In the presence of asymmetric information between a lender and a borrower with costly monitoring, we investigate a contract form and an optimal loan size are investigated. The optimal contract has the features of a standard debt contract. However, the optimal loan size does not always take the form of maximum equity participation. If per unit of return from lending is a decreasing function of the loan size, the contract takes the form of maximum equity participation. Conversely, if per unit of return from lending is an increasing function of the loan size, overinvestment or overborrowing may arise.

1. Introduction

Under an environment of the costly state verification in which there is asymmetric information between a lender and a borrower-entrepreneur on the ex post return of the entrepreneur’s project and monitoring of an lender is costly, the optimal contractual arrangement has the features of a standard debt contract: the payment is state-contingent if the return is less than a certain threshold and is constant if the return is greater than it. The endogenous derivation of the standard debt contract begins with the seminal work of Townsend (1979). Afterward, several elaborations are developed by Diamond (1984), Gale and Hellwig (1985), and Williamson (1987).\(^{1}\)

However, almost all of the studies in this field has not explicitly solved the optimal loan size.\(^{2}\) They have developed many contributions associated with the debt contract without paying attention on the optimal loan size, taking it for granted that “maximum equity participation” is satisfied according to the terminology of Gale and Hellwig. Maximum equity participation is defined by the following. If the indivisible project requires one unit of the consumption good as an input, borrowers would be willing to receive only one unit if they have no equity. If they have an equity, they borrow only the remaining loan required for this project.\(^{3}\)

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1) Diamond (1984) shows that financial intermediation arises endogenously as an incentive-compatible mechanism to efficiently control the agency problem associated with debt contracts. Gale and Hellwig (1985) develops a model in which the debt contract induces underinvestment. Williamson (1987) shows that an equilibrium associated with debt contracts may generate equilibrium credit rationing.

2) At least to my knowledge, two exceptions are Gale and Hellwig (1985) and Seward (1990). Both papers examine an optimal loan size under a divisible production technology.

3) This terminology of “maximum equity participation” results immediately from Gale and Hellwig (1985). Alternatively, Williamson (1987) and de Meza and Webb (1987) denote it as “full equity participation.”
M. Sakuragawa: Overinvestment and Overborrowing under the Debt Contract

In this paper we investigate the optimal loan size in an environment in which the debt contract is endogenously derived. Suppose an environment where there are no other investment opportunities than an indivisible project whose rate of return is strictly above the loan interest rate. An optimal contracting arrangement involves the joint determination of the form of a contract and the optimal loan size. The optimal contract has the features of a standard debt contract, but the optimal loan size does not always satisfy maximum equity participation. We find that whether maximum equity participation is satisfied or not owes entirely to the property of a lender’s profit function.

Two types of economies are distinguished according to whether or not a lender prefers to supply a greater loan. In one economy in which per unit of return from lending is a decreasing function of the loan size, the debt contract satisfies maximum equity participation. Each entrepreneur would raise loans required for his indivisible investment project. Allowing for different levels of endowments among borrowers, equilibrium may exhibit credit rationing. In equilibrium borrowers with relatively large endowments receive loans while others with relatively small ones cannot.

In another economy in which per unit of return from lending is an increasing function of the loan size, maximum equity participation is not always satisfied. The existence of another project and the specific form of monitoring technology allows lenders to prefer to supply a larger loan than would be required for the indivisible project. In spite of the fact that the rate of return of another opportunity is strictly below the loan interest rate, entrepreneurs may raise loans not only for the indivisible project but also for the project with “low” return. The behavior of entrepreneurs who would make overinvestment arises because they may be more beneficial by choosing the pair of a contract of a smaller loan interest rate and a larger loan than the pair associated with maximum equity participation. Furthermore, we show that even in an environment where entrepreneurs have access only to the indivisible project, they may overly borrow to finance this project. The contract may be involved with overborrowing.

This paper is organized as follows. Section 2 describes the model. Section 3 derives the optimal contract. Section 4 explores the properties of the contract and examines the loan size. Section 5 derives an environment in which credit rationing arises. Section 6 derives overinvestment and section 7 discusses on overborrowing. Finally, some concluding remarks are stated.

2. The Model

There are two periods. Period zero is a planning period and in period one consumption takes place. There is a single consumption good which is perishable between periods. There is a continuum of agents. Agents are either lenders or entrepreneurs. \(\alpha (0 < \alpha < 1)\) is the fraction of agents who are lenders and \(1 - \alpha\) is the fraction of agents who are entrepreneurs. Equilibrium conditions are written in per capita terms.\(^4\)

Entrepreneurs can potentially have access to two types of investment projects. The first project is indivisible and the return is uncertain and “high.” The second project is divisible and the return is certain and “low.” The first project yields an uncertain \(\omega\) units of the consumption

\(^4\) Boyd and Prescott (1986) and Williamson (1987) take a similar approach.
good in period one for an input of one unit of the consumption good in period zero. \( \omega \) is a random variable which is distributed according to the probability density function \( f(\omega) \) which is positive and continuously differentiable on the support of \([0, \omega^+]\) with a mean of \( \mu \). Let \( F(\omega) \) denote the associated probability distribution function. Individual returns of the first project are independently and identically distributed across entrepreneurs. The second project yields certain \( r \) units of the consumption good per unit invested between periods.\(^5\) We assume that \( \mu > r \), that is, the expected return from the first project is strictly greater than the return from the second project. Entrepreneurs are ordered in terms of endowment, \( y \), in period zero. \( y \) is distributed according to the probability density function \( g(y) \). \( g(y) \) is positive and continuously differentiable on the support of \([y^-, y^+]\), where it is assumed that \( 0 < y^- \leq y \leq y^+ < 1 \). The associated probability distribution function is \( G(y) \). The distributions of \( \omega \) and \( y \) are independent.

Lenders are endowed with one unit of the consumption good in period zero which may be lent to entrepreneurs or be invested in an alternative opportunity. Lenders have access to the same project as the second project faced by entrepreneurs.

Focus attention on an economy with \( y < 1 \) so that all entrepreneurs must borrow from lenders for their first projects.\(^6\) Next from the assumption of \( \mu > r \), the following inequality is satisfied such that

\[
(1) \quad \mu - (1 - y)r > yr.
\]

The left-hand side in (1) is the expected return when an entrepreneurs endowed with \( y \) units undertakes the first project under perfect information. The right-hand side is the return when he invests his endowment in the second project and never borrow. (1) implies that all entrepreneurs would be willing to raise loans for the first project at least under perfect information.

The realized return of the first project is costlessly observable only to the individual entrepreneur who operates their own projects. Any other agent must incur a cost of \( \beta \) units of effort to monitor the realized return on any one entrepreneur although all agents know \( f(\omega) \). This specification of a monitoring cost as a fixed cost is to capture the lumpy expenses associated with information acquisition.\(^7\) Monitoring is deterministic.\(^8\)

Both agents are risk-neutral maximizers. Entrepreneurs maximize the expected period one consumption \( E[c] \), where \( c \) is the consumption in period one and \( E \) is the expectation operator conditional on information available at period zero. Lenders are endowed with an unbounded

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5) The case of \( r = 0 \) is discussed below in section 7. \( r \geq 1 \) is unnecessary because the consumption good is assumed to be perishable between periods.

6) The examination of an economy with \( y \geq 1 \) is trivial. If \( y \geq 1 \), the allocation achieved is the same as that under perfect information. All entrepreneurs operate the first project and the remaining quantity of \( y - 1 \) units are invested in the second project.

7) This specification of the monitoring cost is broadly used for many papers in this field. See Townsend (1979), Diamond (1984), Williamson (1987), Seward (1990), and etc.

8) As in Townsend (1988) or Mookherjee and Png (1989), allowing for stochastic monitoring in more general setups may yield an optimal arrangement which bears little resemblance to a simple debt contract. In contrast, Bernanke and Gertler (1989) shows that in a risk-neutral setup, the essential features of a debt contract are preserved even in an environment with stochastic monitoring.
quantity of effort and maximize $E[c - e]$, where $e$ is the quantity of effort expended for monitoring. Both agents are protected by limited liability constraints, such that $c \geq 0$ and $e \geq 0$. Entrepreneurs who are endowed with zero units of effort cannot lend their endowment to other agents nor form a coalition between them because they cannot verify the ex post return of other members' projects. Finally, we make the following assumption:

**Assumption 1**

$$\alpha + (1 - \alpha) \int_{y^+}^{y} vg(y) \, dy > 1 - \alpha,$$

so that the demand for credit is at least potentially satisfied under perfect information.

Under perfect information, all entrepreneurs undertake the first project from (1), but are indifferent as to whether to invest in the second project. Without loss of generality, assume that entrepreneurs never undertake any project if the net additional return is zero. This implies that under perfect information they undertake only the first project.

### 3. Optimal Contract

Given that a loan of $b$ units is raised, an entrepreneur endowed with $y$ units would invest one unit in the first project and the remaining $b + y - 1$ units in the second project.

The entrepreneur emits a signal $\omega_s$ to a lender to satisfy $\omega_s \in [0, \omega^+]$. The contract specifies that monitoring occurs if $\omega_s \in S$, while it does not if $\omega_s \not\in S^C$, where $S$ is a monitoring region which is a set of realizations of $\omega_s$ such that there is monitoring (with complement $S^C$). The payment to a lender will be $m(\omega)$ if $\omega_s \in S$, and $z(\omega_s)$ if $\omega_s \in S^C$, where $m(\omega)$ and $z(\omega_s)$ are functions which must obey the feasible conditions, $0 \leq m(\omega) \leq \omega + (y + b - 1)r$, and $0 \leq z(\omega_s) \leq \omega + (y + b - 1)r$, respectively. If the entrepreneur chooses $\omega_s \in S^C$, he will always choose $\omega_s^* = \arg \min z(\omega_s^*)$. Therefore, when monitoring does not occur, the payment the entrepreneur makes to a lender is a constant denoted by $b \cdot x$ given that a loan of $b$ units is raised. Now the optimal payment schedule in the monitoring region, $m(\omega)$, must be determined. Incentive compatibility requires that

$$(2a) \quad \omega_s \in S \quad \text{if} \quad m(\omega) < bx - (b + y - 1)r,$$

and

$$(2b) \quad \omega_s = \omega_s^* \quad \text{if} \quad m(\omega) \geq bx - (b + y - 1)r,$$

where the return from the second project $(b + y - 1)r$, would be used as collateral if it exists. It reflects the fact that the return from the second project is always observable because the raised loan size is known to the lender. Conditions (2a) and (2b) allow us to determine the realizations of $\omega$ over which monitoring occurs, given $m(\omega)$.

Let $B$ and $B^C$ be non-intersecting subsets of $[0, \omega^+]$, with $B \cup B^C = [0, \omega^+]$, we derive $B = \{\omega: m(\omega) < bx - (b + y - 1)r\}$, and $B^C = \{\omega: m(\omega) \geq bx - (b + y - 1)r\}$. That is, for $\omega \in B$, monitoring occurs and for $\omega \in B^C$, monitoring does not occur. The optimal contract is a payment schedule $\{m(\omega), x\}$, given that a loan of $b$ units is raised, which maximizes the entrepreneur's expected return, while giving each lender a level of expected return of at least $r$ per unit...
invested:

\[
(3a) \quad \max_{m(\omega), x} \int_B \{\omega - m(\omega)\} f(\omega) \, d\omega + \int_C \{\omega - bx + (b + y - 1)r\} \, f(\omega) \, d\omega,
\]

subject to

\[
(3b) \quad \int_B \{m(\omega) - \beta + (b + y - 1)r\} \, f(\omega) \, d\omega + \int_C b \times f(\omega) \, d\omega \geq br.
\]

**Proposition 1**

The optimal payment schedule satisfies \(m(\omega) = \omega\).

The proof appears in Appendix. If \(\omega \geq bx - (b + y - 1)r\), it is in the interest of the entrepreneur to pay \(b \cdot x\) to a lender, while, if \(\omega < bx - (b + y - 1)r\), the entrepreneur defaults, monitoring occurs, and the lender receives the entire returns on the projects. Therefore, \(x\) is interpreted as a gross loan interest rate and the state when monitoring occurs as the state of bankruptcy.

### 4. Debt Contract with Maximum Equity Participation

Let \(\pi^e(x, b, y)\) denote the expected profit of an entrepreneur with an endowment of \(y\) units who would raise a loan of \(b\) units such that

\[
(4a) \quad \pi^e(x, b, y) = \int_{bx-(b+y-1)r}^{\omega^+} \{\omega - bx\} f(\omega) \, d\omega + \int_{bx-(b+y-1)r}^{\omega^\ast} \{b + y - 1\} \, f(\omega) \, d\omega,
\]

for \(b \geq 1 - y\). The first term is the expected return from the first project and the second term is the one from the second project. Let \(\pi(x, b, y)\) denote the lender’s expected profit from a loan of one unit to an entrepreneur with an endowment of \(y\) units who would receive \(b\) units, such that

\[
(4b) \quad \pi(x, b, y) = \frac{1}{b} \int_0^{bx-(b+y-1)r} \{\omega + (b + y - 1)r\} \, f(\omega) \, d\omega
\]

\[
+ x\{1 - F[bx - (b + y - 1)r]\} - \frac{\beta}{b} F[bx - (b + y - 1)r].
\]

The first term is the return when the entrepreneur defaults, the second is the expected interest payment per unit invested, and finally the third is the expected monitoring cost measured in per unit of the consumption good.

Differentiating (4) with respect to \(x\) gives

\[
(5) \quad \pi_1(x, b, y) = 1 - F[bx - (b + y - 1)r] - \beta f[bx - (b + y - 1)r].
\]

**Assumption 2**

\[f(\cdot) + \beta f'(\cdot) > 0.\]

Assumption 2 ensures that the lender’s profit function \(\pi(\cdot)\) is strictly concave in the loan interest rate. Substituting the maximum \(x^* = [\omega^* + (b + y - 1)r]/b\) into (4) gives

\[
(6) \quad \lim_{x \to x^*} \pi(x, b, y) = -\beta f(\omega^*) < 0.
\]
Assumption 2 and (6) ensure that $\pi(x, b, y)$ reaches a maximum for some interior value. Without Assumption 2 multiplicity of equilibria may emerge.

Lemma 1

The loan interest rate is strictly above the rate of return of the second project.

The proof appears in Appendix.

Differentiating (4) with respect to $b$ gives

$$
\pi_2(x, b, y) = -\frac{1}{b^2} \int_0^{bx-(b+y-1)r} \left\{ \omega + (b + y - 1)r \right\} f(\omega) \, d\omega 
+ \frac{r}{b} F\{bx - (b + y - 1)r\} - \frac{x - r}{b} \beta f\{bx - (b + y - 1)r\} 
+ \frac{\beta}{b^2} F\{bx - (b + y - 1)r\} \geq 0.
$$

The first term captures that the per unit of the expected return in the event of default is decreasing in loan size. The second term captures that the safe asset funded by borrowing works as collateral, leading to an increase in the expected return in default as the loan size increases. The third term captures that the expected monitoring cost is increasing in loan size because the probability of bankruptcy is increasing in loan size. Finally, the forth term captures that the per unit of the expected monitoring cost is decreasing in loan size because the monitoring cost is incurred as a fixed cost per entrepreneur. The final term implies that lenders can exploit the economies of scale in monitoring.

The first and the third terms are negative. These two terms tend to operate to reduce the lender's profit as the loan size increases. Conversely, the second and the forth terms are positive. These two terms tend to operate to raise the profit as the loan size increases. As a whole, the sign of $\pi_2(\cdot)$ is ambiguous, depending on parameter values.\(^9\)

Assumption 3

$\pi_2(x, b, y)$ is monotonically increasing or decreasing for any $y$.\(^{10}\)

Under this assumption two cases can be distinguished according to the sign of $\pi_2(\cdot)$. In this and the next section the economy with $\pi_2(\cdot) < 0$ is investigated. In sections 6 and 7 the economy with $\pi_2(\cdot) > 0$ is analyzed.

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9) The sign of $\pi_2(\cdot)$ depends on the functional form of the probability distribution function and on the value of a monitoring cost $\beta$. Suppose that $F(\omega) = \omega^a(a > 0)$ and $r = 0$ only for entrepreneurs. We obtain

$$
\pi_2(\cdot) = \frac{1}{b^2} (xb)^a \left\{ \frac{a}{a + 1} xb + (a - 1)\beta \right\}.
$$

If $a \geq 1$, $\pi_2(\cdot) < 0$ is always satisfied, but if $0 < a < 1$, its sign is ambiguous. Given $0 < a < 1$, the more likely an economy with $\pi_2(\cdot) > 0$ to occur the greater is $\beta$. Intuitively, in more general cases with $r > 0$ a broader class of functions may permit us to obtain the latter economy.

10) Unless Assumption 3 is imposed, there may be a complicated and difficult case to analyze, where $\pi_2(\cdot)$ is positive for some entrepreneurs and is negative for others.
Two concepts are defined. One is overinvestment. Over investment is defined to occur if entrepreneurs operate not only the first but also the second project. The intuition behind this definition is as follows. As shown in Lemma 1, the loan interest rate is strictly above the return of the second project, and thus financing the second project by borrowing implies that, given that the first project is financed, entrepreneurs would be willing to invest a project whose net present value is negative.

Another concept is maximum equity participation. A contract is defined to exhibit maximum equity participation if a loan of the smallest units required for the first project is supplied. Thus if maximum equity participation is satisfied, an entrepreneur with an endowment of \( y \) units chooses a loan of \( 1 - y \) units. He operates only the first project.

**Proposition 2**

If \( \pi_2(x, b, y) < 0 \) is satisfied, the debt contract takes the form of maximum equity participation.

The proof appears in Appendix. The interpretation behind this proposition is intuitive. The profit function of the entrepreneur with an endowment of \( y \) is rewritten as

\[
\pi^*(x(b, y), b, y),
\]

where \( x \equiv x(b, y) \) is implicitly derived from \( \pi(x, b, y) = r \). We derive \( \partial x(b, y)/\partial b < 0 \). The intuition behind Proposition 2 is as follows. An increase in the loan size reduces the entrepreneur's profit in two ways, directly and indirectly. First, an increase in the loan size reduces the entrepreneur's profit directly because, other things being equal, it raises the probability of bankruptcy, the effect of which is captured by the second argument of (8). Second, it reduces the profit indirectly through an increase in the loan interest rate because, other things being equal, it also raises the probability of bankruptcy, the effect of which is captured by the first argument of (8).

5. Different Endowments and Credit Rationing

By substituting \( b + y = 1 \) into (4a) and (4b), these two functions are rewritten, respectively, as

\[
\Pi^*(x, b, y) = \int_{bx}^{\omega^+} (\omega - bx) f(\omega) \, d\omega \quad \text{for } b \geq 1 - y,
\]

and

\[
\Pi(x, b) = x - \frac{1}{b} \int_0^{bx} F(\omega) \, d\omega - \frac{\beta}{b} F(bx).
\]

where (9b) is rewritten using integration by parts.

Let \( \Pi^*(b) = \max_x \Pi(x, b) \) denote the maximum expected return a lender can earn from a loan of one unit to an entrepreneur who would fund a loan of \( b \) units. From an application of the envelope theorem we obtain \( \Pi^*(b) < 0 \) because of \( \Pi_2(\cdot) < 0 \). Denote the maximum interest rate \( x^* \) which is determined by \( \Pi_1(x^*, b) = 0 \) given \( b \). There is a cutoff level of loan size \( b^* \) such that \( \Pi(x^*(b^*), b^*) = r \) is ensured. Denote the entrepreneur with an endowment of \( y^* = (1 - b^*) \) units as the marginal entrepreneur. Entrepreneurs with an endowment of greater
than or equal to \( y^* \) units receive loans and undertake the first project. However, those with less than \( y^* \) can not receive loans since the expected return from a loan of one unit would be below the opportunity return.

There are two types of equilibria, one with credit rationing (RA equilibrium) and one without rationing (NRA equilibrium).

First, NRA equilibrium occurs if \( b^* + y^- \geq 1 \), and is characterized by

\[
\Pi(x, b) = r, \\
b + y = 1 \quad \text{for each } y \in [y^-, y^+],
\]

and

\[
a\eta = (1 - a) \int_{y^-}^{y^+} (1 - y) g(y) \, dy,
\]

where (12) is the equilibrium condition for the loan market and \( \eta(0 \leq \eta \leq 1) \) is the fraction of lenders who actually lend to entrepreneurs. All entrepreneurs undertake their first projects and never invest in the second one. The pair \( \{x, b\} \) is determined to satisfy maximum equity participation for all entrepreneurs. As implicitly derived from (10), the greater is their endowment, the smaller is their loan interest rate. The residual \( 1 - \eta \) invests in their alternative opportunities.

Second, RA equilibrium is viable if \( b^* + y^- < 1 \), and is characterized in terms of the marginal entrepreneur, by

\[
\Pi(x^*, b^*) = r, \\
\Pi_1(x^*, b^*) = 0, \\
b^* + y^* = 1,
\]

and

\[
a\eta = (1 - a) \int_{y^-}^{y^*} y \, dG(y) = (1 - a)[1 - G(y^*)],
\]

where the marginal entrepreneur with an endowment of \( y^* \) receives \( b^*(=1 - y^*) \) units and promise \( x^* \) as a loan interest rate. Entrepreneurs with an endowment of greater than or equal to \( y^* \) receive loans to ensure maximum equity participation. For the set of entrepreneurs who actually receive loans, the greater is their endowment, the smaller is their loan interest rate. Conversely, entrepreneurs with an endowment of smaller than \( y^* \) are denied loans. Those who are rejected loans invest their endowment in the second project. The latter set of borrowers are credit rationed in the sense that they would be willing to pay higher than market interest rates for their first project, but are rejected loans. There is equilibrium credit rationing where some borrowers receive loans, but others cannot.

Strictly speaking, the condition (1) is not sufficient in order to obtain the two types of equilibria. Under asymmetric information, for some parameter values the inequality of (1) may be reversed because of the existence of the monitoring cost. Some entrepreneurs might invest his endowment in the second project and never have access to lenders. They are then never credit rationed in any sense even if he does not raise loans for the first project. Here we give a necessary condition under which RA equilibrium is viable by using an example.

Suppose that \( F(\omega) = \omega \) on the support of \([0, 1]\). The mean of \( \omega \) is 0.5. From (13) and (14)
we obtain

\[ \Pi(x^*, b^*) = x^* - \frac{1}{2} b^* x^*^2 - \beta x^* = r, \tag{e-1} \]

and

\[ \Pi_1(x^*, b^*) = 1 - b^* x^* - \beta = 0. \tag{e-2} \]

For the offered pair \( \{x, b\} \), any entrepreneur strictly prefers to raise loans if and only if

\[ \Pi^*(x, b) = \mu - \beta bx - br > (1 - b)r, \tag{e-3} \]

which is equivalent to

\[ \mu - \beta bx - r > 0. \tag{e-4} \]

One necessary condition for RA equilibrium to be viable is that (e-4) is satisfied for the marginal entrepreneur with the pair of \( \{x^*, b^*\} \). If (e-4) is satisfied there exist a neighboring entrepreneur to the marginal entrepreneur who would be willing to raise loans but cannot. It arises since entrepreneurs are assumed to be continuously ordered in terms of endowment. From (e-2) and (e-4) we derive

\[ \mu - r > \beta (1 - \beta). \tag{e-5} \]

(e-5) implies that if the difference of the returns between the two projects is sufficiently large relative to the value of a monitoring cost \( \beta \), RA equilibrium may be viable. Notice that the condition of (e-5) does not depend on \( y \), so that (e-5) can be satisfied for any \( y \). From (e-1) and (e-2) we derive

\[ x^* = \frac{2r}{1 - \beta}, \quad b^* = \frac{(1 - \beta)^2}{2r}. \]

Suppose that \( \beta = 0.7 \), (e-5) obtains when \( r < 0.29 \) since \( \mu = 0.5 \). \( x^* = 0.666 \ldots \) and \( b^* = 0.45 \). The equilibrium is NRA equilibrium if \( y^* \geq 0.55 \), and is RA equilibrium if \( y^* < 0.55 \).

Our credit rationing is similar to Stiglitz and Weiss (1981) and Williamson (1987) in the sense that some borrowers cannot raise loans in spite of the fact that they would be willing to pay higher than market determined interest rates. However, it sharply differs from theirs in that who would be credit rationed is known in the sense of ex ante in ours but is random in theirs.

6. Overinvestment

In previous sections it was shown that the optimal loan size satisfies maximum equity participation if \( \pi_2(x, b, y) < 0 \) is satisfied. Now consider another case with \( \pi_2(x, b, y) > 0 \). If per unit of return from lending is increasing in loan size, we demonstrate that the debt contract does not always take the form of maximum equity participation. Some entrepreneurs may choose debt contracts which specify a loan of greater quantity than is required for the first project. Entrepreneurs may operate also the second project. Overinvestment arises.

Assumption 4

\[ \frac{\partial}{\partial x} \left( \frac{db}{dx} \bigg|_{x^*} \right) < \frac{\partial}{\partial x} \left( \frac{db}{dx} \bigg|_{x} \right). \]
Assumption 4 guarantees the existence of a solution. Formally an entrepreneur with $y$ satisfies

$$\max_{x,b} \pi'(x, b, y)$$

subject to $\pi(x, b, y) = r,$
and $b + y \geq 1.$

The optimal conditions are

$$\pi_2^2 - \pi_1^2 \pi_2 + \theta = 0,$$

and $\theta(b + y - 1) \geq 0$ with complementary slackness,
where the second-order conditions are ensured from Assumption 4.

**Proposition 3**
Under Assumption 4 any lender offers a smaller loan interest rate the greater quantity of loan any entrepreneur is willing to receive under the competitive loan market.

Proposition 3 is obtained by implicitly differentiating $\pi(x, b, y) = r$ for any $y.$

**Proposition 4**
Under Assumption 4, if $\pi_2(x, b, y) > 0$ is satisfied, some entrepreneurs may undertake both projects.

The proof appears in Appendix. Two cases are to be distinguished. In one case the constraint is not binding. The pair $\{x, b\}$ is determined by $\pi(x, b, y) = r,$ and

$$\pi_2^2 - \pi_1^2 \pi_2 = 0,$$

for any $y$ to satisfy $b \geq 1 - y.$ In another case the constraint is binding. For an entrepreneur with $y,$ the loan size is determined to ensure $b = 1 - y.$ The loan interest rate is implicitly obtained from $\pi(x, b, y) = r.$

**Proposition 5**
The indifference curves for any lender, $\pi(x, b, y) = r,$ have a unique turning point and slope downward for any $x$ less than the turning point and upward for any $x$ larger than it.

The proof appears in Appendix.

**Lemma 3**
The indifference curves are strictly convex.

This immediately follows from Assumption 4.

Overinvestment may arise for some set of entrepreneurs. Figure 1 illustrates two entrepre-
neurs with different levels of endowments, $y_1$ and $y_2$, ($y_1 < y_2$). $E_1(E_2)$ is the point of the contract of the entrepreneur with a smaller (a greater) endowment of $y_1(y_2)$. The quantity invested in the second project, which is one measure of overinvestment, is $E_2S$ and $E_1T$, respectively. The entrepreneur with a greater endowment makes a debt contract with a smaller loan interest rate than the other. However, which entrepreneur receives a loan of greater quantity is ambiguous, depending on the shapes of the indifference curves and the iso-profit curves.

Entrepreneurs raise loans not only for the first but also for the second project although the rate of return of the second project is strictly below the loan interest rate. The intuition behind the result is that total interest payments $b \cdot x$ may be decreasing as the loan size increases because the loan interest rate is decreasing in loan size as seen in Proposition 3. Our overinvestment is characterized by the behavior of entrepreneurs who would borrow more than under perfect information.

In an alternative setting, de Meza and Webb (1987) obtains the result of overinvestment under asymmetric information. Our overinvestment differs from them in two respects. First, our overinvestment implies that borrowers operate more projects under asymmetric information, while their overinvestment implies that under asymmetric information borrowers are willing to undertake projects whose net present value is negative. Second, our overinvestment is closely linked to the debt contract which is endogenously derived, while in their model the contract form is given exogenously.

7. Overborrowing

In the previous section it was shown that overinvestment is always linked to the behavior of
entrepreneurs who would borrow more than under perfect information. In this section it is shown that the behavior of entrepreneurs, which we call hereafter as overborrowing, can arise. Overborrowing is defined to occur if borrowers receive loans to satisfy $b + y > 1$ although they face only the first project. If overborrowing actually occurs, it follows that entrepreneurs receive $b$ units and waste $b + y - 1$ units away.

The model is modified in such a manner that entrepreneurs cannot have access to the second project. Entrepreneurs do not have any other method of yielding returns than the first project. Lenders have access to the second project as before.

Profit functions of both types are obtained by incorporating $r = 0$ into (9a) and (9b), which leads to the same as (4a) and (4b). The formulations predict that the optimal loan size might be independent of the endowment level of each entrepreneur.

Lemma 4

\[
\frac{\partial}{\partial x} \left( \frac{db}{dx} \bigg|_{x^*} \right) < \frac{\partial}{\partial x} \left( \frac{db}{dx} \bigg|_{x} \right).
\]

Lemma 4 results immediately from Assumption 4. Lemma 4 guarantees the existence of a solution.

Proposition 6

Under Lemma 4, if $\Pi_2(x, b) > 0$ is satisfied, overborrowing may arise.

The proof is almost the same as in Proposition 4 and is deleted. The optimal conditions are

(18) \[ \Pi_2^* - \frac{\Pi_1}{\Pi_2} \Pi_2 + \theta = 0, \]

and

\[ \theta(b + y - 1) \geq 0 \quad \text{with complementary slackness.} \]

Allowing for the possibility that investment for the first project is unattractive for some entrepreneurs, entrepreneurs are partitioned into three sets. Define two points. First, denote the pair which jointly satisfies $\Pi(x', b') = r$, $\Pi_e(x', b') = 0$, and $b' + y = 1$ as $\{x', b'\}$. Second, denote the pair which jointly satisfies (18), $\Pi(x, b) = r$, and $b^{**} + y > 1$ as $\{x^{**}, b^{**}\}$ (see Figure 2).

(I) \[ 1 - b^{**} < y \leq y^* \]

Entrepreneurs write debt contracts associated with overborrowing. The contracts are written at the point $E$, where the indifference curve and the iso-profit curve are tangential with each other. They overly borrow in spite of the fact that they have access only to the first project. The chosen loan size $b^{**}$ is independent of their endowment because the only accessible project is indivisible. It arises since, given that per unit of return from lending is increasing in loan size, borrowers may be more beneficial in choosing the pair of a smaller loan interest rate and a larger loan than the pair associated with maximum equity participation.

(II) \[ 1 - b' \leq y \leq 1 - b^{**} \]

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Entrepreneurs write contracts with maximum equity participation. For example, for an entrepreneur with $y''$ such that $1 - b' \leq y'' \leq 1 - b''$, the contract is written at the point $E''$ on the horizontal line $b'' = 1 - y''$.

(III) $y^- \leq y < 1 - b'$

Entrepreneurs never have access to any project because the expected return from the project is non-positive.
The results could be, alternatively, explained by replacing the entrepreneur’s profit function by

\[ \Pi'(x(b), b) = \mu - br - \beta F(bx(b)), \]

where \( x \equiv x(b) \) is implicitly derived from \( \Pi(x, b) = r \). The expected cost is minimized at \( b^{**} \) to ensure

\[ b^{**} = \arg \min_b br + \beta F(bx(b)). \]

unless there would be no other constraint. As shown in Figure 3, if \( 1 - y < b^{**} \), the optimality is achieved at \( b^{**} \), while if \( 1 - y > b^{**} \), the chosen loan size must exceed \( b^{**} \). Beyond \( b^{**} \) the expected payment is increasing in loan size so that there may be a cutoff level of the loan size under which the expected profit is zero. It occurs for an entrepreneur with \( y = 1 - b' \). Given that other parameters are held constant, the greater is an endowment and thus the smaller is a loan requirement units for his project, the more likely the entrepreneur is to make a debt contract with overborrowing.

8. Conclusion

In this paper the optimal loan size is carefully investigated in an environment where the debt contract is endogenously derived. We find that the optimal loan size does not always satisfy maximum equity participation, depending on a shape of the lender’s profit function. If per unit of return from lending is decreasing in loan size, the debt contract takes the form of maximum equity participation, however, if it is increasing in loan size, overinvestment or overborrowing may arise. For further research whether there is empirical evidence to support our results is to be investigated.

Appendix

The proof of Proposition 1

Proof. Suppose to the contrary that, \( \{m'(\omega), x'\} \) is the optimal contract. Let

\[ B' = \{\omega: m'(\omega) < bx' - (b + y - 1)r\}, \]

and

\[ B'^c = \{\omega: m'(\omega) \geq bx' - (b + y - 1)r\}. \]

(3b) is rewritten as

\[ \int_{B'} \{m'(\omega) - \beta + (b + y - 1)r\} f(\omega) d\omega + \int_{B'^c} bx' f(\omega) d\omega = br, \]

where the equality is satisfied. Now consider another payment schedule \( m''(\omega) \) with \( m''(\omega) \geq m'(\omega) \) for all \( \omega \in [0, \omega^+] \), \( m''(\omega) > m'(\omega) \) for some \( \omega \in B' \), and \( m''(\omega) \) is continuous and monotone increasing on \([0, \omega^+]\). There exists some \( x'' \) with \( 0 < x'' < x' \) such that

\[ \int_{B''} \{m''(\omega) - \beta + (b + y - 1)r\} f(\omega) d\omega + \int_{B''^c} bx'' f(\omega) d\omega = br, \]

where

\[ B'' = \{\omega: m''(\omega) < bx'' - (b + y - 1)r\}, \]

and

\[ B''^c = \{\omega: m''(\omega) \geq bx'' - (b + y - 1)r\}. \]
The change in the objective function (3a) as a result of changing the contract from \( m'(\omega), x' \) to \( m''(\omega), x'' \) is 
\[
\beta \left[ \int_{B} f(\omega) \, d\omega - \int_{B'} f(\omega) \, d\omega \right] > 0
\]
as \( B'' \subset B' \) and \( B' - B'' \neq \emptyset \). We have a contradiction. (Q.E.D.)

The proof of Lemma 1

Proof. In the monitoring region \( S \), \( \omega + (b + y - 1)r < b \cdot x \). We obtain
\[
x > \frac{1}{b} \int_{b}^{\omega+(b+y-1)r} \{ \omega + (b + y - 1)r \} f(\omega) \, d\omega + x \left[ 1 - F \{ bx - (b + y - 1)r \} \right].
\]
Using the fact that \( \pi(x, b, y) = r \), we obtain
\[
x > r + \frac{\beta}{b} F \{ bx - (b + y - 1)r \}
\]
The second term of the right-hand side is positive so that we have \( x > r \). (Q.E.D.)

The proof of Proposition 2

Proof. An entrepreneur with an endowment of \( y \) chooses
\[
\max_{x, b} \pi^*(x, b, y), \quad (a-1)
\]
subject to
\[
\pi(x, b, y) = r, \quad (a-2)
\]
and
\[
b + y \geq 1, \quad (a-3)
\]
where (a-3) indicates that at least one unit of the consumption good is required for the first project. The Lagrangian function \( L \) is denoted as
\[
L = \pi^*(x, b, y) + \lambda [\pi(x, b, y) - r] + \theta (b + y - 1), \quad (a-4)
\]
where \( \lambda \) and \( \theta \) are Lagrangian multipliers associated with constraints (a-2) and (a-3), respectively.

Differentiation with respect to \( b \) and \( x \), respectively, gives
\[
L_b = \pi^*_b(x, b, y) + \lambda \pi^*_b(x, b, y) + \theta = 0, \quad (a-4)
\]
\[
L_x = \pi^*_x(x, b, y) + \lambda \pi^*_x(x, b, y) = 0, \quad (a-5)
\]
and
\[
\theta (b + y - 1) \geq 0. \quad (a-6)
\]
We distinguish between two cases.

(i) \( \pi_1 = 0 \)

An entrepreneur, who is promised to pay \( x^* \) which is determined by \( \pi_1(x^*, b, y) = 0 \), would receive \( b^* \) units. This can be satisfied with an entrepreneur with \( y^* = 1 - b^* \). On the contrary, suppose that \( b = b^* \) for some entrepreneur with \( y > y^* \). He can make profits by decreasing the amount of loan. This contradicts the optimality at \( b = b^* \).

(ii) \( \pi_1 \neq 0 \)

Substituting (a-5) into (a-4) to eliminate \( \lambda \) gives
\[
\pi^*_x - \frac{\pi^*_1}{\pi^*_1} \pi^*_2 + \theta = 0. \quad (a-7)
\]

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From the concavity of the lender's profit function with respect to \(x\), given the value of \(b\), there exist two values of \(x(x_{\text{min}} < x_{\text{max}})\) which satisfy \(\pi(x, b, y) = r\) for any \(y\). Apparently, profit-maximizing entrepreneurs choose the lower value \(x_{\text{min}}\), which implies \(\pi_1(\cdot) > 0\) in equilibrium. From Lemma 1 we obtain \(\pi_2(\cdot) < 0\) and we derive \(\pi_1(\cdot) < 0\). In addition, \(\pi_2(\cdot) < 0\) obtains from assumption. Thus

\[
\pi_1^2 - \pi_2^2 \frac{\pi_1}{\pi_1} < 0,
\]

is strictly satisfied so that the equality (a-7) implies \(\theta > 0\). We obtain \(b + y = 1\) from complementary slackness. From (i) and (ii), for any entrepreneur receiving loans, the debt contract takes the form of maximum equity participation.

(Q.E.D.)

Lemma 2
\[\pi_1(x, b, y) > 0\] at the optimum.

Proof.
We distinguish between two cases: \(\pi_1 \neq 0\) and \(\pi_1 = 0\).

(i) \(\pi_1 \neq 0\)
Implicit differentiation of \(\pi(x, b, y) = r\) gives
\[x = x(b, y)\]
where \(\partial x(b, y)/\partial b < 0\).
Substituting this into the entrepreneur's profit function gives
\[\pi(x, b, y) = \pi(x(b, y), b, y),\]
which is equivalent to (8). Define \(b^*\) to satisfy \(\pi(x^*, b^*, y) = r\) and \(\pi_1(x^*, b^*, y) = 0\) jointly. Differentiating with respect to \(b\) evaluating at \(b = b^*\), yields.
\[
\frac{d\pi}{db}\bigg|_{b = b^*} = \pi_1^* \cdot \partial x(\cdot)/\partial b + \pi_2^* > 0,
\]
since \(\pi_1(\cdot)\) and \(\pi_2(\cdot)\) are both negative and bounded from below, that is, \(-\infty < \pi_1(\cdot), \pi_2(\cdot) < 0\), and \(\lim_{b \to b^*} \partial x(\cdot)/\partial b = -\infty\).

(ii) \(\pi_1 = 0\)
From the concavity of the lender's profit function with respect to \(x\), there exist two values of \(x(x_{\text{min}} < x_{\text{max}})\) to satisfy \(\pi(x, b, y) = r\), given the value of \(b\) for any \(y\). Obviously, a profit-maximizing entrepreneur chooses the lower value of \(x\), which implies \(\pi_1(\cdot) < 0\) in equilibrium.
From (i) and (ii) we obtain \(\pi_1 > 0\).

(Q.E.D.)

The proof of Proposition 4
Proof. \(\pi_1^2(x, b, y) < 0\) from Lemma 1. \(\pi_1^2(x, b, y) < 0\). By assumption, \(\pi_2(x, b, y) > 0\). Finally, \(\pi_1(x, b, y) > 0\) from Lemma 2. If \(\theta > 0\), \(b + y = 1\) and if \(\theta = 0\), \(b + y \geq 1\) from complementary slackness. For the latter case with \(\theta = 0\), for some \(y\), generally there exists a pair \(\{x, b\}\) to satisfy

\[
\pi_2^* - \pi_1^* \frac{\pi_2}{\pi_1} = 0,
\]
with $b + y > 1$. (Q.E.D.)

The proof of Proposition 5

Total differentiation of $\pi(x, b, y) = r$ given the value of $y$, yields

$$\frac{db}{dx}\bigg|_{\pi} = -\frac{\pi_1}{\pi_2}$$

$\pi_2(\cdot) > 0$, $\pi_1(\cdot) > 0$ for $x < x^*$ and $\pi_1(\cdot) < 0$ for $x > x^*$. Thus $db/dx < 0$ for $x < x^*$ and $db/dx > 0$ for $x > x^*$. Finally, we derive $db/dx = 0$ for $x = x^*$ since $\pi_1(\cdot) = 0$ at this point. (Q.E.D.)

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REFERENCES


