END-OF-PERIOD AND BEGINNING-OF-PERIOD SPECIFICATIONS OF ASSET EQUILIBRIUM AND BALANCE SHEET IDENTITY*

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This paper analyzes the structures of end-of-period and beginning-of-period specifications of asset equilibrium under portfolio adjustment costs and intertemporal optimization motives, by distinguishing effective from notional asset demands and planning horizon from market period length. We show: (1) The expectations functions in discrete time converge to continuously differentiable functions of time in the continuous time limit. (2) Quasi end-of-period equilibrium is a relevant notion to deal with effective asset demands. (3) The two specifications are equivalent in continuous time only if the perfect foresight condition is satisfied. (4) The balance sheet identity and the flow budget constraint are equivalent.

1. Introduction

A major problem in macro/monetary economics concerns a choice between end-of-period (EOP) and beginning-of-period (BOP) equilibrium specifications of asset market equilibrium.1) Since Foley's work (1975), the structures of the two specifications have been scrutinized both in discrete and continuous time together with corresponding conservation laws (e.g., Buiter and Woglom (1977), Turnovsky and Burmeister (1977), Turnovsky (1977), Karni (1978), Buiter

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1) According to Foley's characterization, the end-of-period equilibrium refers to a situation in which demands and supplies are offered as of the end of the period with contracts made for labor, capital services, and consumption during the period and with asset deliveries made at the end (1975, p. 309). On the other hand, the beginning-of-period equilibrium refers to a situation in which trading in and delivery of assets take place at the beginning of the period with within-period production of new capital carried out on a kind of speculation and with within-period consumption contracted for (1975, p. 310). That is, while the end-of-period equilibrium deals exclusively with future trading, the beginning-of-period equilibrium is based on spot trading in assets. Karni (1978), however, views this characterization unnecessarily restrictive, and suggests that spot trading in existing assets be allowed in studying the implication of the existence of future markets. He thus considers two alternative institutional setups, one allowing spot trading in assets and future trading in consumption goods and the other allowing in addition future trading in investment goods. We share Karni's view.
Yet, all of the works in the literature have not distinguished the planning horizon of a decision maker from the market period length, hence leaving open the implications of a multi-period planning horizon on the equivalence question; nor have they examined the implications of portfolio adjustment costs on how EOP and BOP equilibrium conditions are to be specified except Foley (1975). Likewise, May's interpretation of a balance sheet identity, which is now so commonly accepted, has not been subjected to rigorous scrutiny for its validity.

Thus, the purpose of this paper is four-fold: First, we argue that the planning horizon of a decision maker should be clearly distinguished from market period length. A decision maker, participating in market transactions in discrete periods, is best viewed as an intertemporal optimizer with his planning horizon significantly longer than the market period length. This means that price expectations must be extended beyond current and immediate future periods.

Second, if portfolio adjustment costs are present, it is essential to distinguish between effective and notional asset demands. The former quantities refer to what is intended to be held by the end of the period when portfolio adjustment costs are taken into account, whereas the latter to what would be intended if such costs were absent. By the requirement that the physical market period length be irrelevant to essential outcomes of period analysis, we suggest that effective asset demands be used in specifying conservation laws and EOP equilibrium conditions. Hence, we introduce the notion of quasi EOP equilibrium to deal with equilibrium under portfolio adjustment costs.

Third, the equivalence of BOP and EOP equilibrium specifications is a valid question only in the context of no portfolio adjustment costs; this is because BOP equilibrium is possible only if such costs do not exist. In this regard, we establish that the two specifications are equivalent in continuous time only if the perfect foresight condition is satisfied. Thus, we verify Foley's conjecture in his reply to Buiter and Woglom (1977) that "these two forms are consistent with each other only when there is perfect foresight" (1977, p. 402).

Fourth, while admitting that an ex-ante household budget constraint transforms, in the continuous time limit, into two separate constraints, i.e., a balance sheet identity and a flow budget restraint, we question May's (1970) claim that the former contains additional information not included in the latter in that not all asset stock equilibrium conditions can be independent. We show that a balance sheet identity contains exactly the same information as a household flow budget constraint. May's interpretation of a balance sheet identity leads to an impasse that the system would not contain enough independent equilibrium relations to determine its equilibrium position. This helps us identify some sources of confusion in May's (1970) critique and Karni's (1978) defense of Patinkin's macro model; we claim that Patinkin's conclusions can be reinstated in continuous as well as discrete time.

2. Expectations and Quasi EOP Equilibrium

Suppose that adjustment costs are incurred in altering the composition of asset portfolios so that they are not generally balanced at any given point in time. Because the notion of BOP equilibrium precludes the existence of portfolio imbalances that cannot be corrected instantly, only the notion of EOP equilibrium is relevant to deal with this case. Foley (1975), however, warns...
that in such a case any EOP equilibrium specification, which is written with notional demands for assets and with an arbitrary market period length, is ‘ill-formed’ in the sense that a continuous time analogue of such a specification does not exist.2) Behind this warning lies a methodological precept that if a natural period is not identified, then a period model should be well defined in the sense (1) that its outcomes are not affected by the period length, (2) that a continuous time analogue exists and is meaningful, and (3) the outcomes are preserved in the continuous time limit. Foley suggests that this market period irrelevance test be applied to period models on a routine basis to see how well they are defined. We, therefore, introduce here the notion of effective asset demands and of quasi EOP equilibrium.

Our analysis is carried out in the context of Foley’s (1975) model with two goods: consumption goods, $Q_C$, and investment goods, $Q_I$, and with two assets: money, $M$, and capital, $K$. The prices of money and investment goods and the rental rate of capital at time $t$, expressed in terms of consumption goods, are denoted by $p_m(t)$, $p_k(t)$, and $r(t)$, respectively. Under the assumption of full employment and full mobility of factors between the consumption and the investment goods sectors, the amounts of the two types of goods to be produced are determined as functions of the total stock of capital and the price of the investment goods, i.e., $Q_C = Q_C[K(t), p_k(t)]$ and $Q_I = Q_I[K(t), p_k(t)]$; the rental rate of capital is also determined as a function of the price of investment goods, i.e., $r(t) = f[p_k(t)]$ (see Foley and Sidrauski (1971, chapter 2)).

The market period is $h$ time units in length. For comparison purposes, we work with a simple time structure of asset demands employed by Foley (1975), Turnovsky (1977), Buiter and Woglom (1977), Turnovsky and Burmeister (1977), and Hayakawa (1982, 1984), and not with a more elaborate structure proposed by Buiter (1980) with distinctions among market period, delivery interval, and forecast (or planning) interval. Hence, asset demand functions are planned at the beginning of each period with reference to the end of the same period.

We distinguish the planning horizon of an agent from the market period length. To avoid any arbitrary cut-off point, the planning horizon is assumed to be infinite. We will not specify any particular intertemporal optimization scheme underlying an agent’s demands for assets. Dwelling on the property that the optimal solution to an intertemporal optimization problem is a function of the current values and the future expected paths of prices (as well as those of other relevant parametric variables), we take a general approach of considering price expectations and expected rates of return that extend indefinitely into the future in asset demands; these expectations consist of the expected prices of money and investment goods, the expected rental rate of capital, and the real expected rates of return from holding money and capital.

The time structure of price expectations should be consistent with that of asset demand functions, but with one important difference. Because the planning horizon is infinite, expectations

2) Foley’s concern in this respect is explicit in the following statement:

The problem can be seen intuitively. The household, in planning its end-of-period stock of capital, has two goals in mind: first, to correct its current portfolio imbalances and, second, to allow for its planned saving over the period. The hidden assumption is that the household adjusts its portfolio imbalance completely within the period. If the period becomes short, the size of this adjustment term, since it stays constant in absolute terms, overwhelms the flow of investment and saving.

—Foley (1975, p. 312)—
of the prices of money and investment goods are formed at time $t$ (the time of decision making) with reference to the beginning time points of all future periods, $t + nh$, $n = 1, 2, 3, \ldots$; such expectations are denoted, respectively, by $p^*_m(t + nh, t)$ and $p^*_k(t + nh, t)$. Note that the expected rental rate of capital formed at time $t$ in reference to time $t + nh$, i.e., $r^*_k(t + nh, t)$, is related to the expected price of investment goods, $p^*_m(t + nh, t)$, via $r^*_k(t + nh, t) = f[p^*_k(t + nh, t)]$, $n = 1, 2, 3, \ldots$. Likewise, the real expected rates of return from holding money and capital are formed at time $t$ with reference to the beginning time points of all future periods; they are denoted by $\rho^*_m(t + nh, t)$ and $\rho^*_k(t + nh, t)$, $n = 1, 2, 3, \ldots$, and are defined by

1. $\rho^*_m(t + nh, t) = [p^*_m(t + nh, t) - p^*_m(t + (n - 1)h, t)]/[hp^*_m(t + (n - 1)h, t)]$,
2. $\rho^*_k(t + nh, t) = r^*_k(t + nh, t)/p^*_k(t + nh, t)$

$\rho^*_k(t + nh, t) = p^*_k(t + nh, t) - p^*_k(t + (n - 1)h, t)]/[hp^*_k(t + (n - 1)h, t)]$.

Because expectations for each of the future periods, by construction, remain constant throughout that period, the expectations of price variables may be viewed as step functions of time. For instance, the expectations of the price of money can be considered as a step function, $P^*_m(\tau, t)$, $\tau \geq t$ (see Figure 1), defined as

$P^*_m(\tau, t) = \begin{cases} p^*_m(t, t) & \text{for } t \leq \tau < t + h \\ p^*_m(t + h, t) & \text{for } t + h \leq \tau < t + 2h \\ p^*_m(t + 2h, t) & \text{for } t + 2h \leq \tau < t + 3h \\
\vdots & \end{cases}$

We can define the expectations of the price of investment goods, the rental rate of capital, and the real rates of return from holding money and capital in a similar manner. They are denoted by $P^*_k(\tau, t)$, $R^*_k(\tau, t)$, $D^*_m(\tau, t)$, and $D^*_k(\tau, t)$, respectively. Such expectations, treated as functions of future time, shall be called the expectations functions.

Notice that the expectations functions do not collapse to a single value at time $t$ as the market period length is taken to zero. What happens is the following: As the market period is refined, expectations will have to be formed with reference to each of the refined periods, so that steps or discontinuities in the expectations functions will gradually smooth out; in the limit as $h \to 0$,
such functions will, most likely, become continuous functions of future time $\tau$. It is conceivable for the expectations functions still to have some discontinuities in the limit as $h \to 0$, but we assume that they transform to continuously differentiable functions of future time $\tau$.\(^3\)

The expectations functions, $P_m(\tau, t)$, $P_k(\tau, t)$, $R(\tau, t)$, $D_m(\tau, t)$, and $D_k(\tau, t)$, are thus seen to transform to the following functions in the continuous time.

\[
\begin{align*}
(4) \quad \lim_{h \to 0} P_m(\tau, t) &= p_{m}(\tau, t), \quad \tau \geq t, \quad \text{with} \quad p_{m}(t, t) = p_m(t), \\
(5) \quad \lim_{h \to 0} P_k(\tau, t) &= p_{k}(\tau, t), \quad \tau \geq t, \quad \text{with} \quad p_{k}(t, t) = p_k(t), \\
(6) \quad \lim_{h \to 0} R(\tau, t) &= r_{m}(\tau, t), \quad \tau \geq t, \quad \text{with} \quad r_{m}(t, t) = r(t), \\
(7) \quad \lim_{h \to 0} D_m(\tau, t) &= p_{m}^{*}(\tau, t) = p_{m1}(\tau, t)/p_{m}(\tau, t), \quad \tau \geq t, \\
(8) \quad \lim_{h \to 0} D_k(\tau, t) &= p_{k}^{*}(\tau, t) = r_{m}(\tau, t)/p_{k}(\tau, t) + p_{k1}(\tau, t)/p_{k}(\tau, t), \quad \tau \geq t.
\end{align*}
\]

The weak consistency axiom is assumed to hold in the sense that the values at time $t$ of the limiting expectations functions, $p_{m}^{*}(\tau, t)$, $p_{k}^{*}(\tau, t)$, and $r_{m}(\tau, t)$, are equated to their actual values at time $t$.

Assuming that the supply plans of money and capital, $M^S(t + h, t)$ and $K^S(t + h, t)$, are always realized, we let the growth of money and capital be governed, respectively, by

\[
\begin{align*}
(9) \quad M^S(t + h, t) &= M(t + h) = M(t) + D(t)h, \\
(10) \quad K^S(t + h, t) &= K(t + h) = K(t) + Q[K(t), p_k(t)]h.
\end{align*}
\]

where money, issued by the government, is transferred to the household sector at an exogenously determined rate of $D(t)$.

If capital is not perfectly malleable, the stock of capital demanded at time $t$ to be held at time $t$ is generally different from the stock existing at time $t$. So long as adjustment costs are real, it takes time to have an existing stock imbalance corrected, and how much to close in a single period will be a function of the costs involved. Hence, it is essential to distinguish effective asset demands from notional ones. The former quantities for money and capital are denoted by $M^d(t + h, t)$ and $K^d(t + h, t)$, and their notional counterparts without bars.

The effective demands for money and capital must be defined such that they become consistent with their notional counterparts and the notional saving as the adjustment costs are reduced.

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\(^3\) We might work with a more general case of the expectations functions converging to only piecewise differentiable functions and consider the right hand derivatives wherever such functions are not differentiable. Since such a treatment only adds to complication without changing the nature of our argument, we have chosen to take a more manageable route. It should be pointed out, nonetheless, that such an assumption is not completely out of line with our guiding methodological precept. That is, if a period model is well defined, it should transform, in the continuous time limit, into a meaningful limiting form while its essential outcomes are preserved. Likewise, if a continuous time model is well defined, it should be representable in the form of a period model whose outcomes are invariant to the market period length. Hence, if one has a well-defined continuous time model based on some intertemporal optimization considerations, the relevant expectations functions may be assumed to be differentiable from the outset. Then as we move from the continuous to the discrete time, it should be possible to break them into step functions of time (in which the market period length determines the frequency of steps). Whether the reality is abstracted in the continuous or discrete time form, a model is an abstraction of the reality; in this sense the methodological precept cuts both ways.
to zero. This requirement is met by the following construction: The effective additions to the existing stocks of money and capital by the end of the period are given by $\Delta M(t + h, t) - M(t)$ and $\Delta K(t + h, t) - K(t)$. Such additions should consist of two parts: (i) accumulations of assets through saving which are represented by $\Delta M(t + h, t) - \Delta M(t, t)$ and $\Delta K(t + h, t) - \Delta K(t, t)$, and (ii) corrections of portfolio imbalances existing at time $t$ which are given by $\Delta M(t, t) - M(t)$ and $\Delta K(t, t) - K(t)$. Because of the adjustment costs, only certain fractions of $\Delta M(t, t) - M(t)$ and $\Delta K(t, t) - K(t)$ can be corrected within a single period. Let such fractions be represented by $k_1(h)$ and $k_2(h)$, which, as shown below, should be determined with respect to some cost structure and are to be viewed generally as functions of market period length $h$; that is,

\begin{align}
(11) & \quad \frac{\Delta M(t, t) - \Delta M(t, t)}{\Delta M(t, t)} = k_1(h), \\
(12) & \quad \frac{\Delta K(t, t) - \Delta K(t, t)}{\Delta K(t, t)} = k_2(h).
\end{align}

Now, in the spirit of the Keynesian investment theory based on the notion of cost-of-change (e.g., Eiser and Strotz (1963), Lucas (1967), Gould (1968), and Treadway (1969)), let the structure of the portfolio adjustment costs be represented as a continuously differentiable function of $k_1(k_1, k_2; h)$:

\begin{align}
(13) & \quad H = H(k_1, k_2; h) \text{ assumed convex in } k_1 \text{ and } k_2 \\
& \quad \text{with } \lim_{k_1 \to -\infty} H(k_1, k_2; h) = \lim_{k_2 \to -\infty} H(k_1, k_2; h) = \infty, \\
& \quad \lim_{k_1, k_2 \to 0} H(k_1, k_2; h) = \text{some positive value } a(h), \\
& \quad H_{k_1} < 0 \text{ and } H_{k_2} < 0 \text{ in the neighborhood of } (0, 0).
\end{align}

A general shape of this cost function is shown in Figure 2. Notice that the cost structure is generally a function of $h$ as well. Minimizing this cost function with respect to $k_1$ and $k_2$ yields the

![Figure 2](image_url)

Figure 2 The General Shape of the Cost Structure $H(k_1, k_2; h)$.

Note: The optimal fractions are determined at $k_1(h)$ and $K_2(h)$. 

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optimal fractions of portfolio imbalances to be corrected as functions of $h$:

$$k_1 = k_1(h)$$ and $$k_2 = k_2(h).$$

We assume plausibly that $k_1(0) = k_2(0) = 0$ (i.e., no adjustment if $h = 0$).

To avoid the problem of overadjustment that arises when $k_1(h)$ and $k_2(h)$ happen to be greater than one (see Foley (1975, p. 318), it is necessary to redefine these fractions by $z_1$ and $z_2$, which take the value of 1 if $k_1(h) \geq 1$ and $k_2(h) \geq 1$ and take the values of $k_1(h)$ and $k_2(h)$ if they lie between zero and one. Hence,

$$z_1 = \begin{cases} 1 & \text{if } k_1(h) \geq 1 \\ k_1(h) & \text{otherwise} \end{cases}$$

$$z_2 = \begin{cases} 1 & \text{if } k_2(h) \geq 1 \\ k_2(h) & \text{otherwise} \end{cases}$$

Thus, the effective demands for money and capital can be defined as follows:

$$\tilde{M}^d(t+h, t) = M(t) + [M^d(t+h, t) - M^d(t, t)] + z_1[M^d(t, t) - M(t)],$$

$$\tilde{K}^d(t+h, t) = K(t) + [K^d(t+h, t) - K^d(t, t)] + z_2[K^d(t, t) - K(t)].$$

Because adjustment costs keep notional demands from being realized by the end of the period, it is important to specify the household budget constraint in terms of effective demands. Otherwise, this constraint would not pass the market period irrelevance condition. Hence, we write this constraint as

$$Y(t+h, t) + p_m(t)D(t)h + [p_m^*(t+h, t) - p_m(t)][M(t) + D(t)h] + [p_k^*(t+h, t) - p_k(t)][K(t) + Q_k[K(t), p_k(t)]h] - C(t+h, t) = W_d(t+h, t) - W(t),$$

in which

$$W(t) = p_m(t)M(t) + p_k(t)K(t),$$

$$\tilde{W}^d(t+h, t) = p_m^*(t+h, t)\tilde{M}^d(t+h, t) + p_k^*(t+h, t)\tilde{K}^d(t+h, t),$$

$$Y(t+h, t) = \{Q_c[K(t), p_k(t)] + p_k(t)Q_k[K(t), p_k(t)]\}h,$$

where $W(t)$ is the existing stock of assets at time $t$; $\tilde{W}^d(t+h, t)$ is the effective demand for assets at time $t$ for time $t+h$; $\tilde{C}(t+h, t)$ is the effective consumption during period $(t, t+h)$.

The effective saving for period $(t, t+h)$, $\tilde{S}(t+h, t)$, must be consistent with the effective wealth accumulation plans; hence,

$$\tilde{S}(t+h, t) = \tilde{W}^d(t+h, t) - W(t).$$

Also, because $\tilde{C}(t+h, t)$ and $\tilde{S}(t+h, t)$ must add up to the expected disposable income during period $(t, t+h)$, $Y^d(t+h, t)$, this income should be defined as

$$Y^d(t+h, t) = Y(t+h, t) + [p_m^*(t+h, t) - p_m(t)]M(t) + [p_m(t) + [p_m^*(t+h, t) - p_m(t)]D(t)h + [p_k^*(t+h, t) - p_k(t)][K(t) + Q_k[K(t), p_k(t)]h].$$

With (20), (21), and (22) substituted in, the household budget constraint (19) yields Walras’ Law in the sense that the values of the effective excess supplies of goods and assets evaluated in...
terms of consumption goods add up to zero (all measured in stock dimension)\(^4\):

\[
(25) \quad Q_c[K(t), p_k(t)]h - \bar{C}(t + h, t) + p^*_m(t + h, t)[M(t) + D(t)h] - \bar{M}^d(t + h, t)
+ p^*_k(t + h, t)[K(t) + Q I[K(t), p_k(t)]h] - \bar{K}^d(t + h, t) = 0.
\]

Also, the so-called balance sheet constraint must hold in the sense that the effective expected wealth at time \(t\) for time \(t + h\), \(\bar{W}^*(t + h, t)\), equals the effective demand for wealth, \(\bar{W}^d(t + h, t)\). Again, a distinction is necessary between the effective expected wealth and the notional expected wealth, \(W^*(t + h, t)\), on the same rationale. With the effective expected stocks of money and capital denoted by \(\bar{M}^*(t + h, t)\) and \(\bar{K}^*(t + h, t)\), this constraint can be expressed as

\[
(26) \quad \bar{W}^*(t + h, t) = p^*_m(t + h, t)\bar{M}^*(t + h, t) + p^*_k(t + h, t)\bar{K}^*(t + h, t)
= p^*_m(t + h, t)\bar{M}^d(t + h, t) + p^*_k(t + h, t)\bar{K}^d(t + h, t) = \bar{W}^d(t + h, t).
\]

We now introduce the notion of quasi end-of-period equilibrium. If notional asset demands are not satisfied by the end of the period, it is necessary to consider an alternative equilibrium concept in which only what is effectively demanded under portfolio adjustment costs is realized by the end of the period. We shall call this notion of equilibrium quasi EOP equilibrium. In our model such equilibrium conditions are given by

\[4\) An alternative statement of Walras' Law may be derived by considering the effective expected accumulation of wealth in the way suggested by Foley (1975, p. 308, eq. (6)). Suppose that this accumulation over period \((t, t + h)\), \(\bar{W}^*(t + h, t) - W(t)\), comes from three sources: (1) saving, whose value at current market prices is given by \(Y(t + h, t) + p_m(t)D(t)h - \bar{C}(t + h, t)\), (2) capital gains or losses on asset holdings at the beginning of the period due to expected changes in prices, i.e., \([p^*_m(t + h, t) - p_m(t)]M(t)\) and \([p^*_k(t + h, t) - p_k(t)]K(t)\), and (3) capital gains or losses on asset accumulations due to changes in prices, i.e., \([p^*_m(t + h, t) - p_m(t)][M^d(t + h, t) - M(t)]\) and \([p^*_k(t + h, t) - p_k(t)][K^d(t + h, t) - K(t)]\). Then, identifying \(\bar{W}^*(t + h, t)\) with \(\bar{W}^d(t + h, t)\) by the balance sheet constraint (26), one can obtain the following statement of Walras' Law:

\[
(Q_c[K(t), p_k(t)]h - \bar{C}(t + h, t) + p^*_m(t)[M(t) + D(t)h] - \bar{M}^d(t + h, t)
+ p^*_k(t)[K(t) + Q I[K(t), p_k(t)]h] - \bar{K}^d(t + h, t) = 0.
\]

In this statement, the excess stock supplies of money and capital as of the end of the period are evaluated at \(p_m^*(t + h, t)\) and \(p_k^*(t + h, t)\), and not at \(p^*_m(t + h, t)\) and \(p^*_k(t + h, t)\) as in (25). In general, however, Walras' Law is derived from ex-ante budget/financing constraints of separate sectors of the economy; the law is viewed as an aggregation of such constraints (see e.g., Turnovsky (1977, pp. 10-11, eq. (22)) and Buiter (1980, pp. 6-7, eq. (16)). In our model Walras' Law is derived from the ex-ante household budget constraint. But, even under this conventional procedure, the excess stock supplies of assets as of the end of the period may or may not be evaluated at their expected prices in Walras' Law depending on whether expected prices, hence expected capital gains/losses are considered in ex-ante budget constraints. If expected prices are considered in such constraints, one obtains a statement of Walras' Law with the excess stock supplies of assets evaluated at their expected prices; this is what one sees in our statement (25) as well as in Turnovsky's (1977, pp. 10-11, eq. (22)). If, on the other hand, expected prices are not considered in ex-ante budget constraints, the excess stock supplies of assets must be evaluated at their actual market prices; this is what Buiter (1980, pp. 6-7, eq. (16)) obtains with one important modification that Walras' Law holds only under homogeneous expectations by relevant agents with respect to current and future variables. The difference between the two forms of Walras' Law was noted by Buiter (1980, pp. 5-6) in comparison with Turnovsky's. Note that the disparity disappears in the continuous time limit as Walras' Law with actual prices transforms into the same limiting form as Walras' Law (25); see (45).\]
The \([27]\) \(M^S(t + h, t) = M(t) + D(t)h = \bar{M}^d(t + h, t),\)
\([28]\) \(K^S(t + h, t) = K(t) + Q_s[K(t), p_s(t)]h = \bar{K}^d(t + h, t),\)
or, equivalently,
\([29]\) \(D(t)h = M^d(t + h, t) - M^d(t, t) + z_1[M^d(t, t) - M(t)],\)
\([30]\) \(Q_s[K(t), p_s(t)]h = K^d(t + h, t) - K^d(t, t) + z_2[K^d(t, t) - K(t)].\)

Note that by Walras’ Law (25) the quasi EOP equilibrium prices of money and capital, \(\bar{p}_m\) and \(\bar{p}_k\), also achieve quasi equilibrium in the consumption goods market:
\([31]\) \(Q_c[K(t), \bar{p}_s(t)]h = \bar{C}(t + h, t).\)

Hence, one may choose any two of these three conditions from Walras’ Law to determine \(\bar{p}_m\) and \(\bar{p}_k\). This is an important feature of equilibrium pertaining to the end of the period.

With the planning horizon being infinite, all of the expectations functions must be considered in specifying the notional demands, \(M^d(t + h, t)\) and \(K^d(t + h, t)\). Let them be specified as
\([32]\) \(M^d(t + h, t) = \bar{L}[Y^D(t + h, t)/h, Y^D(t + h, t), W^*(t + h, t), P^*_m(\tau, t), P^*_k(\tau, t), R^*(\tau, t), D^*_m(\tau, t), D^*_k(\tau, t), \tau \geq t]/p^*_m(t + h, t),\)
\([33]\) \(K^d(t + h, t) = \bar{J}[Y^D(t + h, t)/h, Y^D(t + h, t), W^*(t + h, t), P^*_m(\tau, t), P^*_k(\tau, t), \tau \geq t]/p^*_k(t + h, t).\)

Because \(D^*_m(\tau, t)\) can be derived from \(\bar{P}^*_m(\tau, t)\), and \(\bar{D}^*_m(\tau, t)\) from \(\bar{R}^*(\tau, t)\) and \(\bar{P}^*_m(\tau, t)\), and \(\bar{R}^*(\tau, t)\) from \(\bar{P}^*_k(\tau, t)\) [recall \(r^*(\tau + nh, t) = f[\bar{p}^*_k(t + nh, t)], n = 0, 1, 2, \ldots\)], these specifications can be reduced to
\([34]\) \(M^d(t + h, t) = \bar{L}[Y^D(t + h, t)/h, Y^D(t + h, t), W^*(t + h, t), P^*_m(\tau, t), P^*_k(\tau, t), \tau \geq t]/p^*_m(t + h, t),\)
\([35]\) \(K^d(t + h, t) = \bar{J}[Y^D(t + h, t)/h, Y^D(t + h, t), W^*(t + h, t), P^*_m(\tau, t), P^*_k(\tau, t), \tau \geq t]/p^*_k(t + h, t).\)

Notice that two income variables, \(Y^D(t + h, t)/h\) and \(Y^D(t + h, t)\), are entered into specifications (34) and (35); the former is intended to capture the transactions motives of the demand for money whereas the latter to represent accumulation of assets through saving. If only one of these two variables is used, a serious difficulty arises. For instance, while it may capture the asset accumulation aspect via saving, \(Y^D(t + h, t)\) does not adequately represent the transactions motives of liquidity preferences; this is because \(Y^D(t + h, t)\), approaching to zero as \(h \to 0\), disappears as an argument in \(M^d(t + h, t)\) and \(K^d(t + h, t)\). Alternatively, if instead \(Y^D(t + h, t)/h\) alone is considered, it invites a curious consequence that the effective consumption and saving are affected by income which is independent of the market period length. That does not make sense because the consumption and saving motives during any given period should be affected, to a great extent, by income earned during that period.

Observe that as \(k_1(h)\) and \(k_2(h) \to \infty\), it holds that \(W^*(t, t) = W(t)\) from (17), (18), and (26). Hence, with this substituted in, \(\lim_{h \to 0} M^d(t + h, t)\) and \(\lim_{h \to 0} K^d(t + h, t)\) can be expressed as
\([36]\) \(\lim_{h \to 0} M^d(t + h, t) = M^d(t, t)\)
\(= L[y^D(t), Y^D(t), W(t), P^*_m(\tau, t), P^*_k(\tau, t), \tau \geq t]/p^*_m(t, t),\)
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\[ \lim_{h \to 0} K^d(t + h, t) = K^d(t, t) \]

\[ = J[y^D(i), Y^D(t, t), W(t); p^*_m(\tau, t), p^*_k(\tau, t), \tau \geq t]/p^*_k(t, t), \]

where

\[ y^D(t) = Y^D(t, t) = y(t) + p^*_m(t, t)M(t) + p_m(t)D(t) + p^*_k(t, t)K(t), \]

\[ \left[ y(t) = Y(t, t) = Q_c[K(t), p(t)] + p(t)Q_l[K(t), p(t)] \right] \]

\[ p^*_m(\tau, t) = \lim_{h \to 0} P^*_m(\tau, t), \tau \geq t; p^*_k(\tau, t) = \lim_{h \to 0} P^*_k(\tau, t), \tau \geq t, \]

\[ Y^D(t, t) = \lim_{h \to 0} Y^D(t + h, t) = 0 \text{ [from (24)]} \]

where \( p^*_m(t, t), p^*_k(t, t), Y^D(t, t), \) and \( Y(t, t) \) are the partial derivatives of \( p^*_m(t, t), p^*_k(t, t), Y^D(t, t), \) and \( Y(t, t) \) with respect to the first time index.

Thus, our effective demands for money and capital are well defined with (34), (35), (36), and (37) substituted into (17) and (18). It follows that our quasi EOP equilibrium conditions, (27) and (28) (or (29) and (30)) are equally well defined.

Turning now to the metamorphosis of our period model in the continuous time limit, we first observe that the household budget constraint (19) transforms into two limiting forms, one in flow form and the other in stock form. If the constraint is divided by \( h \), it transforms to a flow constraint of the following form:

\[ W^d_I(t, t) = y(t) + p^*_m(t, t)M(t) + p_m(t)D(t) + p^*_k(t, t)K(t) - c(t) = y^D(t) - c(t) = s(t) \]

where \( W^d_I(t, t), c(t), \) and \( s(t) \) are the partial derivatives of \( W^d(t, t), C(t, t), \) and \( S(t, t) \) with respect to the first time index, respectively. Note that, in reference to (17), (18) and (21), because \( k_1 \) and \( k_2 \) become less than one as \( h \to 0 \), \( W^d_I(t, t) \) is equal to

\[ W^d_I(t, t) = \left[ p_m(t)\bar{M}^d_I(t, t) + p^*_m(t)M(t) \right] + \left\{ p_k(t)\bar{K}^d_I(t, t) + p^*_k(t)K(t) \right\} \]

\[ = \left[ p_m(t)\bar{M}^d_I(t, t) + \gamma_1[M^d(t, t) - M(t)] \right] + \left[ p_k(t)\bar{K}^d_I(t, t) + \gamma_2[K^d(t, t) - K(t)] \right] + p^*_m(t)M(t), \]

where \( \bar{X}^d(t, t), X = M, K, \bar{M}, \bar{K}, \) and \( \bar{W} \) is the partial derivative of \( X^d(t, t) \) with respect to the first time index; \( \gamma_1 = \lim_{h \to 0} k_1(h)/h = k'_1(0) \) and \( \gamma_2 = \lim_{h \to 0} k_2(h)/h = k'_2(0) \) (recall our assumption that \( k_1(0) = k_2(0) = 0 \)). The meaning is clear: The instantaneous effective saving rate must equal the effectively desired instantaneous rate of wealth accumulation.

On the other hand, if \( h \) is taken to zero directly, the household budget constraint transforms to a stock constraint:

\[ W(t) = W^d(t, t), \]

which may also be obtained from the balance sheet constraint (26). We shall call this constraint the balance sheet identity. That is, the effective stock demand for wealth at time \( t \) for the same moment must equal the stock of wealth existing at time \( t \).

Observe from (17) and (18) that \( \lim_{h \to 0} \bar{M}^d(t + h, t) = M(t) \) and \( \lim_{h \to 0} \bar{K}^d(t + h, t) = K(t); \)

hence, the balance sheet identity holds only with \( \bar{M}^d(t, t) = M(t) \) and \( \bar{K}^d(t, t) = K(t) \). This point is equally valid when there are no portfolio adjustment costs \( i.e., W(t) = W^d(t, t) \) holding only with \( M^d(t, t) = M(t) \) and \( K^d(t, t) = K(t) \). This means that the balance sheet identity
should not be interpreted to bind the stock equilibrium conditions of the two assets to be equivalent.

This observation raises a serious question as to whether the flow budget constraint (40) and the balance sheet identity (42) contain different information, or, more generally, whether any additional information may be gained as we move from the discrete to the continuous time. May (1970) claims that the latter contains additional information not included in the former. We contend ‘no.’

We stress that May’s contention is in sharp contradiction with Foley’s methodological precept that the essential outcomes of a well-defined period model should not be affected by the real time length of the market period and should be preserved in the continuous time limit. It is indeed difficult to conceive how any additional information of critical importance may be obtained by merely refining the market period length to the limit. We claim that the flow budget constraint (40) and the balance sheet identity (42) are equivalent in information and that Foley’s methodological precept withstands.

To see our claim, rewrite the household budget constraint (19) as:

\[
W(t+h, t) - W(t, t) - \left[ (YD(t+h, t) - YD(t, t)) - (C(t+h, t) - C(t, t)) \right] = W(t) - Wd(t, t)
\]

(note \(YD(t, t) = C(t, t) = 0\)) which may be rewritten as

\[
\{Wd1(t, t) - [yD(t) - c(t)]\}h = W(t) - Wd(t, t).
\]

This expression is valid for sufficiently small \(h\). It shows that \(W(t) - Wd(t, t) = 0\) implies \(\overline{Wd}(t, t) - [yD(t) - c(t)] = 0\) for any \(h > 0\); conversely, \(\overline{Wd}(t, t) - [yD(t) - c(t)] = 0\) implies \(W(t) - Wd(t, t) = 0\). Thus, the flow budget constraint (40) and the balance sheet identity (42) are really equivalent in information. We should also point out that because the flow budget constraint (40) and Walras’ Law (45) below are equivalent, we have actually three equivalent expressions of the same underlying constraint. Thus, taking the household budget constraint into the continuous time limit does not yield any additional information to what is already contained in any one of the three equivalent expressions. In section 3 we shall present yet another strong argument in support of our claim; namely, any attempt to read more into the balance sheet identity leads to an impasse that the system becomes under-determined.

Walras’ Law (in flow form) holds equally in the continuous time limit; that is, dividing Walras’ Law (25) by \(h\) and taking \(h \rightarrow 0\) yields

\[
Qd[K(t), p_k(t)] - \overline{c}(t) + p_m(t)[M(t) - M^d(t, t) + \gamma_1(M^d(t, t) - M(t))]
+ p_s(t)[Qs[K(t), ps(t)] - K^d(t, t) + \gamma_2(K^d(t, t) - K(t))] = 0.
\]

That is, the sum of the values of the effective instantaneous excess flow supply rates of goods and assets evaluated in terms of consumption goods must be zero.

Also, the continuous time analogues of the quasi end-of-period equilibrium conditions, (29) and (30), are well defined; they are obtained by dividing these conditions by \(h\) and taking \(h \rightarrow 0\).

\[
\begin{align*}
D(t) &= M^d(t, t) + \gamma_1(M^d(t, t) - M(t)), \\
Qs[K(t), p_s(t)] &= K^d(t, t) + \gamma_2(K^d(t, t) - K(t)).
\end{align*}
\]

By Walras’ Law (45), these flow equilibrium conditions imply flow equilibrium in the consumption goods market, i.e., \(Qc[K(t), p_s(t)] = \overline{c}(t)\). Thus, as in discrete time, the quasi EOP equili-
3. No Adjustment Costs and Equivalence of EOP and BOP Equilibrium

Suppose that capital is perfectly malleable and portfolio adjustments incur no real costs in both assets as in May (1970), Buiter and Woglom (1977), Turnovsky and Burmeister (1977), and Turnovsky (1977). Here, this case emerges when $k_1(h)$ and $k_2(h)$ are taken to infinity. Hence, our discussion in the preceding section carries over with the effective quantities replaced with their notional counterparts and with $M^d(t, t)$ and $K^d(t, t)$ equated to $M(t)$ and $K(t)$.

Because portfolios of assets are now continuously balanced, both BOP and EOP equilibrium specifications are possible. Hence, a legitimate question arises as to what their equivalence implies. The EOP equilibrium specifications are stated as

$$M(t+h) = Md(t+h, t),$$
$$K(t+h) = Kd(t+h, t),$$

whereas the BOP equilibrium specifications can be written as

$$M(t) =Md(t; h),$$
$$K(t) =Kd(t; h),$$

where $Md(t; h)$ and $Kd(t; h)$ denote the demands for money and capital that are formed at the beginning of the period for that same moment while the market period is still discrete.

We specify $Md(t; h)$ and $Kd(t; h)$ as follows:

$$Md(t; h) = L[YD(t+h, t)/h, YD(t+h, t), W(t); P_m(\tau), P_k(\tau), \tau \geq t]/p_m(t),$$
$$Kd(t; h) = J[YD(t+h, t)/h, YD(t+h, t), W(t); P_m(\tau), P_k(\tau), \tau \geq t]/p_k(t).$$

The expectations functions, $P_m(\tau)$ and $P_k(\tau)$, $\tau \geq t$, must be considered as in the case of EOP equilibrium. Also, they must be distinguished from the limiting forms, $Md(t, t)$ and $Kd(t, t)$, of EOP asset demand functions, $Md(t+h, t)$ and $Kd(t+h, t)$, as $h \to 0$. $Md(t; h)$ and $Kd(t; h)$ are formed in discrete time in which the market period is still of length $h > 0$, while $Md(t, t)$ and $Kd(t, t)$ are the limiting forms as $h \to 0$. But, because $Md(t; h)$ and $Kd(t; h)$ are based on the same underlying optimization considerations as are $Md(t+h, t)$ and $Kd(t+h, t)$, we require that $\lim_{h \to 0} Md(t; h) = \lim_{h \to 0} Md(t+h, t)$ and $\lim_{h \to 0} Kd(t; h) = \lim_{h \to 0} Kd(t+h, t)$, i.e.,

$$Md(t; 0) = Md(t, t)$$
$$Kd(t; 0) = Kd(t, t).$$

In the continuous time limit, the EOP equilibrium conditions take two limiting forms. The stock forms are obtained by taking $h \to 0$ directly in (48) and (49):

$$M(t) = Md(t, t),$$
$$K(t) = Kd(t, t).$$

The flow forms are obtained by subtracting (54) and (55) from (48) and (49), respectively, dividing the differences by $h$, and then taking $h \to 0$:

$$D(t) = Md^1(t, t),$$
$$Q_{M}[K(t), p_k(t)] = K^1(t, t).$$

Subtracting the EOP equilibrium conditions, (48) and (49), from the stock equilibrium conditions, (54) and (55) holding at time $t + h$ and applying the mean value theorem, we have

$$Md^2(t+h, \xi) = 0, t < \xi < t + h,$$
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(59) \[ K_d^2(t + h, \eta) = 0, \ t < \eta < t + h, \]
which, in the limit as \( h \to 0 \), yield

(60) \[ M_d^2(t, t) = 0, \]
(61) \[ K_d^2(t, t) = 0, \]

where \( X_d^2(z, t), X_d = M_d, K_d \), is the (left-hand) partial derivative of \( X_d(z, t) \) with respect to the second time index \( i.e. X_d^2(z, t) = \lim_{h \to 0, h < 0} \frac{[X_d(z, t + h) - X_d(z, t)]}{h} \). \( M_d^2(z, t) \) and \( K_d^2(z, t) \), for arbitrary \( z \) and \( t \), measure the rates at which the desired holdings of money and capital (in real value terms) as of a given future date \( z \) change when the planning date \( t \) is altered (see Turnovsky and Burmeister (1977, p. 385)). With (60) and (61) combined with (56) and (57), it follows that

(62) \[ D(t) = \dot{M}_d(t, t), \]
(63) \[ Q[K(t), p_m(t)] = \dot{K}_d(t, t), \]

where the dot denotes the time derivative.

Turning to the BOP equilibrium, we see that conditions (50) and (51) also take two forms in the continuous time limit. Taking \( h \to 0 \) directly in (50) and (51) yields their stock forms:

(64) \[ M(t) = \tilde{M}_d(t; 0), \]
(65) \[ K(t) = \tilde{K}_d(t; 0), \]

And, taking the differences of (50) and (51) between \( t + h \) and \( t \) and dividing them by \( h \) gives us their flow forms as \( h \to 0 \):

(66) \[ D(t) = \dot{\tilde{M}}_d(t; 0), \]
(67) \[ Q[K(t), p_m(t)] = \dot{\tilde{K}}_d(t; 0). \]

It is worth pointing out that the fact that both EOP and BOP equilibrium conditions have limiting forms in stock and flow forms indicates that it is not proper to associate the notion of flow equilibrium with EOP equilibrium and the notion of stock equilibrium with BOP equilibrium.

Now on the equivalence of EOP and BOP equilibrium specifications in the continuous time:

We need to establish the equivalence not only in stock form but also in flow form.

We first observe from (52) and (53) that \( \tilde{M}_d(t; 0) \) and \( \tilde{K}_d(t; 0) \) are given by

(68) \[ \tilde{M}_d(t; 0) = \lim_{h \to 0} \tilde{M}_d(t; h) = \tilde{L}[Y^d(t), Y^D(t, t), W(t); \theta_m(t, \tau), \theta_k(t, \tau), \tau \geq t]/p_m(t), \]
(69) \[ \tilde{K}_d(t; 0) = \lim_{h \to 0} \tilde{K}_d(t; h) = \tilde{J}[Y^d(t), Y^D(t, t), W(t); \theta_m(t, \tau), \theta_k(t, \tau), \tau \geq t]/p_k(t). \]

Comparing these with \( M^d(t, t) \) and \( K^d(t, t) \) obtained from (34) and (35), we see that the equivalence of the two in stock form requires that the weak consistency axiom \( i.e. \theta_m(t, \tau) = p_m(t) \) and \( \theta_k(t, \tau) = p_k(t) \) be satisfied (note that when this axiom is satisfied, \( W^*(t, \tau) = W(t) \) as shown in (26)).

It remains to study the equivalence in flow form, \( i.e. \)

(70) \[ M^d(t, t) = \tilde{M}_d(t; 0), \]
(71) \[ K^d(t, t) = \tilde{K}_d(t; 0). \]

This requires that we examine the partial and the total derivatives of the price expectations functions; the former are involved in \( M^d(t, t) \) and \( K^d(t, t) \) and the latter in \( \tilde{M}_d(t; 0) \) and \( \tilde{K}_d(t; 0) \). Thus, it is necessary for this equivalence that the partial derivatives of such functions equal their total derivatives, \( i.e. \) \( \theta_m(t, \tau) = \dot{\theta}_m(t, \tau) \) and \( \theta_k(t, \tau) = \dot{\theta}_k(t, \tau), \tau \geq t \), or, equivalently,
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\[ p_m^*(\tau, t) = 0 \text{ and } p_k^*(\tau, t) = 0, \text{ at any given point in time } t. \] Applying the mean value theorem and the weak consistency axiom to \( p_m^*(\tau, t) \) and \( p_k^*(\tau, t), \tau \geq t, \) yields:

\[(72) \quad p_m(\tau) - p_m^*(\tau, t) = p_m^*(\tau, \xi)(\tau - t), \quad t < \xi < \tau \text{ (by the mean value theorem)}, \]

\[(73) \quad p_k(\tau) - p_k^*(\tau, t) = p_k^*(\tau, \epsilon)(\tau - t), \quad t < \epsilon < \tau \text{ (by the mean value theorem}). \]

But, replacing \( t \) with any arbitrary \( t' \) in (70) and (71) yields \( p_m^*(\tau, t') = p_k^*(\tau, t') = 0, \tau \geq t'; \) hence, it must hold that \( p_m^*(\tau, \xi) = 0 \text{ and } p_k^*(\tau, \epsilon) = 0, \tau \geq \epsilon, \) at time \( \epsilon. \) It follows that \( p_m^*(\tau, t) = p_m(\tau) \) and \( p_k^*(\tau, t) = p_k(\tau), \tau \geq t, \) at all \( t; \) that is, the expectations functions satisfy the perfect foresight condition. In view of the fact that the perfect foresight condition implies the weak consistency axiom, it is, therefore, established that the perfect foresight condition is necessary for the equivalence of the EOP and BOP equilibrium specifications in the continuous time limit. Thus, we have verified Foley's conjecture that this equivalence holds only along perfect-foresight-equilibrium paths (1977, p. 402).

We now turn to the now traditional interpretation of the balance sheet identity under the absence of portfolio adjustment costs: Recall that the household budget constraint in discrete time (19) transforms in the continuous time limit into two constraints, the balance sheet identity (42) and the flow budget constraint (40). May (1970) and others, however, have interpreted the balance sheet identity to imply that if there are \( n \) assets in the economy, only \( n - 1 \) stock equilibrium conditions can be independent; with only two assets, it means that stock equilibrium in one asset must be equivalent to stock equilibrium in the other. The interpretation has been the source of adding-up restrictions on the coefficients of asset stock demand functions. We show that the interpretation leads to an impasse that the system would not possess enough independent equilibrium relations with which to determine its endogenous position.

The impasse can be shown as follows: Note first that the stock equilibrium conditions, \( M(t) = M^d(t, t) \) and \( K(t) = K^d(t, t) \), are equivalent to the flow equilibrium conditions, \( D(t) = M^d_1(t, t) \) and \( Q_1[K(t), p_k(t)] = K^d_1(t, t). \) The necessity part of this equivalence proposition follows immediately by differentiating totally the stock equilibrium conditions and applying \( M_{m2}(t, t) = K_{d2}(t, t) = 0. \) The sufficiency part, on the other hand, is seen by integrating the flow equilibrium conditions with respect to the first time index from \( t \) to \( t + h; \) that is,

\[(74) \quad \int_t^{t+h} D(\tau) \, d\tau = \int_t^{t+h} M^d_1(\tau, t) \, d\tau, \]

\[(75) \quad \int_t^{t+h} Q_1[K(\tau), p_k(\tau)] \, d\tau = \int_t^{t+h} K^d_1(\tau, t) \, d\tau, \]

i.e.,

\[(76) \quad M(t + h) - M^d(t + h, t) = M(t) - M^d(t, t), \]

\[(77) \quad K(t + h) - K^d(t + h, t) = K(t) - K^d(t, t). \]

But, under the requirement of EOP equilibrium, the left hand side is zero in (76) and (77). Hence, the stock equilibrium conditions, \( M(t) = M^d(t, t) \) and \( K(t) = K^d(t, t), \) hold.

Then, take whichever of the two flow equilibrium conditions, (56) and (57), say, the flow equilibrium condition for money, \( D(t) = M^d_1(t, t). \) As seen above, this condition is equivalent to
the stock equilibrium condition, \( M(t) = M^d(t, t) \). If the traditional interpretation of the balance sheet condition is valid, then this stock equilibrium condition has to be equivalent to that of capital, \( K(t) = K^d(t, t) \). But, this, in turn, is equivalent to the corresponding flow equilibrium condition, \( Q_c[K(t), p_s(t)] = K^f(t, t) \). Hence, starting with either of the two flow equilibrium conditions, we have \( p_m(t)[D(t) - M^d(t, t)] + p_s(t)[Q_c[K(t), p_s(t)] - K^f(t, t)] = 0 \). Consequently, via Walras' Law (45), it follows that the flow equilibrium condition of the goods market, \( i.e., \, Q_c[K(t), p_s(t)] - c(t) = 0 \), is also satisfied. Thus, under the traditional interpretation of the balance sheet identity, the system becomes under-determined in that it allows only one independent equilibrium relation, from which evidently \( p_m(t) \) and \( p_s(t) \) cannot be determined. Extending this to a more general situation with goods and \( n - 1 \) assets, we see that the traditional interpretation of the balance sheet identity leads to an impasse of under-determination, that the system would possess only \( n - 2 \) independent equilibrium relations, which are not enough to determine the equilibrium values of \( n - 1 \) prices (with the price of goods or any one asset used as a numeraire) (this is quite consistent with what I have shown elsewhere using a different two-asset model (Hayakawa (1984)). This fact provides another strong argument in support of our claim that the balance sheet identity contains no more information than what is already contained in the flow budget constraint or Walras' Law in flow form.

4. Patinkin's Macroeconomic Model

May (1970) argued that transforming a discrete time model into its continuous time analogue could yield additional information of critical importance to comparative statics/dynamics, demonstrating such possibilities in the context of Patinkin's macroeconomic model. Karni (1978), however, raised a serious doubt on this contention, arguing that when May transformed a discrete version of Patinkin's model into its continuous time analogue, the nature of asset equilibrium was changed from the end-of-period equilibrium to the beginning-of-period equilibrium. Karni, therefore, claimed that this subtle change was the only cause of the differences in May's outcomes.

Karni's critique, however, is misleading and in clear contradiction with Foley's methodological precept. May allows the household budget constraint and the EOP equilibrium conditions to be written in terms of notional asset demands. Therefore, his analysis can be interpreted to deal with the case in which the non-presence of adjustment costs allows portfolios of assets to be balanced at every point in time. In this case, as shown in the preceding section, the EOP equilibrium conditions cast in discrete time transform, in the continuous time limit, into two limiting conditions, one in stock form and the other in flow form. May works with the limiting conditions in stock form, but his analysis remains in the framework of EOP equilibrium. Hence, the nature of asset equilibrium is not changed in the continuous time limit as claimed by Karni. The problem with Karni's critique is that he associated the limiting EOP equilibrium conditions in stock form with the limiting BOP equilibrium conditions in stock form (\( i.e., \, M^d(t, t) = M(t) \) with \( \hat{M}^d(t, 0) = \bar{M}(t) \) and \( K^d(t, t) = K(t) \) with \( \hat{K}^d(t, 0) = \bar{K}(t) \) in our model). As we have shown, these two conditions are not equivalent unless the perfect foresight condition is satisfied.

Unlike May, Karni writes his market clearing conditions (his eq. (6)) after Foley (1975); they are based on the notion of what we have called quasi EOP equilibrium above. This is the only
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feature that distinguishes the two works. Then, Karni’s argument that Patinkin’s conclusions can be reinstated under quasi EOP equilibrium is founded on the point that Walras’ Law in flow form, in the continuous time limit, does not require that the flow equilibrium conditions of the two assets be identical. But, since this is also the case when portfolio adjustment costs are not present, it follows that Patinkin’s conclusions should hold regardless of such costs.

May’s difficulty lies in his claim that the limiting EOP equilibrium conditions in stock form contain, by way of his interpretation of the balance sheet identity, additional information to what is already contained in the limiting EOP equilibrium conditions in flow form. Thus, if the balance sheet identity is interpreted in the way suggested here, May’s analysis should lead to the same conclusions as reached by Patinkin and reconfirmed by Karni.

5. Conclusion

If adjustment costs keep portfolios of assets from being balanced continuously, it is essential, particularly from the standpoint of the market period irrelevance test, to distinguish between effective and notional asset demands and also between proper EOP equilibrium involving notional quantities and quasi EOP equilibrium involving effective quantities. Equally essential is a distinction between the market period length and the planning horizon of a decision maker in assessing the role of expectations on the equivalence of the two specifications; this has led to the expectations as a step function of future time, which converges to a continuously differentiable function of this time as the market period length is refined to the continuous time limit. An individual, whether under BOP or EOP equilibrium, is likely to be motivated intertemporally so that the length of the market period should not dictate the planning horizon. The former is an institutional constraint while the latter is a matter of individual motivation.

Two propositions have been established: First, if portfolio adjustment costs do not exist, asset equilibrium can be specified either in terms of EOP or BOP equilibrium. We have then verified Foley’s conjecture by showing that the two specifications are equivalent in continuous time only if the perfect foresight condition is satisfied. Second, we have also demonstrated that transforming a well-defined period model into its continuous time analogue does not yield any additional information of critical importance. In particular, the balance sheet identity and the flow budget constraint, both of which are derived from the household budget constraint in the continuous time limit, contain the same information. This proposition invalidates May’s interpretation of the balance sheet identity; in fact, it leads to an impasse of under-determination of system. Moreover, the proposition clarifies the nature of confusion in May’s critique and Karni’s defense of Patinkin’s propositions. Our interpretation of the balance sheet identity reinstates Patinkin’s conclusions in continuous time.

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