OPTIMAL TAXATION AND PRODUCTION EFFICIENCY RECONSIDERED*

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This paper elucidates the question of whether production efficiency (in the weak sense) obtains in an optimally taxed economy, which is posed by the seminal paper of Diamond and Mirrlees. This paper presents a complete conclusion and a proof for that the (desirability of) production efficiency holds, except (i) a not interesting case where private production is wholly less efficient than public production, or (ii) a case where some good(s) is exempt from tax imposition.

1. Introduction

P. Diamond and J. Mirrlees in their seminal work (1971) posed a problem of whether the (weakly) efficient production obtains and be desirable in optimally taxed economies. The problem asks if it is desirable to organize productive activities efficiently in the sense that they lie on the production frontier. The question comes from the fact that consumption taxes imply difference between MRS and MRT and Pareto-optimality does not necessarily follow, and then that the 'second-best' theorem suggests a question of whether efficient production is desirable from a viewpoint of an economy as a whole.

The problem, after them, is further analyzed in different manners and differing situations by J. Stiglitz and P. Dasgupta (1971, 1972), Mirrlees (1972, 1986), F. Hahn (1973), E. Sadka (1977), K. Munk (1978, 1980), R. Guesnerie (1979), J. Weymark (1979) and M. Yamada (1985). All these contributions have clarified some condition(s) for production efficiency. However, the problem does not seem to have been settled, most importantly, because all the preceding contributions assume that producers' profit does not change when their production configuration varies, and thus do not analyze the problem in the case which is most common and important in the actual economy.2)

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1) They refer this as the 'weak efficiency' distinguishing from Pareto efficiency. Here we however use a simple expression of 'efficiency' after their followers.

2) The following also shows insufficiency of the preceding results. (i) Some of their results leave gaps to each other. On the one hand it is said that the production efficiency holds if production frontier is smooth (and strictly convex) (Dasgupta and Stiglitz (1972), Hahn (1973, Proposition 3, Corollary 3), Sadka (1977)). On the other it is said that the efficiency follows with some producer prohibited from production (Mirrlees (1972, p. 607)). (ii) Some proofs of (desirability of) production efficiency remain incompletely. E.g., Stiglitz and Dasgupta start by assuming production functions and therefore that all
The problem will have relevance to the design of tax system as well as to the analysis. If production inefficiency in an optimally taxed economy is so common, then the tax authority may not make a plan of tax system simply depending on a usual assumption that production is carried out efficiently. The aim of this paper is, reviewing the present state of the arguments, to present a more complete and unified answer by reconsideration of the problem taking account also of the case with variable production profits.

Our argument below is organized as follows. The next section presents a complete model, which intends to clarify and facilitate the analysis. Section 3 summarizes the preliminary results and preceding contributions, which makes our main argument clear. Section 4 presents our main argument to the problem, i.e., shows that: (i) the production efficiency in the optimally taxed economy obtains in the most ordinary cases if all goods are taxable. But, (ii) the inefficiency may be brought into if the private production is wholly inefficient compared to the public production, or some goods are exempt from taxation. The final section concludes the paper.

2. The Model

We consider an economy where there are \( N \) households, \( K \) firms and the government, as well as \( J \) private goods. Firm \( k \) is owned by households in the ratio \( \{ \theta^{k1}, \ldots, \theta^{ki}, \ldots, \theta^{kN} \} \) \( (\theta^{ki} \geq 0, \sum_{i=1}^{N} \theta^{ki} = 1 \) for any \( i \) and/or \( k \)).

The production possibility set of the firm \( k \) is denoted by \( Y^k \) (with element \( y^k \)). \( Y^k \) is assumed to be closed, convex and contains the nonpositive orthant \( RJ \) (free disposability). The private producers are assumed to be the profit-maximizer. The public production possibility set \( Y^G \) (with element \( y^G \)) is supposed to be closed and with free disposability (not necessarily includes the origin). We use also the notations: \( Y^p \equiv \sum_{k=1}^{K} Y^k \) (with element \( y^p \)), and \( Y \equiv Y^p + Y^G \) (with element \( y \)).

Preference of the household \( i \) is represented by a continuous, strictly quasi-concave and strictly increasing function \( u^i(\cdot) \) defined on the closed and convex consumption set \( X^i \subset R^J_+ \). \( w^i \) (\( \in X^i \)) is initially owned by household \( i \), and its net demand is denoted by \( x^i \), thus \( x^i + w^i \in X^i \). The households maximize their utility under their respective budget restraint.

Thirdly, the government has the social welfare function defined on the every household's utility, \( W(u^1, \ldots, u^N) \), which is strictly increasing with regard to each \( u^i \). We here assume that government’s demand is zero and thus tax is redistributive, based on the following understanding: First, this is common to all the literature cited above. Second and critically, assuming both the public production and government purchase of a specified amount of goods \( g \) is equivalent to supposing only \( Y^G = Y^G - g \) instead. This treatment does not lose any generality since we assume simply the free disposability on \( Y^G \). The government aims to maximize the social welfare through commodity taxation and public production.

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producers lie on their respective production set, and ask if MRT’s for each producer equal. Then they go on to considering desirability and non existence of the optimum with inefficient production making use of examples.
M. Yamada: Optimal Taxation and Production Efficiency Reconsidered

Fourthly, let \( p(\in \mathbb{R}^k) \) denote producer prices (prices that producers face with) and \( t \) be (specific) tax rates levied on consumption. Consumer prices \( q(\in \mathbb{R}^k) \) are by definition equal to \( p + t \). Here, we understand that commodity taxes are taxes imposed on consumer, and therefore no tax is levied on firms' purchase or on interfirm transactions (cf. eq. (1) below)\(^3\). The first good is considered as numeraire and \( p_1 \equiv 1 \) unless otherwise specified. As Munk (1978) discussed we can impose commodity tax on all goods in cases where there exist production profits and therefore a lump-sum income for households.

Fifthly, some other symbolic definitions are as follows: \( r^k \) means firms \( k \)'s profit. Thus

\[
r^k = p y^k \quad \text{when firm } k \text{ choose to produce } y^k \in Y^k \text{ facing with } p, \quad \tilde{r} \equiv (r^1, \ldots, r^K), \quad \tilde{r} = \sum_{k=1}^{K} r^k;
\]

\[
\tilde{x} \equiv (x^1, \ldots, x^K), \quad x \equiv \sum_{i=1}^{N} x^i, \quad w = \sum_{i=1}^{N} w^i; \quad \tilde{y}^p \equiv (y^1, \ldots, y^K), \quad y^p \equiv \sum_{k=1}^{K} y^k, \quad y = y^p + y^G.
\]

Lastly we state a usual assumption on the indirect social welfare function \( V(q, \tilde{r}) \). To state this, note that household's demand is a function of consumer prices and firms' profits, and denote this as \( x^t(q, \tilde{r}) \). \( x^t(q, \tilde{r}) \) is unique and continuous with respect to \((q, \tilde{r}) \) for \( q > 0.4\). Then, the following indirect social welfare function

\[
V(q, \tilde{r}) \equiv W(u^1(x^1(q, \tilde{r})), \ldots, u^N(x^N(q, \tilde{r})))
\]

is defined. We pose the following assumption on \( V \);

**Assumption 1.** \( V(q, \tilde{r}) \) is locally nonsatiated with respect to \( q \).

Diamond and Mirrlees (1971, Lemma 1) demonstrated that this condition is sufficient for the production efficiency in the optimally taxed economy only with public production. All contributors to the problem thus follow this and we do too.

The optimum tax problem here is

\[
\max_{t,\tilde{r}} V(p + t, \tilde{r}) \quad \text{sub. to} \quad t \sum_{i=1}^{N} x^t(p + t, \tilde{r}) + p y^G \geq 0, \quad y^G \in Y^G,
\]

where it is supposed that demand and supply balance under \( p \), \( q \) and \( y^G \), and that the

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\(^3\) We shall deal with the production efficiency problem in the case where only the commodity tax uniform on every household is available, and assume away (differential) producer taxes and/or poll taxes. This is the problem originally posed by Diamond and Mirrlees, we believe. This point has much relevance to the problem because: Imagine such examples presented by Mirrlees (1972, p. 107) and Sadka (1977, p. 386), and suppose that differential producer taxation is possible. Then, an allocation which is inefficient and gives a higher social welfare than any allocation under efficient production can be always attainable. (This is true even if all producers are assumed to have smooth production sets, differing from Mirrlees and Sadka's assertion.)

Note further that Mirrlees (1972), Munk (1980) and Stiglitz and Dasgupta (1972) analyze also the production efficiency problem in the situation where taxation on producer is possible as well as that on consumer. Though it is an interesting and relevant problem, it is left for another discussion. Also, though Mirrlees (1986, Theorem 4.1) says production efficiency always prevails if poll tax (subsidy) is available, this will not be the problem due to Diamond and Mirrlees (1971). Besides, R. Guesnerie (1977) and A. Smith (1983) analyzed the same problem as here in the process of tax reform.

\(^4\) Vector (in)equalities follow the usual notation as: \( a, b \in \mathbb{R}^n, a \geq b \iff a_j \geq b_j, \forall j; a > b \iff a_j > b_j, \forall j; \) and \( a \geq b \iff a \geq b \) and \( a \neq b \).
government just avoids its budget deficit. We may note here that \( p \) and thus \( \hat{t} \) are determined in general dependent upon \( t \) and \( y^G \).\(^5\) We should repeat that (1) reveals that taxes other than the commodity tax are not available by the government as well as that the government maximizes the social welfare through the commodity tax and public production.

In the succeeding arguments the solution to (1), the optimum pair of taxes and public production, is assumed to exist and written as \( \{t^*, y^G*\} \), and the other variables at the optimum are denoted with asterisk (*). The problem of production efficiency in the optimal commodity tax economy is to ask whether the optimum allocation \( y^* = x^* \) is attained on the frontier of \( Y \), i.e., if \( y^* \in \text{bdry } Y \).\(^6\) or not.

3. The Preliminary and Preceding Results

It will make easy and clear the following argument to summarize preliminary and preceding results on the present production efficiency problem in the optimal tax economy. We shall present the results relevant and important for our argument below as in the following.

Basic and Preceding Results

1. When public production is assumed away ( \( Y^G = \{0\} \)), the production efficiency always prevails \( (y^* \in \text{bdry } Y) \) irrespective of (the optimum) commodity taxes.

2. When production is carried out only by the public process, the production efficiency obtains under the optimum commodity tax, if Assumption 1 is satisfied. (Diamond and Mirrlees, 1971, Lemma 1).

3. Both the private and public productions are carried out efficiently in the sense that they are on the boundary of the respective production set \( (y^P* \in \text{bdry } Y^P \text{ and } y^G* \in \text{bdry } Y^G) \) under the optimum commodity tax, if Assumption 1 is satisfied.

4. The aggregate production (composed of the private and public production) efficiency obtains, if Assumption 1 is fulfilled and private firms' profit is kept constant when the firms face demand changes.

Some remarks on these results will be appropriate. First, Result 1 is direct from that \( y^P* \in \text{bdry } Y^P \) by the firms' profit maximization, and \( y^P* = y* \) and \( Y^P = Y \) by assumption. Though this result is simple, we may note the following implication of this: (i) Even if there remains possibility to attain a higher social welfare with inefficient production, such a configuration is ruled out through profit-maximizing behavior of the private producers. (ii) This result depends on that we have excluded the possibility of introducing differential producer taxation and thus of controlling private production, differing from Stiglitz and Dasgupta (1971), Mirrlees

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5) The equilibrium price \( p \) (and therefore \( \hat{t} \)) may not be uniquely defined for given \( t \) and \( y^G \). However, this neither causes any trouble nor leaves ambiguity to the problem here because the problem (1) by definition supposes and requires to choose one that maximizes the social welfare if there exist plural equilibria. \( p^* \) defined below, thus, denotes the equilibrium price assuring the highest social welfare when \( t^* \) and \( y^G* \) are set forth by the government.

6) \( \text{bdry } S \) means the set of boundary points of \( S \), and \( \text{int } S = S \setminus \text{bdry } S \).
(1972, p. 107) and Sadka (1977, pp. 386-387). However, it seems that the consideration of differential producer taxation blurs what and where is the production efficiency problem under the optimum commodity taxation. (iii) Because this is derived only from producers' profit-maximizing behavior, the result holds even in a case with public production if public producers seek maximum profit by such a reason as being on a self-paying basis. (However we here assume the public producer does not behave so, see eq. (1).)

Second, Result 3 also is straightforward from Results 1 and 2. That is, that \( y^p \in \text{bdry } Y^p \) always holds by profit maximization, and that \( y^G \in \text{bdry } Y^G \) is shown by following the same reasoning as Diamond and Mirrlees in Result 2. This result is natural, and says that the optimum is not attained with an interior public production, and the aggregate production inefficiency due to that by the public producer is not caused.

Third, Result 4 is a slight generalization of Diamond and Mirrlees' Lemma (Result 2). (Thus, it covers Result 2.) The assumption of the constancy of the private producers' profit is important for this result, and when noting the assumption, a proof for this result is obtained following the same way as Diamond and Mirrlees. We should note also that this condition of constant profits is meant that it is determined exogenously, and is satisfied if one of the following is met

(a) Each of \( Y^k \)'s is a cone,

(\( \beta \)) such a profit tax that keeps after tax profit constant, including one with the rate of 100%, is imposed on all private producers, or

(\( \gamma \)) every private production \( y^k, k = 1, \ldots, K \), is on a hyperplane in a range relevant for the argument.

First two of these are the cases often discussed in the literature referred to in the introduction.

4. Production Efficiency and Inefficiency in Optimally Taxed Economy

The arguments in Sections 1-3 above show that the problem of production efficiency to be analyzed lies in the case where both the private and public producers exist and private producers' profits vary when private producers face different demand situation. That is, the problem in this case is left unanswered in the preceding literature. Here we will first present an answer to this question, and second explain by examples how the assumptions in the result below play an indispensable role.

4.1. It will make the subsequent argument clear to start with stating the result to be proved.

Proposition

Suppose that Assumption 1 is satisfied and that the optimum tax equilibrium exists. Then, the aggregate production (composed of the private and public production) efficiency obtains, if (i) \( Y^p \not\subset Y^G \) and (ii) all goods are taxable.

A note on this proposition is in order. This proposition includes not only the case with smooth and strictly convex production sets but also the case with such production sets having kinked points or cusps. Also, the variability of the private producers' profits is supposed, though not stated explicitly, since the constant case is dealt with in Result 4 above.
Now let us turn to a proof of this proposition. We here have to follow a reasoning very different from Diamond and Mirrlees’ in the Results summarized above, as found below. To prove this, firstly define the following two sets based on the optimum allocation \( \tilde{x}^* = (x_1^*, \ldots, x_N^*) \),

\[
D_i(x^*) = \{ x_i + w_i \in X_i \mid u'(x_i + w_i) > u'(x_i^* + w_i) \}, \quad i = 1, \ldots, N, 
\]

\[
D(x^*) = \{ x \in R^l \mid x = \sum_{i=1}^N x_i, \quad x_i \in D_i(x^*) \}
\]

\( D_i \) and \( D \) are convex by the quasi-concavity of \( u' \)'s. Note that from the definition of \( \tilde{x}^* \), \( x_i^* \) and \( x^* \) are tangent to the hyperplane \( H^* \) with normal \( q^* \).

Secondly, note Results 2 and 3 imply that \( D(x^*) \) must not join \( y^{p*} + Y^G \) (at least in the neighborhood of \( x^* = y^* \)). This is because, if \( D(x^*) \) and \( y^{p*} + Y^G \) share a set of points around \( y^* \), there will exist a consumer price that induce households' demands being in the set, assuring a higher social welfare than that at \( \tilde{x}^* \) as in Result 3 above. It is a contradiction. Thus, possible relationship between \( y^{G*} + Y^P \) and \( D(x^*) \) must be such that:

(a) each set is separated from the other by \( H^* \) at \( y^* \), or,

(b) the two sets join.

We shall show in turn that \( y^* \) must be on the frontier of \( Y \) in both cases.

(A) First consider the case (a) above. The result is easily ascertained here as follows: It is recalled first that both \( Y^P \) and \( Y^G \) are tangent to a hyperplane with normal \( q^* \) at \( y^{p*} \) and \( y^{G*} \) respectively. This means that there is no possible production \( y^{p'} \) and \( y^{G'} \) more efficient than \( y^{p*} \) and \( y^{G*} \) respectively, i.e., no \( y^{p'} \) and \( y^{G'} \) satisfying \( y^{p'} \geq y^{p*} \) and \( y^{G'} \geq y^{G*} \), which is equivalent to say \( y^* = y^{p*} + y^{G*} \) be on the boundary of \( Y^P + Y^G \). (See Fig. 1.a.)

(B) We turn to the case (b). We below proceed supposing the optimum-tax equilibrium is at point \( A \) in Fig. 1.b, though it can be at point \( B \). This is because the proving arguments are parallel in both situations. We shall show that if \( A \) is assumed to be interior to \( Y \), then there exist a
set of equilibrium allocation and prices \( \{(\hat{x}, \hat{y}); \hat{x}; \hat{y}; \hat{p}\} \) satisfying

\[
W(u^1(\hat{x}^1 + w^1), \ldots, u^N(\hat{x}^N + w^N)) > W(u^1(x^1\prime + w^1), \ldots, u^N(x^N\prime + w^N))
\]

against \( \{(i*, y^G*); \hat{x}^*; \hat{y}^*; p*\} \), which shows a contradiction and thus that the point \( A \) cannot be interior to \( Y \) in the case (b) too.

At the first place, let us ascertain the following two points. One is that household \( i \)'s demand change after income changes of \( \Delta I_i \), be on the hyperplane with normal \( q_i^* \) distant by \( \Delta I_i \) from \( x_i^* \). This is direct from differentiation of the household \( i \)'s budget with respect to income

\[
\sum_{j=1}^J q_j^* x_{ji}^* \Delta I_i = \Delta I_i
\]

where \( x_{ji}^* = \partial x_{ji}^*/\partial I_i \). This means that such demand change is small when \( \Delta I_i \) is so (especially if demands are normal).

The other is that if the point \( A = y^* \) is interior to \( Y \), then a north-east point is attainable by both a north-west change in private production \( \Delta y^P \) along the frontier of \( Y^P \) and a south-east change in public production \( \Delta y^G \) along the frontier of \( Y^G \), i.e., \( \Delta y = \Delta y^P + \Delta y^G > 0 \) is possible. (See Fig. 2. \( y^{**} \neq 0 \) is assumed without loss of generality since \( Y^P \not\subset Y^G \).) Further, since \( y^{**} + \Delta y^P \in \text{bdry} Y^P \), there exists a producer price \( p \) supporting \( y^{**} + \Delta y^P \). Thus let us define \( \Delta p = p - p^* \), denote by \( \Delta r^k \) the change of firm \( k \)'s profit which is brought by the production change \( \Delta y^k \) which forms \( \Delta y^P \), and designate by \( \Delta I_i \) the change of household \( i \)'s income due to the change of \( \Delta r^k \)'s. This \( \Delta I_i \) is given as

\[
\Delta I_i = \sum_{k=1}^K \theta_{ki} \Delta y^k = \sum_k \theta_{ki} \Delta r^k
\]

since \( (p^* + \Delta p) \Delta y^k = 0 \) for any \( k \) in both cases where \( Y^P \) is smooth or not. (Firm \( k \)'s profit change \( \Delta r^k \) brought by the production change above is necessarily positive at point \( A \). So we shall assume \( \Delta I_i > 0 \) in the subsequent argument.)
We show that these two properties mean existence of an equilibrium allocation and prices satisfying (4), a contradiction to that \( x^* \) is an optimal allocation. We have to demonstrate that there exist a supply change \( \Delta y \) and the corresponding income change \( (\Delta I^1, \ldots, \Delta I^N) \) satisfying

\[
W(u^*, \ldots, u^N) > W^* \quad \text{and} \quad \sum_{i=1}^{N} x_i^* \Delta I^i \leq \Delta y
\]

where \( x_i^* = (x_{i1}^*, \ldots, x_{iJ}^*) \), \( u^i = u^i(x^* + x^*_i \Delta I^i + w^i) \) and \( \Delta I^i = \Delta I^i/(1 + \alpha) \) with a non-negative scalar \( \alpha \) denoting degree of proportional increase of commodity tax rates.

To prove (6), we have to show that \( \Delta I^i \) or \( \Delta I^i' \) is sufficiently small (compared to \( \Delta y \)). The arguments below differ depending on size of profit change \( \Delta r \)'s. So we differentiate the cases to be analyzed depending on that \( Y^p \) and \( Y^g \) are smooth or not, and consider only the cases where both are smooth and both are not. (The remaining cases also are proved in parallel.)

(B-i) First, consider the case that \( Y^p \) and \( Y^g \) are smooth at \( y^p \) and \( y^g \) respectively. Here, if supply change \( \Delta y \) is small, \( A^p \) and \( y^* \Delta p \) also are small. It follows that corresponding household income change is small, and so are their demand changes by (5). On the other hand, the amount of supply change measured by the distance from hyperplane \( H^* \) is given as (see Fig. 2)

\[
q^* \Delta y = q^* \Delta y^p + q^* \Delta I^G.
\]

While the second term in the right-hand side is nearly zero by assumption of smoothness of \( Y^G \) (see Guesnerie (1979), and also Figs. 1.b and 2), the first term is absolutely positive. These facts mean that supply change of \( \Delta y^p \) causes small demand change \( \Delta x = \sum_{i=1}^{N} x_i^* \Delta I^i \) (due to its effect on firms' profit), while \( \Delta y \) is relatively large, and thus \( \Delta y \) suffices \( \Delta x \) (i.e., \( \Delta y > \Delta x \)). This means that (6) holds for this \( \Delta x \) and \( \Delta y \) (with \( \alpha = 0 \)).

(B-ii) Consider then the case where both \( Y^p \) and \( Y^g \) are not smooth. Here it is not assured that the change of firms' profits and therefore that of households' demand caused by the supply alteration of \( \Delta y^p \) are small. However, we can show that real increase of households' income can be made small by proportionately raising commodity taxes (or consumer prices), and thus the same conclusion as above follows.

To show this, first, recalling the possibility to proportionately increase consumer prices, note the upper limit of this proportionality that assures a higher social welfare than \( W^* \). Suppose that original increase of households' income is \( (\Delta I^1, \ldots, \Delta I^N) \) and that consumer prices are made \((1 + \alpha)\) times large. Then real increment of household \( i \)'s income \( \Delta I^i \) is

\[
\Delta I^i = \frac{I^i + \Delta I^i}{1 + \alpha} - I^i = \frac{1}{1 + \alpha}(\Delta I^i - \alpha I^i).
\]

The upper limit of \( \alpha \) that assures higher social welfare than \( W^* \) is known from the condition

\[
\sum_{i=1}^{N} W_i^i \lambda^i \Delta I^i > 0,
\]

where \( \lambda^i \) denotes household \( i \)'s marginal utility of income. That is,

\[
\alpha < \begin{cases} \frac{\Sigma \sigma^i \Delta I^i}{\Sigma \sigma^i I^i} \equiv \alpha' & \text{if } \Sigma I^i > 0 \\ \infty & \text{if } \Sigma I^i = 0 \end{cases}
\]

where \( \sigma^i \equiv W^i \lambda^i \).

Second, if \( \Sigma I^i = 0 \), there is no upper limit on \( \alpha \), and increase of households' real income \( \Delta I^i \)...
and their demands can be made arbitrarily small. Then the result wanted easily follows.

Third, consider the case of \( \Sigma r^i > 0 \), and note the amount of total income increase when the degree of proportionate price increase is \( \alpha' \). This is given by

\[
\Sigma_i \Delta I^i = \frac{1}{1 + \alpha'}(\Sigma_i \Delta I^i - \alpha' \Sigma I^i)
\]

\[
= \frac{\Sigma \sigma^i I^i}{\Sigma \sigma^i (I^i + \Delta I^i)} \left[1 - \frac{\Sigma I^i}{\Sigma \sigma^i I^i} \right] \Sigma \Delta I^i
\]

Now we shall show that \( \Sigma \Delta I^i \) is near zero or much smaller than \( \Sigma \Delta I^i \), because: (i) The first term at RHS of the above is less than 1, and becomes less as \( \Delta I^i \)'s get large. (ii) The second term, that in the bracket, is near zero, or much smaller than 1. The second point (ii) is ascertained as follows. The value of the bracket term depends on relations among \( \sigma^i \)'s, \( I^i \)'s and \( \Delta I^i \)'s. The relation between \( \sigma^i \)'s and \( I^i \)'s is classified as positively or negatively correlating, or being independent, and when they correlate negatively or are independent, the bracket term is less than or near zero, if negative correlation between \( \sigma^i \)'s and \( \Delta I^i \)'s is not strong. Since we can suppose that there is no definite correlations between \( \sigma^i \)'s and \( I^i \)'s or \( \Delta I^i \)'s from their definitions, it follows that the value of the term is near zero. This, together with the point (i) above, implies that \( \Sigma \Delta I^i \) is much smaller than \( \Sigma \Delta I^i = \Sigma \Delta r^i \). This proves that (6) holds for some \( \Delta y = \Delta y^p + \Delta y^G \) also in the case (b) with both \( Y^p \) and \( Y^G \) not smooth.

This completes our proof for proposition.

4.2. The role of the condition (i) in Proposition above may not be apparent, while that of the condition (ii) is seen in the preceding proof. We shall here present two examples which are completely modeled and show that these conditions are indispensable for Proposition.

(A) First, to explain the necessity of the condition (i) in Proposition, let us suppose the contrary, i.e., that \( Y^p \not\subset Y^G \), and show by exemplification that the production inefficiency may be led under this situation.

Before entering the argument, note the following two points as to this condition. First, note that example below implies that the smoothness of \( Y^p \) is not sufficient for production efficiency (cf. note 2)). Second, in the situation supposed here efficient production means \( r(p) = 0 \) for any \( p \) supporting \( Y^G \) since private production must be zero. Stiglitz and Dasgupta and Mirrlees (1972) argued, by illustration, that an allocation with an inefficient aggregate production can attain a higher social welfare than that with efficient production. However, the former said also the optimum may not exist, and the latter's argument relies on a special assumption that the government initially holds a good to be distributed. Our example below intends to show that it is completely modeled and that an optimum with inefficient production exists without any specific assumption.

A complete example is as follows: There are the representative household, the representative firm, and two commodities \( (N = K = 1, J = 2) \). Household's preference is given by the Cobb-Douglas utility function,

\[
U = (x_1 + w) x_2^\beta \quad (\alpha + \beta = 1, \alpha, \beta > 0)
\]

where \( w = (w_1, 0) \) is the initial endowment of the household and thus the society. It maximizes its utility under the budget \( q_1 x_1 + q_2 x_2 = r \) where \( r \) is profit distributed from firm. Both the
firm and government have productive technology, which produces good 2 using good 1 as input. Firm’s production possibility is assumed as

$$
\begin{align*}
\left( y_1^p + \frac{a}{\sqrt{2}} \right)^2 + \left( y_2^p + \frac{a}{\sqrt{2}} \right)^2 & \leq a^2 \quad \text{if} \quad y_1^p \geq -\frac{a}{\sqrt{2}}, \\
\left( y_2^p \right)^2 & \quad \text{if} \quad y_1^p < -\frac{a}{\sqrt{2}}.
\end{align*}
$$

(a > 0, y_1^g \leq 0, y_2^g \geq 0) \quad (8)

and that of government is

$$
\begin{align*}
by_1^g + y_2^g & \leq 0 \quad \text{if} \quad -d \leq y_1^g \leq 0, \quad y_2^g \leq 0, \\
y_1^g + y_2^g & \leq \frac{a(b-1)}{2} \quad \text{if} \quad y_1^g \leq -d.
\end{align*}
$$

(b > 1, d > 0) \quad (9)

These production functions mean the production sets $Y^p$, $Y^g$ and $Y = Y^p + Y^g = Y^g$ as in Fig. 3 and that $Y^p$ is inefficient at all.

As Fig. 4 display, we can show in this setting a higher utility level (denoted by the indifference curve $I' I'$) is attainable by an inefficient allocation than the highest (corresponding to the indifference curve $I I$) attained through an efficient allocation.\(^7\)

(B) Lastly, reconsider the role of full taxability assumption from a different viewpoint, while it is apparent in Section 4.1. that this assumption played an indispensable role to the arguments. That is, we shall present a concrete model where production inefficiency results if the full taxability condition is assumed away. Though some literature as Munk (1980) pays attention to this case, our argument differs from him in that we deal only with commodity taxes while he considers the problem assuming producer taxes.

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\(^7\) A detailed argument is given in Yamada (1988), and is obtainable on request.
The following model satisfies our present purpose. The model is the same as above except that production possibilities are simplified. The firm’s production possibility is simplified as

\[
\begin{align*}
(y_1^f + a)^2 + (y_2^f)^2 & \leq a^2 & \text{if } -a \leq y_1^f \leq 0 \\
y_2^f &= a & \text{if } y_1^f < -a
\end{align*}
\]

and that of government is

\[
y_1^g + y_2^g \leq 0, \quad y_1^g \leq 0
\]

The production sets $Y^f$, $Y^g$ and $Y = Y^f + Y^g$ are as in Fig. 5 and $Y^f \not\subset Y^G$. The government is assumed to consume $g = (g_1, 0)$, which has to be financed through the commodity tax. (As stated above, we may suppose $Y^G - g$ instead of $Y^G$ and $g$.) Assume that the good 2 is exempted from tax and $p_2 = q_2 = 1$. Assume further that household's preference is represented by indifference curve $II$ in Fig. 5. Then, consumption frontier for the household is given by $Y - g$, and there is no household's indifference curve which is tangent to a budget line through the point $(0, r(1))$ on the frontier of $Y - g$. ($r(1)$ denotes the profit the firm obtains at $p = 1$. Note that aggregate production can be on $bdry Y$ only when $p = 1$ prevails.) This means that there is no equilibrium attained on the frontier of $Y$, and the optimum allocation must thus be interior to $Y$. See Fig. 5.

5. Concludings
Let us summarize the arguments thus far. First, Result 1 says that if there exist only private producers as in the most of arguments, no room of production inefficiency under the optimum commodity taxation is left. Second, Results 1 and 2 imply production inefficiency can be
brought only when both the private and public producers exist under Assumption 1. Third, even then, Proposition shows that production inefficiency under the optimum tax is caused only in the very restricted and not interesting case where production technology of the private producers, under variable returns and thus with positive profits, are wholly less efficient than the public producer, assuming all goods are taxable.  

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REFERENCES


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8) As the last but a little different point, let us note also that both the terms, “production efficiency” and “desirability of production efficiency” are used, and that they reflect a difference in understanding of, or attitude to, the problem. I.e., when the former is used, the analysts ask whether the efficiency obtains in an economy with the optimum commodity tax, assuming away another types of taxes even if their introduction (especially that of producer and/or poll taxes) could increase social welfare. On the other hand, the latter is used when asking if there is no room to attain a higher social welfare through any type of taxation. We here considered the former corresponds to the problem due to Diamond and Mirrlees.
M. Yamada: Optimal Taxation and Production Efficiency Reconsidered


