  \[ dy = (a_k)' dz - (\eta_{ij}) dp - (\rho_{ij}) dw. \]

Now from Eq. (A.3.3), it follows that \( (dz/dL_j) = (dz/dL) = N, \)
\( (dw_i/dL_j) = (dw/dL) = M. \)

Hence from Eq. (A.3.4) by letting \( dp = 0, \)

\[ dy/dL = A'N - \rho M = (dw/dp)'. \]

REFERENCES


