A modified pulse-coupled spiking neuron circuit with memory threshold and its application

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Abstract: A modified pulse-coupled spiking neuron circuit (MPSNC) with memory threshold is presented. It is constructed with two single modified spiking neuron circuits (MSNCs). In MPSNC, one MSNC stores the potential of its threshold signal at the latest firing time of another MSNC as threshold, implying a short-term cross memory. We extend the definition of least common multiple to obtain the firing phase of MPSNC. By studying the iterative map of firing time and firing phase, the dynamics and statistics are revealed. The chaotic map of firing phase is applied to image encryption. Besides, the hardware circuit and simulation are built.

Keywords: modified pulsed-coupled spiking neurons circuit, chaos, image encryption, circuit implementation

Classification: Integrated circuits

References

1 Introduction

As well known, all the activities of neural network can be transformed into electrical signals, thus artificial neural network (ANN) can be built by mapping out these signals. Nowadays, lots of ANN models with a variety of mappings have been put forward and applied in different fields [1, 2, 3, 4]. In 1990, Kaihara et al. first proposed the construction of chaotic neural network [5]. The introduction of chaos theory makes ANN much closer to actual biological neural network. Over the last few decades, the researchers in the field of ANN pay more and more attention to spiking neural networks (SNNs). If neural network models are classified according to their component units, SNNs are regarded as the third generation of neural networks [6]. In comparison with the previous two generation models, SNNs are computationally more powerful and substantially more realistic [6, 7, 8]. There are many models of spiking neurons, and Integrate-and-Fire model (IFM), known for its low computational cost and simple structure, is one of the most widely used models [9]. Because of the extremely simple structure, IFM can not have some properties like threshold variability, spike latencies, chaos dynamics, etc. Furthermore, a series of circuit model of spiking neurons are designed. In [10], Yoshiho Kon’no, et al. proposed a spiking neuron circuit (SNC) and a pulse-coupled spiking neuron circuit (PCSNC), which are both in IFM. The SNC and PCSNC successfully exhibit rich dynamics and model the working of IFM. However, the SNC and PCSNC in [10] only have the constant firing threshold, while threshold of biological neurons is
variable. Moreover, the study of the circuit model that has multiple input signals with different periods is not sufficient in existing literature. In [11], although a circuit model with two periodic inputs is mentioned, yet this study focuses on the record of circuit simulation and testing to clarify how the inputs vary the shape of return map, and lacks the clear mathematical analysis in the case of multiple periodic inputs. And its threshold just changes with time without any memory.

Chaos-based encryption [12, 13, 14, 15] is one of the original applications of nonlinear dynamics in the chaotic regime. It has been widely used in the encryption of image [12, 13], video [14], communication [15], etc. Many properties of chaotic systems, such as ergodicity, sensitivity, deterministic dynamics and structure complexity, have their corresponding counterparts in encryption systems [16], such as confusion, diffusion, deterministic pseudo-randomness and algorithm complexity. It supports the suitability of the combination of chaos and cryptography. Besides chaos-based encryption can provide a good combination of advantages such as high speed, large key space, high-level security, etc.

In this paper, we modify the SNC to get memory threshold. The modified spiking neuron circuit (MSNC) has two switches that are controlled by the output pulse to decide the reset of membrane potential and threshold potential. The MSNC not only has the advantages of IFM, but also add some properties like threshold variability, spike latencies, chaos dynamics, memory threshold, etc. By cross-switching of two MSNCs, we construct a small SNN called modified pulse-coupled spiking neuron circuit (MPSNC). In the event of an output pulse from one MSNC, the switches of another MSNC are closed to reset the membrane potential and threshold potential to the voltages of their corresponding resetting signal and threshold signal at the moment, respectively. The threshold of one MSNC does not change continuously with time forward, but remains unchanged until the coming of the next pulse yielded by another MSNC. Thus it can be found that the threshold of the MPSNC not only performs variable but also has activity-dependence and short-term cross memory. The MPSNC can output a rich set of pulse trains, through integrating and firing, resetting circuit potential, repeatedly. Since the input signals of the MPSNC have different periods, some processing should be done to calculate the firing phase. By extending the definition of the least common multiple (LCM) into the field of rations, we can get the common period of the different input signals, and further derive the iterative map of firing phase. It should be noted that this method is only valid with the rational periods.

Bifurcation behavior is an important part of nonlinear dynamics. And it is a significant feature of ANN. It can be described clearly, after plotting bifurcation diagram and graph of Lyapunov exponent (LE). Moreover, there is a strong link between dynamics and the slop of maps of firing time and firing phase [17]. Thus the circuit parameters can be divided into three ranges including one range assuring periodic/quasi-periodic dynamics, one range ensuring chaos, and one range that chaos and periodic/quasi-periodic behavior are distributed confusedly. And we give several plots of the iterative maps of firing time and return map of firing phase in these parameters ranges to illustrate the dynamics of the MPSNC in detailed. It is well known that chaotic sequence is pseudo-random, and each sequence is independent from each other. As the standard Central Limit Theorem (CLT) affirms, the
probability density of an enormous number of independent identically distributed (iid) random variable is close to Gaussian [18]. The probability density of the sum of the iterative firing phases in chaotic region is calculated, and it should comply with Gaussian.

Moreover, the hardware circuit is done in our laboratory using the components like LM339, CD4009, CD4538, CD4066 and some resistors, capacitors. The results from both simulation and actual circuit verify the validity of the design.

Furthermore, the chaotic iterative map of firing phase characterizing the MPSNC is extremely sensitive to parameters and initial value, which meets the basic requirements of cryptosystem [19]. Also, the chaos is able to fill the whole definition interval of the selected control parameter, and chaotic iteration can traverse the entire valid domain, after setting some other parameters. In this way, the chaotic map is very adaptable and can overcome the shortcoming of 1D chaotic system, the limited and discontinuous rang of chaos [20, 21]. Accordingly, we apply it to encrypt image. The image encryption scheme is in a typical encryption structure including bit-level permutation and pixel-level diffusion. These two operations are both based on chaos. The encryption scheme is easy to realize and can provide noise-like encryption results.

2 Modified spiking neuron circuit

The MPSNC shown in Fig. 1(a) is composed of two single MSNCs which are called MSNC$_1$ for the left part and MSNC$_2$ for the right. Using the method of cross-switching, these two parts reset the base level and threshold of each other. The capacitor voltages $v_1$ and $v_2$ model the membrane potentials, and the capacitor voltages $Th_1$ and $Th_2$ represent the threshold potentials. The working mechanism of MPSNC is plotted in Fig. 1(b). The capacitor $C_{12}$ changes with constant current $I_1$, until capacitor voltage $v_1$ reaches firing threshold level $Th_1$. And MSNC$_1$ outputs a pulse signal $Y_1$. When receiving $Y_1$, the switches $SW_{21}$ and $SW_{22}$ of MSNC$_2$ are closed, and $v_2$ and $Th_2$ are reset to $B_{21}(t)$ and $B_{22}(t)$, respectively. The capacitors $C_{12}$ and $C_{22}$ continue to charge until capacitor voltage $v_2$ reaches $Th_2$. And MSNC$_2$ generates a pulse signal $Y_2$ to close $SW_{11}$ and $SW_{22}$, and reset $v_1$ and $Th_1$ to $B_{11}(t)$ and $B_{12}(t)$, respectively. The MPSNC can generate a pulse train by repeating such behavior.

In Fig. 1, resetting signals $B_{11}(t)$ and $B_{21}(t)$ are triangular signals with period of $T_1$, and threshold signals $B_{12}(t)$ and $B_{22}(t)$ are sinusoidal signals with period $T_2$, as given in Eq. (1).
\begin{equation}
B_{11}(t) = \begin{cases} 
-K_{11}(t - T_1/4) - V_D & 0 \leq t < T_1/2 \\
K_{11}(t - 3T_1/4) - V_D & T_1/2 \leq t < T_1 
\end{cases}, \quad \text{and } B_{11}(t + T_1) = B_{11}(t) \\
B_{12}(t) = -K_{12} \sin(2\pi t/T_2) + V_{Th} \\
B_{21}(t) = \begin{cases} 
-K_{21}(t - T_1/4) - V_D & 0 \leq t < T_1/2 \\
K_{21}(t - 3T_1/4) - V_D & T_1/2 \leq t < T_1 
\end{cases}, \quad \text{and } B_{21}(t + T_1) = B_{21}(t) \\
B_{22}(t) = -K_{22} \sin(2\pi t/T_2) + V_{Th}
\end{equation}

Moreover, the following condition needs to be obeyed.

\begin{equation}
\begin{cases}
B_{12}(t) > B_{11}(t) \\
B_{22}(t) > B_{21}(t)
\end{cases}, \quad \text{i.e., } \begin{cases}
-|K_{12}| + V_{Th} > |K_{11}|T_1/4 - V_D \\
-|K_{22}| + V_{Th} > |K_{21}|T_2/4 - V_D
\end{cases}
\end{equation}

The working state of the MPSNC is given by Eq. (3). \(t_{p1}\) and \(t_{p2}\) denote the latest firing time of MSNC_1 and MSNC_2. As can be seen, the threshold of one MSNC has short-term memory about the last firing time of another MSNC. So the coupled network MPSNC has memory which is not only activity-dependent.

\begin{equation}
\begin{cases}
v_2(t^+) = B_{21}(t^+), Y_1(t^+) = E, t_{p1} = t, \quad \text{and } Th_2(t^+) = B_{22}(t_{p2}^+) \quad \text{for } v_1(t) = Th_1(t) \\
C_{22} \frac{dv_2(t)}{dt} = I_2, Y_1(t) = -E, \quad \text{and } Th_2(t) = Th_2(t_{p2}) \quad \text{for } v_1(t) \neq Th_1(t)
\end{cases}
\end{equation}

The dimensionless quantities in Eq. (4) are used to transform Eqs. (1) and (3) into dimensionless forms fit for numerical simulation.

\begin{align}
\tau &= \frac{t}{T_1}, \quad \rho = \frac{T_2}{T_1}, \quad x_1 = \frac{v_1 - V_{Th}}{V_{Th} + V_D}, \quad \dot{x}_1 = \frac{dx_1}{d\tau}, \quad y_1 = \frac{Y_1 + E}{2E}, \quad \tau_{p1} = \frac{t_{p1}}{T_1}, \\
s_1 &= \frac{I_1 T_1}{C_{22}(V_{Th} + V_D)}, \quad b_{11}(\tau) = \frac{B_{11}(t) - V_{Th}}{V_{Th} + V_D}, \quad k_{11} = \frac{K_{11} T_1}{V_{Th} + V_D}, \\
b_{12}(\tau) &= \frac{B_{12}(t) + V_D}{V_{Th} + V_D}, \quad k_{12} = \frac{K_{12}}{V_{Th} + V_D}, \quad \dot{th}_1 = \frac{Th_1 - V_{Th}}{V_{Th} + V_D}, \quad x_2 = \frac{v_2 - V_{Th}}{V_{Th} + V_D}, \\
\dot{x}_2 &= \frac{dx_2}{d\tau}, \quad y_2 = \frac{Y_2 + E}{2E}, \quad \tau_{p2} = \frac{t_{p2}}{T_1}, \quad s_2 = \frac{I_2 T_1}{C_{22}(V_{Th} + V_D)}, \quad b_{21}(\tau) = \frac{B_{21}(t) - V_{Th}}{V_{Th} + V_D}, \\
k_{21} &= \frac{K_{21} T_1}{V_{Th} + V_D}, \quad b_{22}(\tau) = \frac{B_{22}(t) + V_D}{V_{Th} + V_D}, \quad k_{22} = \frac{K_{22}}{V_{Th} + V_D}, \quad \dot{th}_2 = \frac{Th_2 - V_{Th}}{V_{Th} + V_D}
\end{align}

The dimensionless forms of Eqs. (1) and (3) are Eqs. (5) and (6). The condition for \(|k_{11}|/4 + |k_{12}| < 1\) and \(|k_{21}|/4 + |k_{22}| < 1\) must be met.

\begin{equation}
\begin{cases}
b_{11}(\tau) = \begin{cases} 
-k_{11}(\tau - 1/4) - 1 & 0 \leq \tau < 1/2 \\
k_{11}(\tau - 3/4) - 1 & 1/2 \leq \tau < 1 
\end{cases}, \quad \text{and } b_{11}(\tau + 1) = b_{11}(\tau) \\
b_{12}(\tau) = -k_{12} \sin(2\pi \rho \tau) + 1
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
b_{21}(\tau) = \begin{cases} 
-k_{21}(\tau - 1/4) - 1 & 0 \leq \tau < 1/2 \\
k_{21}(\tau - 3/4) - 1 & 1/2 \leq \tau < 1 
\end{cases}, \quad \text{and } b_{21}(\tau + 1) = b_{21}(\tau) \\
b_{22}(\tau) = -k_{22} \sin(2\pi \rho \tau) + 1
\end{cases}
\end{equation}

\begin{align}
x_2(t^+) = b_{21}(t^+), Y_1(t^+) = 1, \quad \tau_{p1} = \tau, \quad \text{and } th_2(t^+) = b_{22}(t_{p2}^+) - 1 \quad \text{for } x_1(t) = th_1(t) \\
\dot{x}_2 = x_2, y_1(t) = 0, \quad \text{and } th_2(t) = th_2(t_{p2}) \quad \text{for } x_1(t) = th_1(t)
\end{align}

\begin{align}
x_1(t^+) = b_{11}(t^+), y_2(t^+) = 1, \quad \tau_{p2} = \tau, \quad \text{and } th_1(t^+) = b_{12}(t_{p1}) - 1 \quad \text{for } x_2(t) = th_2(t) \\
\dot{x}_1 = x_1, y_2(t) = 0, \quad \text{and } th_1(t) = th_1(t_{p2}) \quad \text{for } x_2(t) = th_2(t)
\end{align}
The pulse train is generated due to the alternate firing of MSNC$_1$ and MSNC$_2$. Let $f_1$ and $f_2$ be the iterative maps on firing time of MSNC$_1$ and MSNC$_2$, and $\tau(n)$ be the $n$-th firing time. We assume the case that $0 < s \equiv s_1 = s_2$ to avoid repetitive work. A iterative map $f$ of firing time $\tau(n)$ is defined by Eq. (7).

$$
\tau(n + 1) = f(\tau(n)) = \begin{cases} 
  f_1(\tau(n)) = \tau(n) + (b_{12}(\tau(n)) - b_{11}(\tau(n)) - 1)/s & \text{for odd } n \\
  f_2(\tau(n)) = \tau(n) + (b_{22}(\tau(n)) - b_{21}(\tau(n)) - 1)/s & \text{for even } n 
\end{cases}
$$

(7)

Then, we extend the definition of the LCM for calculating the common period of $b_{11}(\tau), b_{12}(\tau), b_{21}(\tau)$ and $b_{22}(\tau)$.

**Definition 1**: The positive rational number $l$ is the LCM of two positive rational numbers $a$ and $b$, if, and only if, there exist two positive integers $m$ and $n$ such that $l = m \times a = n \times b$, and there does not exist $n' < m$, $n' < n$, $l' < l$ such that $l' = m' \times a = n' \times b$.

**Lemma 1**: There must exist the LCM of positive integer 1 and positive rational number $1/r$.

Proof: It is obvious that the positive rational number $1/r$ can be written in the fraction in lowest term as $p/q$, where $p$ and $q$ are two relatively-prime positive integers. And 1 is equal to $q/q$. Thus the LCM of $1/r$ and 1 is the LCM of $p/q$ and $q/q$. As is known to all, the LCM of two relatively-prime positive integers is their product, implying that the LCM of $p$ and $q$ is $pq$. So $p/q \ast q = q/q \ast p = p$ holds, and $p$ is the LCM of $1/r$ and 1.

If $p$ is the LCM of 1 and $1/r$, $b_{11}(\tau), b_{12}(\tau), b_{21}(\tau)$ and $b_{22}(\tau)$ have the common period $p$. Hence the case that $f(\tau + p) = f(\tau + p)$ holds. A return map $F$ of firing phase $\theta(n)$ can be given by Eq. (8).

$$
\theta(n + 1) = F(\theta(n)) = f(\theta(n)) \mod p
$$

$$
= \begin{cases} 
  f_1(\theta(n)) \mod p & \text{for odd } n \\
  f_2(\theta(n)) \mod p & \text{for even } n 
\end{cases}, \text{ and } F : I \rightarrow I, I \equiv [0, p),
$$

(8)

$$
\begin{align*}
  f_1(\theta(n)) &= \begin{cases} 
    (-k_{12} \sin(2\pi\theta(n)) + k_{11}(\theta_m(n) - 1/4) + 1)/s + \theta(n), & 0 \leq \theta_m(n) < 1/2 \\
    (-k_{12} \sin(2\pi\theta(n)) - k_{11}(\theta_m(n) - 3/4) + 1)/s + \theta(n), & 1/2 \leq \theta_m(n) < 1
  \end{cases}
\end{align*}
$$

$$
\begin{align*}
  f_2(\theta(n)) &= \begin{cases} 
    (-k_{22} \sin(2\pi\theta(n)) + k_{21}(\theta_m(n) - 1/4) + 1)/s + \theta(n), & 0 \leq \theta_m(n) < 1/2 \\
    (-k_{22} \sin(2\pi\theta(n)) - k_{21}(\theta_m(n) - 3/4) + 1)/s + \theta(n), & 1/2 \leq \theta_m(n) < 1
  \end{cases}
\end{align*}
$$

where $\theta(n) = \tau(n) \mod p$, $\theta_m(n) = \theta(n) \mod 1$.

### 3 Analysis

#### 3.1 Dynamics

Fig. 2. (a) and (b) are bifurcation diagram and graph of Lyapunov exponent with $k_{12} = 0.6$, $k_{21} = 0.8$, $k_{22} = 0.4$, $r = 2$ and $s = 1.5$.

In Fig. 2(a), we plot the bifurcation diagram of firing phase, where $k_{12} = 0.6$, $k_{21} = 0.8$, $k_{22} = 0.4$, $r = 2$ and $s = 1.5$, to exhibit the bifurcation of the MPSNC. It is clear that the MPSNC has rich dynamics such as period-doubling bifurcation,
inverse period-doubling bifurcation, chaos, etc. And the Lyapunov exponent (LE) is plotted in Fig. 2(b) to distinguish the distributions of periodic/quasi-periodic behavior and chaos, more clearly.

There is a strong link between dynamics and slope of \( f \) and \( F \). We adopt \( k \) to represent the slope, and \( k \) is calculated by Eq. (9).

\[
k = \begin{cases} 
(-2\pi r k_{12} \cos(2\pi r \theta_2(n)) + k_{11})/s + 1, & 0 \leq \theta_0(n) < 1/2 \\
(-2\pi r k_{12} \cos(2\pi r \theta_2(n)) - k_{11})/s + 1, & 1/2 \leq \theta_0(n) < 1 \\
(-2\pi r k_{22} \cos(2\pi r \theta_1(n)) + k_{21})/s + 1, & 0 \leq \theta_0(n) < 1/2 \\
(-2\pi r k_{22} \cos(2\pi r \theta_1(n)) - k_{21})/s + 1, & 1/2 \leq \theta_0(n) < 1 
\end{cases}
\]  

(9)

1) The slope of MPSNC is bigger than 0 only when MSNC\(_1\) and MSNC\(_2\) both have positive slopes. It is obvious that

\[
k \geq 1 - (2\pi |r| k_{12} + |k_{11}|)/s
\]

Thus if the case that

\[
\begin{align*}
1 - (2\pi |r| k_{12} + |k_{11}|)/s &> 0, \quad i.e., \quad 2\pi |r| k_{12} + |k_{11}| < s \\
1 - (2\pi |r| k_{22} + |k_{21}|)/s &> 0, \quad i.e., \quad 2\pi |r| k_{22} + |k_{21}| < s
\end{align*}
\]

holds, \( k > 0 \) is given, ensuring the generation of periodic/quasi-periodic behavior.

2) The absolute value of slope \( |k| \) is constant bigger than 1, when

\[
\begin{align*}
1 + (2\pi |r| k_{12} - |k_{11}|)/s &< -1, \quad i.e., \quad |k_{11}| - 2\pi |r| k_{12} > 2s \\
1 + (2\pi |r| k_{22} - |k_{21}|)/s &< -1, \quad i.e., \quad |k_{21}| - 2\pi |r| k_{22} > 2s
\end{align*}
\]

In this case, the MPSNC can exhibit rich chaotic behavior.

3) Except the cases above, periodic/quasi-periodic behavior and chaos coexist in the rest parameters region.

![Fig. 3](image-url)

**Fig. 3.** Iterative map \( f \) and return map \( F \) of the MPSNC. The parameter values are \( k_{11} = 1.4, k_{12} = 0.1, k_{21} = 0.3, k_{22} = 0.4, r = 0.5 \) and \( s = 2 \) for (a), \( k_{11} = 1.8, k_{12} = 0.5, k_{21} = 1.48, k_{22} = 0.2, r = 0.25 \) and \( s = 0.5 \) for (b), \( k_{11} = 1.2, k_{12} = 0.6, k_{21} = 1.4, k_{22} = -0.5, r = 0.5 \) and \( s = 1.8 \) for (c), \( k_{11} = 1.5, k_{12} = 0.4, k_{21} = 0.5, k_{22} = 0.8, r = 0.5 \) and \( s = 1.5 \) for (d).

Several examples are presented in Fig. 3 to illustrate the dynamics of MPSNC. In Fig. 3(a), the iterative results only have 4 values, and the parameters are set as \( k_{11} = 1.4, k_{12} = 0.1, k_{21} = 0.3, k_{22} = 0.4, r = 0.5 \) and \( s = 2 \), ensuring \( k > 0 \). Fig. 3(b) shows an example with the parameters setting that \( k_{11} = 1.8, k_{12} = 0.5, \)
$k_{21} = 1.48$, $k_{22} = 0.2$, $r = 0.25$ and $s = 0.5$. This parameters setting meets case 2), and the system presents chaos. The rest two subfigures are two examples belong to case 3). Fig. 3(c) exhibits 2-periodic behavior, while Fig. 3(d) has a different chaotic attractor from Fig. 3(b). As can be seen, the MPSNC has rich dynamics, and its firing time and firing phase iterate between two curves that characterize MSNC$_1$ and MSNC$_2$.

### 3.2 Statistical characteristics

In chaotic system, the correlation among the distribution of variables is close to 0, meaning asymptotic statistical independence [18]. In the chaotic region of $F$, the result of each iteration can be regarded as obeying iid. So the CLT is met in this case. And $y$ has the definition as

$$y = \sum_{i=1}^{m} (\theta_i - \bar{\theta}). \quad (10)$$

In Eq. (10), $y$ will tend to obey $N(0, m\sigma^2)$ for $m \to +\infty$, $\bar{\theta}$ is the mathematical expectation of $\theta(i)$ calculated as average, and $\sigma^2$ is the variance of $\theta(i)$. As shown in Fig. 4, the probability distribution of $y$ is generally fit to a Gaussian distribution, except for some tiny deviations due to finite value of $m$, digital calculation accuracy and other factors. It supports the pseudo randomness of chaotic sequence $\{\theta(i)\}$, meaning that $\{\theta(i)\}$ is difficult to predict from outside and useful in encryption.

**Fig. 4.** (a) Probability distribution function of the sum of iterative results with $k_{11} = 1.2$, $k_{12} = 0.2$, $k_{21} = 0.8$, $k_{22} = 0.4$, $r = 0.5$ and $s = 0.18$ (color online). The corresponding Gaussian $f(y) = e^{-y^2/(2\sigma^2)}/\sqrt{2\pi}\sigma^2$ is marked with black solid line. (b) Logarithmic plot of the same data.

### 4 Results of circuit simulation and implementation

The MPSNC is implemented with components like LM339, CD4009, CD4538, CD4066 and some resistors, capacitors.

**Fig. 5.** Simulation results of the circuit working waveform.
Fig. 5 shows the working waveform of the circuit simulated by Multisim. $v_1$ and $v_2$ repeat integrating and resetting between the base level and threshold. Reaching the threshold can cause the generation of a pulse and reset another MSNC in MPSNC. The threshold of one MSNC stores the voltage of threshold signal at the last firing time of another MSNC, illustrating the memory of MPSNC. The pulse train of the MPSNC is given by putting $Y_1$ and $Y_2$ in the same time axis. It looks more chaotic. This result is consistent with the theoretical analysis and design.

Furthermore, the hardware circuit is built with printed circuit board (PCB), as shown in Fig. 6(a). Figs. 6(b)–(f) give the working waveform. The behavior that the pulses of MSNC$_2$ reset $v_1$ and $Th_1$ in MSNC$_1$ is shown in Fig. 6(b), while situation in Fig. 6(c) is the opposite. Fig. 6(d) demonstrates the process of generating pulse. In Fig. 6(e), the upper pulse train is from MSNC$_1$ while the under one is given by MSNC$_2$. It reflects the fact that MSNC$_1$ yields a pulse when $v_1$ reaches the threshold decided by the pulse of MSNC$_2$. Fig. 6(f) shows the pulse train of the MPSNC by putting the pulse trains of these two MSNCs together. The results verify the theoretical design and analysis.

![Simulation results of the circuit working waveform.](image)

**5 Application in image encryption**

In this section, the image is encrypted using the chaotic sequence from $F$. First, we exhibit several chaotic properties of $F$.

![Bifurcation diagram and graph of Lyapunov exponent](image)

Fig. 7. (a) and (b) are bifurcation diagram and graph of Lyapunov exponent with $k_{11} = 1.5$, $k_{21} = 2.2$, $k_{22} = 0.2$, $r = 0.2$ and $s = 0.3$.

The bifurcation diagram and Lyapunov exponent of $F$ are given by Figs. 7(a) and (b), respectively, after fixing some parameters as $k_{11} = 1.5$, $k_{21} = 2.2$, $k_{22} = 0.2$, $r = 0.2$ and $s = 0.3$. The parameters settings are in accordance with the
relationship between the slop and dynamics mentioned in Section 3.1. As can be seen, the chaos fills the whole definition interval of \( k_{12} \) and its chaotic iterations distribute within the whole range \([0, p)\). Thus, \( k_{12} \) can be set as an arbitrary value within the definition domain instead of the limited piecewise parameter settings.

The sensitivities of \( F \) to parameter and initial value are illustrated in Figs. 8(a) and (b). For the original orbits, parameters are set as \( k_{11} = 1.5, k_{12} = 0.4, k_{21} = 2.2, k_{22} = 0.2, r = 0.2, s = 0.3 \), and initial value \( \theta(1) \) is equal to 0.3. In Fig. 8(a), the two orbits have the same starting point with tiny difference \( 10^{-14} \) in \( k_{12} \), while the compared orbit in Fig. 8(b) has tiny change on initial point with \( 10^{-14} \) bigger. It is clear that, after nearly 20 iterations, the two pairs of orbits in Figs. 8(a) and (b) both separate obviously, implying the high sensitivity of \( F \).

Above all, the map \( F \) characterizing the MPSNC has obvious advantages for cryptosystem. Accordingly, we apply it to image encryption, and the detailed description of encryption follows.

1. Starting from \( \theta(1) = \theta_{0}(1) \), iterate the map \( F \) in chaotic range for \( 16 \times M \times N + 1999 \) times. After removing the first 2000 elements of \( \{\theta(i)\} \), a chaotic sequence \( S \) is constructed with the rest elements. In the sequence \( S \), the elements at the even position belong to sub sequence \( S^e \), and the rest elements construct sub sequence \( S^o \). Divide \( S^e \) into \( M \) sub sequences of equal length, denoted as \( S^e_i, i = 1, 2, \ldots, M \). Sort the elements of \( S^e_i \) in ascending order, and return position-sequences \( P^e_i \) where \( i = 1, 2, \ldots, M \), according to the original position of each element. For \( S^o \), divide it into \( N \) sub sequences of equal length, denoted as \( S^o_i \) where \( i = 1, 2, \ldots, N \), and return position-sequences of \( S^o_i \), denoted as \( P^o_i, i = 1, 2, \ldots, N \).

2. Transform the 8-bit grayscale image \( I \) of size \( M \times N \) into a binary matrix \( Ib_1 \) of size \( M \times 8N \), by decomposing each pixel value of \( I \) into 8 binaries.

3. Sort binaries in the \( i \)-th row of \( Ib_1 \), according to the positions recorded in \( P^e_i \). After processing the binaries in all rows, a binary matrix \( Ib_1^* \) of size \( M \times 8N \) is obtained. And transform \( Ib_1^* \) into a 8-bit grayscale image \( I_1 \) of size \( M \times N \).

4. Transform \( I_1 \) into a binary matrix \( Ib_2 \) of size \( 8M \times N \) by decomposing each pixel value of \( I_1 \) into 8 binaries. Sort the elements of \( Ib_2 \) on column direction with \( P^o_i \), using the method similar to steps 2 and 3, and a resulting image \( I_2 \) is obtained.

5. Pixels in \( I_2 \) is scanned from the upper left to the lower right to form a sequence \( C \), and then diffusion operation is carried on \( C \) using Eq. (11).
\[ \theta_d(n + 1) = F(\theta_d(n)) \]
\[ t(n) = \lfloor \theta_d(n + 2000) \times 10^{10} \rfloor \mod 256 \]
\[ C(n) = (Q(n) + \lfloor 255 \times (-t(n)/255)^2 + 1 \rfloor) \mod 256 \]
\[ \oplus (C(n - 1) + \lfloor 255 \times (t(n)/255)^2 \rfloor) \mod 256 \]
\[ \oplus \lfloor 255 \times (-4 \times (t(n)/256 - 1/2)^2 + 1) \rfloor \]

where \( C(0) = \lfloor 255 \times (Q(MN)/255)^4 \rfloor \), \([\bullet] \) returns the largest integer not exceeding itself.

6. Transform \( C \) into \( M \times N \) matrix, and construct a ciphered image.

The proposed cryptosystem with the chaotic map \( F \) has six control parameters and two initial values, which are regarded as the secret keys in the cryptosystem. And Figs. 8(a) and (b) reveal that a tiny fluctuation of \( 10^{-14} \) in parameters or initial values can lead to completely different chaotic sequences. So the length of each parameter and the initial value can be set to 14 decimals, and the key space can reach \( 10^{14 \times 8} = 1.0112 \) in theory. Furthermore, these eight secret keys can also provide the flexible combinations of secret keys according to the demands. As an example, we can fix parts of the secret keys as \( k_{11} = 1.5, k_{21} = 2.2, k_{22} = 0.2, r = 0.2, s = 0.3 \), and only change \( k_{12}, \theta_1(1), \theta_2(1) \) in each encryption. In this case, the key space of the cryptosystem can still be up to \( 0.625 \times 10^{14} \times 0.3 \times 10^{14} \times 0.3 = 1.5625 \times 10^{43} \), according to Fig. 7. It is bigger than \( 2^{100} \), meaning that it is large enough to resist brute-force attack [16]. Here, we set \( k_{12}, \theta_1(1) \) and \( \theta_2(1) \) as 0.4, 0.3 and 1.7, respectively, and take Lena.bmp of 512 \( \times \) 512 pixels as the plain image. In Figs. 9(a) and (c), the plain image and its noise-like ciphered image are given. Obviously, the uniformity of the histogram is improved dramatically after encryption, as shown in Figs. 9(b) and (d). In the diffusion operation, the influence of each bit of the image is spread all over the cipher image, and the “XOR plus mod” is operated between the pixels and the chaotic sequence which is pseudo random. These can change the statistical property, and the distribution of gray-scale will be balanced [12, 22].

![Fig. 9.](image)

**Fig. 9.** (a) Plain image. (b) Histogram of plain image. (c) Ciphered image. (d) Histogram of ciphered image.

We randomly select 2000 pairs of adjacent pixels (in vertical, horizontal, and diagonal direction) from Figs. 9(a) and (c) to calculate the correlation coefficients. It is clear from Table I that the encryption greatly breaks the correlation of two adjacent pixels in the plain image.

<table>
<thead>
<tr>
<th>Image</th>
<th>Horizontal</th>
<th>Vertical</th>
<th>Diagonal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plain image</td>
<td>0.9721</td>
<td>0.9871</td>
<td>0.9605</td>
</tr>
<tr>
<td>Ciphered image</td>
<td>-0.0057</td>
<td>0.0040</td>
<td>-0.0051</td>
</tr>
</tbody>
</table>
The image can be encrypted successfully by using the chaotic sequences produced by the map $F$. However, if we use the periodic sequences of $F$ instead of the chaotic ones, the encryption would fail. As an counterexample, Fig. 10(b) is a failed encryption, and its keys settings are $k_{11} = 1.2$, $k_{12} = 0.6$, $k_{21} = 1.4$, $k_{22} = -0.5$, $r = 0.5$, $s = 1.8$, $\theta_d(1) = 0.3$ and $\theta_d(1) = 1.7$. As mentioned in Section 3.1, the map $F$ will generate a 2-periodic sequence. This periodic sequence can not sort the binaries randomly. After encryption, the object contour in Fig. 10(b) can still be found easily, compared with Fig. 10(a), the plain image, so the information is leaked, meaning the encryption fails.

Fig. 10. (a) Plain image. (b) Ciphered image.

6 Conclusion

In this paper, the MPSNC is proposed. It is constructed by cross-switching of two MSNCs, and can yield a rich sets of pulse trains by repeating cross integrate-and-fire behavior. In the MPSNC, one MSNC can store the potential of its threshold signal at the last firing time of another MSNC as threshold. It reveals that the threshold has a short-term cross memory. So the MPSNC is closer to the biological neurons than previous SNC and PCSNC. By extending the definition of the LCM into the field of rational number, we have further derived the iterative map of firing phase. This extended definition can be applied to compute the common period of signals with different periods. We use the methods of bifurcation diagram, graph of Lyapunov exponent, plots of iterative firing time and iterative firing phase to exhibit the rich dynamics of the MPSNC. Also, the probabilistic nature of firing phase is considered. The probability density of the sum of the firing phases in chaotic region is calculated, and it complies well with Gaussian. Besides, the hardware circuit of MPSNC is done in our laboratory. The results from simulation and actual circuit are both consistent with the design and analysis. Furthermore, the return map of firing phase is applied to encrypt image, bringing excellent encryption effect.

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