Maximum available efficiency formulation based on a black-box model of linear two-port power transfer systems

Takashi Ohira^{a)}

Toyohashi University of Technology,

1-1 Hibarigaoka, Tempaku, Toyohashi 441-8580, Japan

a) ohira@tut.jp

Abstract: This paper describes how the port voltage and current vectors behave in linear two-port power transfer systems. We find there is an optimum set of them at which the output power reaches its highest while keeping the input power constant or vice versa. Derived maximum available efficiency formulas work with a power source having any impedance e.g. zero- or infinite-ohm. Heuristically discovered angle parameter θ enables us to graphically predict the system's maximum available efficiency from its port parameters i.e. matrix $\mathbf{Y}$, $\mathbf{Z}$, or $\mathbf{S}$ in 50-ohm domain.

Keywords: wireless power transfer, efficiency angle tangent

Classification: Microwave and millimeter wave devices, circuits, and systems

References

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1 Introduction

As a key performance criterion of capacitive- and inductive-coupling wireless power transfer systems, the maximum available efficiency was theoretically discussed from the aspect of coupling coefficient k multiplied by quality factor Q [1]. We therein extended the conventional concept of k - Q product to a sophisticated formula viable on a system having arbitrary coupling structure. The formulas was deduced from an incident-reflected wave perception involving simultaneous two-port conjugate impedance matching conditions, and thus enabled us to estimate the maximum available efficiency without need of

resonance or mutual coupling calculations. After that publication, two stimulating questions came up from majority of the power electronics field. One is on whether the simultaneous two-port conjugate matching condition always gives the maximum efficiency. The other concerns what happens to the formula in case the system is excited by sufficiently low impedance or saliently a zero-ohm voltage source [2]. To answer those questions, this paper theoretically explores the maximum power transfer efficiency of linear reciprocal two-port systems again. This time however we make it regardless of the source available power being finite or not. The formulation also goes without explicitly assuming the source or load impedance, let alone the conjugate matching conditions.

2 Passive linear reciprocal system

Consider that we transfer RF power through a linear reciprocal two-port system. We assume a black box model of the system to make a general formulation for any power transfer scheme. Only what we can observe is the set of voltage vector $[v_1, v_2]$ and current vector $[i_1, i_2]$ at input port #1 and output port #2 shown in Fig. 1. Suppose a sinusoidal-wave power source connected to port #1 and a passive load to port #2, but they are both hidden on purpose from this diagram. See the model reported in Ref. [1] for comparison of the same system. Neither Fig. 1 nor the rest of this paper directly takes source or load properties into account at all. This is based on Kirchhoff's law stating that v_2 -to- i_2 ratio is dominated uniquely by the load. In other words, the ratio fully characterizes the load property and plays its equivalent role instead. Also v_1 -to- i_1 ratio stems uniquely from the load by way of the two-port system provided that its immittance matrix $\mathbf{Y}$ or $\mathbf{Z}$ is known a priori. As what to hold attention, those input and output voltage-to-current ratios are totally invariant against the source impedance. The source impedance variation just affects magnitude of the induced voltage and current vectors while keeping their proportion and relative phase unchanged. This is an essence of the theory and always true as long as the system at least behaves in a linear and reciprocal manner to external stimuli.

Start formulation of the problem with well-known voltage-to-current relation

$$\begin{aligned} i_1 &= y_{11}v_1 + y_{12}v_2 \\ i_2 &= y_{21}v_1 + y_{22}v_2 \end{aligned} \quad (1)$$

where each admittance element is decomposed into its conductive and susceptive parts as

$$\begin{aligned} y_{11} &= g_{11} + jb_{11} \\ y_{12} &= y_{21} = g_{21} + jb_{21} \\ y_{22} &= g_{22} + jb_{22} \end{aligned} \quad (2)$$

Note in this paper that the non-diagonal elements are common since the system responds in reciprocity. For convenient complex voltage notation, we

put the time origin in phase to the input voltage waveform so that v_1 simply falls into a real number without losing generality. Taking v_1 as reference scalar, we observe the rest of variables on a real-imaginary plane as

$$\begin{aligned} v_2 &= v_{21} + jv_{22} \\ i_1 &= i_{11} + ji_{12} \\ i_2 &= i_{21} + ji_{22} \end{aligned} \quad (3)$$

Substituting Eq. (3) into Eqs. (1) and (2), the current components are expressed in linear combinations

$$\begin{aligned} i_{11} &= g_{11}v_1 + g_{21}v_{21} - b_{21}v_{22} \\ i_{21} &= g_{21}v_1 + g_{22}v_{21} - b_{22}v_{22} \\ i_{22} &= b_{21}v_1 + b_{22}v_{21} + g_{22}v_{22} \end{aligned} \quad (4)$$

where we omit i_{12} which does not appear hereafter. We then regard v_1 , v_{21} , and v_{22} as three independent scalar axes, along which we find input power P_1 and output power P_2 expressed as homogeneous quadratic polynomials

$$\begin{aligned} P_1 &= \frac{1}{2} v_1 i_{11} = \frac{1}{2} (g_{11}v_1^2 + g_{21}v_1v_{21} - b_{21}v_1v_{22}) \\ P_2 &= -\frac{1}{2} (v_{21}i_{21} + v_{22}i_{22}) = -\frac{1}{2} (g_{21}v_1v_{21} + b_{21}v_1v_{22} + g_{22}v_{21}^2 + g_{22}v_{22}^2) \end{aligned} \quad (5)$$

In this definition of input power, we should be careful that it is the power actually put into port #1 and not that available from any power source as defined in Ref. [1]. This is just for the moment, and we will find the two differently defined efficiencies reach exactly the same value at their maxima as formulated in a following section.

Fig. 1. A black box transferring RF power from port #1 to port #2

3 Power efficiency maximization

This section is to find the optimum combination of v_1 , v_{21} , and v_{22} that makes the system reach its highest transfer efficiency. The problem is translated into how to maximize the output power at port #2 under constraint of input power at port #1 kept in constant for known matrix $\mathbf{Y}$. Alternatively, it is how to minimize the input power for constant output. In either way, the optimum voltage set must satisfy simultaneous Jacobean requirements

$$\begin{vmatrix} \frac{\partial P_1}{\partial v_1} & \frac{\partial P_1}{\partial v_{21}} \\ \frac{\partial P_2}{\partial v_1} & \frac{\partial P_2}{\partial v_{21}} \end{vmatrix} = \begin{vmatrix} \frac{\partial P_1}{\partial v_1} & \frac{\partial P_1}{\partial v_{22}} \\ \frac{\partial P_2}{\partial v_1} & \frac{\partial P_2}{\partial v_{22}} \end{vmatrix} = 0 \quad (6)$$

Imposing them upon the input and output power in Eq. (5), we obtain

$$\begin{aligned}(g_{21}v_1 + g_{22}v_{21})(g_{11}v_1 - b_{21}v_{22}) + (g_{11}v_1 + g_{21}v_{21})g_{22}v_{21} &= 0 \\ (g_{11}v_1 + g_{21}v_{21})(b_{21}v_1 + g_{22}v_{22}) + (g_{11}v_1 - b_{21}v_{22})g_{22}v_{22} &= 0\end{aligned}\quad (7)$$

Multiplication of each respective term in the two equations yields a single factorized form

$$(g_{11}v_1 + g_{21}v_{21})(g_{11}v_1 - b_{21}v_{22})(g_{21}b_{21}v_1 + g_{22}b_{21}v_{21} + g_{21}g_{22}v_{22}) = 0 \quad (8)$$

This suggests three candidates for the solution to meet the requirements. If the first factor vanishes, the input power must be

$$P_1 = \frac{1}{2}v_1 i_{11} = -\frac{1}{2}g_{11}v_1^2 \quad (9)$$

This makes a negative value since g_{11} always stays positive for passive systems, which is out of the question. The second factor ends up in the same result. We must therefore choose the final factor to vanish as non-trivial equation

$$\frac{v_1}{g_{22}} + \frac{v_{21}}{g_{21}} + \frac{v_{22}}{b_{21}} = 0 \quad (10)$$

Substituting it into Eq. (7), we eliminate v_{22} to reach standalone equation

$$|y_{21}|^2 \frac{g_{22}^2 v_{21}^2}{g_{21}^2 v_1^2} + 2H \frac{g_{22} v_{21}}{g_{21} v_1} + H = 0 \quad (11)$$

for v_{21} over v_1 . Sophisticated symbols represent quadratic terms

$$\begin{aligned}|\mathbf{G}| &= g_{11}g_{22} - g_{21}^2, \quad |y_{21}|^2 = g_{21}^2 + b_{21}^2 \\ H &= |\mathbf{G}| + |y_{21}|^2 = g_{11}g_{22} + b_{21}^2\end{aligned}\quad (12)$$

Under assumption of positive discriminant $H|\mathbf{G}|$ for passive systems, we solve Eq. (11) as

$$v_{21} = \frac{-H \pm \sqrt{H|\mathbf{G}|}}{|y_{21}|^2} \cdot \frac{g_{21}v_1}{g_{22}} \quad (13)$$

associated with

$$v_{22} = -b_{21} \left(\frac{v_1}{g_{22}} + \frac{v_{21}}{g_{21}} \right) \quad (14)$$

Analytically speaking, Jacobean zeros may not always give the global solution but just a local minimal or maximal of the original function. In our case, a physical aspect endorses that the power efficiency must be a continuous single-valued scalar positive real function having only one upper bound involved in its closed range. Appending this condition to what Jacobean exactly brought, we can rigorously conclude that Eqs. (13) and (14) lead the system to its maximum efficiency. It is also worth notifying that the formulas are valid for an arbitrary source impedance because they are deduced without any assumption on source model throughout the formulation process at all. To visit ultimate ends, they yet stay sound with a pure zero-ohm voltage or infinite-ohm current source.

4 Efficiency angle tangent

Feeding the optimum voltages derived in Eq. (13) back into the input power in Eq. (5), we find

$$P_1 = \frac{1}{2} v_1 i_{11} = \frac{1}{2} \sqrt{H|\mathbf{G}|} \frac{v_1^2}{g_{22}} \quad (15)$$

In the same way, we also find the output power

$$P_2 = -\frac{1}{2} (g_{21} v_1 + g_{22} v_{21}) \frac{|y_{21}|^2}{g_{21} g_{21}} v_{21} \quad (16)$$

From these powers, we obtain output-to-input ratio

$$\eta_{\max} = \frac{P_2}{P_1} = 1 + \frac{2}{|y_{21}|^2} \left\{ |\mathbf{G}| - \sqrt{(|\mathbf{G}| + |y_{21}|^2)|\mathbf{G}|} \right\} \quad (17)$$

This is the maximum power transfer efficiency expression in terms of admittance matrix. Duality theorem replaces admittance matrix $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$ with impedance matrix $\mathbf{Z} = \mathbf{R} + j\mathbf{X}$ to deduce alternative expression

$$\eta_{\max} = 1 + \frac{2}{|z_{21}|^2} \left\{ |\mathbf{R}| - \sqrt{(|\mathbf{R}| + |z_{21}|^2)|\mathbf{R}|} \right\} \quad (18)$$

The formula enables us to calculate the maximum efficiency from given two-port immittance matrix elements or S -parameters measured by a vector network analyzer since $\mathbf{Z} = 50(\mathbf{I} + \mathbf{S})(\mathbf{I} - \mathbf{S})^{-1}$ where $\mathbf{I}$ stands for identity matrix. The above result exactly agrees with $\eta_{\max}$ derived from the simultaneous conjugate matching technique [3]. It must be worthy of remark that the conjugate matching was just something convenient for use but none of necessary issue to be primarily imposed upon input or output ports. During the process of efficiency maximization, the system automatically ended up to satisfy conjugate matching. The efficiency definition we use in the denominator of Eq. (17) or (18) is based on the power that is actually fed to port #1, not that the source can maximally supply. Nonetheless, the maximum efficiency results in the same value as derived in Ref. [1] which employed a source-available-power based denominator. That is to say the maximum value is identical for the two different definitions of power transfer efficiency.

We have successfully deduced Eqs. (17) and (18) in explicit forms so that one can directly use them as they are. Even so, the right-hand side of either formula looks a little too complicated to understand what it physically means. To get a clearer vista, we convert the formula by introducing efficiency angle θ defined as

$$\tan 2\theta = \frac{|y_{21}|}{\sqrt{|\mathbf{G}|}} = \frac{|z_{21}|}{\sqrt{|\mathbf{R}|}} \quad (19)$$

Indeed this may look somewhat heuristic [3], we will soon realize how elegantly and insightfully it works. This is because the right-hand side of Eq. (19) implies the system's transfer function in immittance domain divided by the matrix determinant relatively representing its inside power dissipation in total. Simply putting Eq. (19) into Eq. (17) or (18), the maximum efficiency is translated as

$$\eta_{\max} = \tan^2 \theta \quad (20)$$

In this formula, tangent suggests output-to-input ratio. Square corresponds to power rather than just voltage or current. A graphical chart is depicted in Fig. 2 to provide an intuitive comprehension of Eq. (20). One can easily estimate θ by double-angle arc tangent operation from immittance components of the system. Then the maximum efficiency can be predicted as the area of black-painted square. The derived formula is quite useful not only in the system design process but also effective in the prototype measurement stage if we have installed Eqs. (19) and (20) as user-defined functions in an RF vector network analyzer so that it can plot $\eta_{\max}$ on a visual display as a real-time response.

Fig. 2. Graphical chart to estimate the maximum efficiency from two-port immittance matrix elements by way of efficiency angle tangent

5 Conclusion

A linear passive reciprocal two-port system transfers RF power of which efficiency is defined as output-to-input power ratio. Once the system is characterized by its port immittance matrix, the power transfer efficiency can be maximized by adjusting the voltage-current ratio i.e. the load impedance. In case a fixed load is provided beforehand, this adjustment can be alternatively done by inserting an appropriate matching circuit before the load. The maximum available efficiency can be graphically estimated from the immittance matrix elements on a chart of double angle and tangent square area. It never suffers from any source impedance variation even ranging from zero to infinite ohm. The presented efficiency angle concept will uniformly work as an elegant and versatile pilotage in every stage from basic study to hardware prototyping on various and sundry genera of future RF power transfer systems.

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