Novel time-domain asymptotic solution for transient whispering-gallery mode radiation from a cylindrical concave conducting boundary

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Abstract: We derive a novel time-domain asymptotic solution for a transient whispering-gallery (WG) mode radiation from the aperture plane of a cylindrical concave conducting boundary. The WG mode is excited by a Gaussian-type modulated pulse source. We show that the WG mode radiation field propagates with the phase velocity coincident with the speed of light and the newly derived group velocity. The validity of the time-domain asymptotic solution is confirmed by comparing with the reference solution calculated numerically. We show that the asymptotic solution proposed here is easy to understand the physical phenomena of the transient WG mode radiation field.

Keywords: time-domain, asymptotic solution, transient whispering-gallery mode, radiation field, cylindrical concave conducting boundary

Classification: Electromagnetic theory

References

The whispering-gallery (WG) modes have been utilized in the research and design of high-power WG mode type gyrotrons [1] and the laser resonator [2]. In such devices, it is necessary to analyze the radiation field of the WG modes. Analyzing the radiation of the WG modes is also important in the studies on the scattering of electromagnetic fields by a cylindrically curved conducting concave and a concave-to-convex surfaces [3, 4, 5].

In this research, a novel time-domain asymptotic solution is derived for the transient WG mode radiation from the edge aperture plane of a cylindrical concave conducting boundary [6, 7] (see also [11] in the reference [7]). A time-domain uniform geometrical theory of diffraction (TD-UTD) impulse response has been proposed by Rousseau and Pathak [8]. However, to obtain the transient scattered fields excited by a more general source, it is necessary to convolve the TD-UTD impulse response [8] with any given pulsive source. Except for the special cases [8], the numerical convolution may be required furthermore. Also, the paper by Rousseau and Pathak does not include the analysis of the radiation of the WG mode from the cylindrically curved concave boundary [8].

Here, we assume that the WG mode is excited by a high-frequency source with a truncated Gaussian time variation [6, 7]. We will confirm the validity of the time-domain asymptotic solution proposed here by comparing with the reference solution calculated numerically. We will show that the asymptotic solution is easy to understand the physical phenomena of the WG mode radiation field. We will derive newly the group velocity of the transient WG mode radiation field.
2 Formulation and novel time-domain asymptotic solution

2.1 Formulation and frequency-domain asymptotic solution

In this section, we will review the previous results that dealt with the frequency-domain WG mode radiation [5, 6]. Fig. 1 (a) shows a open cylindrical concave conducting boundary with a constant radius \( a \) and the two-dimensional coordinate system \((\rho, \phi)\). The magnetic type \( m \)th order WG mode \( u_m(\rho, \phi) \) defined by

\[
u_m(\rho, \phi) = J_{\nu_m}(k\rho) \exp\{i\nu_m(\phi + \phi_0)\}, \quad J'_{\nu_m}(ka) = 0, \quad m = 1, 2, \cdots, M \quad (1)
\]

is trapped between the concave boundary and the \( m \)th modal caustic \( \rho = \rho_m = \nu_m/k \) (\( k \): wave number) and propagates along the boundary toward the aperture plane near the edge A (see Fig. 1 (a)) [5, 6]. In Eq. (1), \( J_{\nu_m}(k\rho) \) is the Bessel function, \( \nu_m \) is the \( m \)th eigenvalue satisfying \( J'_{\nu_m}(ka) = 0 \), and the time factor \( \exp(-i\omega t) \) is suppressed. We assume that the WG mode is started from the some reference point at the azimuthal angle \( \phi = -\phi_0 \) (\( \phi_0 > 0 \)) (see Fig. 1 (a)).

The WG mode in Eq. (1) is radiated from the aperture plane near the edge A. The frequency-domain asymptotic solution for the WG mode radiation field \( u_m(\rho, \omega) (\equiv u_m(\omega)) \), hereafter the position vector \( \rho = (\rho, \phi) \) is dropped in the notations) may be represented by [5, 6]

\[
u_m(\omega) = U(P)u_{m,go}(\omega) + u_{m,d}(\omega) \quad (2)
\]

\[
u_{m,j}(\omega) = A_{m,j}(\omega) \exp\{i\omega L_{m,j}(\omega)/c\}, \quad j \equiv \text{go or } j \equiv \text{d} \quad (3)
\]

Here, \( u_{m,go}(\omega) \) denotes the geometrical ray \( R \rightarrow P_L \) (see Fig. 1 (b)) converted from the WG mode \( Q_0 \uparrow Q_1 \rightarrow R \) and is observed at \( P_L \) located in the lit region between the shadow boundary (SB) and the reflection boundary (RB) shown in Fig. 1 (b). The unit step function \( U(P) \) is defined as \( U(P) = 1 \) (\( U(P) = 0 \)) for the observation point located in the lit region (in the shadow

![Fig. 1. WG mode propagation and radiation from cylindrical concave boundary.](image-url)
region). While, \( u_{m,d}(\omega) \) denotes the edge-diffracted ray \( A \rightarrow P_L \) (or \( P_S \)) excited by the incident modal ray \( Q_0 \uparrow Q_2 \rightarrow A \). The notation \( Q_0 \uparrow Q_{1,2} \) has been used to indicate the WG mode propagation from the starting point \( Q_0 \) to the point \( Q_{1,2} \) along the arc of the modal caustic defined by \( \rho = \rho_m \) (see Fig. 1 (b)).

In Eq. (3), \( L_{m,go}(\omega) \) and \( L_{m,d}(\omega) \) denote the total propagation distances of the geometrical ray \( Q_0 \uparrow Q_1 \rightarrow R \rightarrow P_L \) and the edge-diffracted ray \( Q_0 \uparrow Q_2 \rightarrow A \rightarrow P_L \) (or \( P_S \)), respectively. Note that the total propagation distances \( L_{m,j}(\omega) \), \( j = go \) and \( j = d \), change as the function of the angular frequency \( \omega \) since the caustic location \( \rho = \rho_m = \nu_m c/\omega \) (\( c \): speed of light) changes as the function of \( \omega \) [5]. While, \( A_{m,j}(\omega) \) denotes the slowly varying amplitude as the function of \( \omega \). The readers may find the explicit frequency-domain uniform asymptotic solution for \( u_m(\omega) \) associated with \( u_{m,go}(\omega) \) and \( u_{m,d}(\omega) \) in the references [5] and [6] (see also [12] quoted in the reference [5]).

### 2.2 Novel time-domain asymptotic solution for WG mode radiation field

The time-domain asymptotic solution \( y_m(\rho, t) (\equiv y_m(t)) \) for the WG mode radiation may be obtained from the inverse Fourier transform [6, 7]:

\[
y_m(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u_m(\omega) S(\omega) \exp(-i\omega t) d\omega. \tag{4}
\]

We assume the truncated Gaussian-type modulated pulse source [6, 7]:

\[
s(t) = \exp\{-i\omega_0(t-t_0) - (t-t_0)^2/(4d^2)\} \text{ for } 0 \leq t \leq 2t_0, \quad s(t) = 0 \text{ : elsewhere} \tag{5}
\]

where \( \omega_0 \) denotes the central angular frequency, and \( t_0 \) and \( d \) are constant parameters. The frequency spectrum \( S(\omega) \) of the source function \( s(t) \) is given by

\[
S(\omega) = 2d\sqrt{\pi} \text{Re}[\text{erf} \beta(\omega)] \exp\{i\omega t_0 - d^2(\omega - \omega_0)^2\}, \quad \beta(\omega) = \frac{t_0}{2d} - id(\omega - \omega_0) \tag{6}
\]

Fig. 2 illustrates the Gaussian-type modulated pulse source. The main portion of the frequency spectrum \( S(\omega) \) is distributed within the high-frequency region.

Then substituting \( u_m(\omega) \) in Eq. (2) and \( S(\omega) \) in Eq. (6) into the inverse Fourier transform in Eq. (4) yields [6] (see also [11] in the reference [7])

\[
y_m(t) = U(P)y_{m,go}(t) + y_{m,d}(t) \tag{7}
\]

\[
y_{m,j}(t) = \frac{d}{\sqrt{\pi}} \int_{-\infty}^{\infty} F_{m,j}(\omega) \exp\left[-d^2 \left\{(\omega - \omega_0)^2 + i h_{m,j}(\omega)\right\}\right] d\omega \tag{8}
\]

\[
F_{m,j}(\omega) = A_{m,j}(\omega) \text{Re}[\text{erf} \beta(\omega)] \tag{9}
\]

\[
h_{m,j}(\omega) = \frac{\omega}{d^2} T_{m,j}(\omega), \quad T_{m,j}(\omega) = t - t_0 - \frac{L_{m,j}(\omega)}{c}, \quad j = go \text{ or } j = d \tag{10}
\]

where \( y_{m,go}(t) \) and \( y_{m,d}(t) \) in Eq. (7) denote, respectively, the time-domain geometrical ray and the time-domain edge-diffracted ray (see Fig. 1 (b)).
Fig. 2. Gaussian-type modulated pulse source. 
\( t_0 = 2.50 \times 10^{-10} \, [s], \quad d = 4.00 \times 10^{-11} \, [s], \) and 
\( \omega_0 = 1.00 \times 10^{11} \, [\text{rad/s}] \) in Eq. (5).

Since \( h_{m,j}(\omega) \) in Eq. (8) varies slowly as the function of \( \omega \) near the central angular frequency \( \omega_0 \), we will use the approximation:

\[
h_{m,j}(\omega) \sim h_{m,j}(\omega_0) + (\omega - \omega_0) h'_{m,j}(\omega_0). \tag{11}
\]

Then, the integral in Eq. (8) may be evaluated asymptotically by applying the saddle point technique [7] after extending the integral in Eq. (8) to the complex \( \omega' \)-plane via \( \omega' = 2d^2 \omega \) and assuming the large \( \Omega = 1/(4d^2) \), i.e., \( \Omega \gg 1 \). One may obtain the following novel time-domain asymptotic solution for the transient WG mode radiation field:

\[
y_{m,j}(t) = A_{m,j}(\omega_{s,m,j}) \text{Re}[\text{erf}(\beta(\omega_{s,m,j}))\exp\left\{-i\omega_0 \left(t - t_0 - \frac{L_{m,j}(\omega_0)}{v_{g,m,j}}\right)\right\} - \frac{1}{4d^2} \left(t - t_0 - \frac{L_{m,j}(\omega_0)}{v_{g,m,j}}\right)^2]\] \quad j \equiv go \ or \ j \equiv d \tag{12}
\]

where the group velocity of the transient WG mode radiation field is given by

\[
v_{g,m,j} = \frac{c}{1 + \omega_0 L'_{m,j}(\omega_0)/L_{m,j}(\omega_0)}, \quad c: \text{speed of light}, \quad j \equiv go \ or \ j \equiv d \tag{14}
\]

The time-domain asymptotic solution in Eq. (7) with Eqs. (12)–(14) is easy to understand the physical phenomena of the transient WG mode radiation. The total propagation distance of the geometrical ray \( Q_0 \uparrow Q_1 \rightarrow R \rightarrow P_L \) (see Fig. 1 (b)) is given by \( L_{m,go}(\omega_0) \), while the total propagation distance of the edge-diffracted ray \( Q_0 \uparrow Q_2 \rightarrow A \rightarrow P_L \) (or \( P_S \)) is given by \( L_{m,d}(\omega_0) \). The geometrical ray \( y_{m,go}(t) \) and the edge-diffracted ray \( y_{m,d}(t) \) propagate, respectively, with the phase velocity \( c \), i.e., the speed of light, and the group velocities \( v_{g,m,j} \).

### 3 Numerical results and discussions

The time-domain WG mode radiation fields have been calculated extensively by using the novel asymptotic solution derived in the previous section and
Fig. 3. Time-domain 2nd order \((m = 2)\) WG mode radiation fields observed at the points \(P_L(r, w)\) and \(P_S(r, w)\) (see Fig. 1 (b)). \((a, -\phi_0) = (0.6 \text{ m}, -120^\circ)\).

the pulse source \(s(t)\) given in Fig. 2 (see also Eq. (5)). We will show here only the typical examples since all the other solutions are similar to those shown here if the numerical parameters used in the calculations satisfy the high-frequency condition and \(\Omega (= 1/(4d^2)) \gg 1\).

Fig. 3 (a) and Fig. 3 (b) show the transient WG mode radiation fields vs. time curves observed at \(P_L\) in the lit region and at \(P_S\) in the shadow region, respectively, when the 2nd-order WG mode with \(m = 2\) started from \(\phi = -\phi_0\) \((\phi_0 > 0)\) at the time \(t = 0 \text{ [s]}\) is radiated from the aperture plane near the edge A (see Fig. 1 (a)). The solid curve (-----) is calculated from the asymptotic solution in Eq. (7) associated with Eqs. (12)–(14) and the open circles (○○○) are obtained numerically from the numerical integration of the integral representing the WG mode radiation field (in the frequency-domain calculation) [5, 6] and the FFT (Fast Fourier Transform) numerical code (in the time-domain calculation) [6, 7]. The numerical solutions (○○○) serve as the reference solution.

It is observed that the asymptotic solution agrees excellently with the reference solution. From these results, it is clarified that the time-domain geometrical ray and the edge-diffracted ray comprising the transient WG mode propagate the total distances \(L_{m,j}(\omega_0), j \equiv go\) and \(j \equiv d\), along the trajectories \(Q_0 \uparrow Q_1 \rightarrow R \rightarrow P_L\) and \(Q_0 \uparrow Q_2 \rightarrow A \rightarrow P_L\) (or \(P_S\)) (see Fig. 1 (b)), respectively, with the phase velocity \(c\) and the group velocities \(v_{g,m,j}, j \equiv go\) and \(j \equiv d\), derived in Eq. (14). Note that the total propagation distances \(L_{m,j}(\omega_0), j \equiv go\) and \(j \equiv d\), are determined at the central angular frequency \(\omega_0\) of the modulated pulse source \(s(t)\) defined in Eq. (5).

Since the time-domain asymptotic solution in Eq. (7) associated with Eqs. (12)–(14) describes the local phenomena, it may be extended to the problem of the WG mode radiation from the varying radius of the concave boundary [7].
4 Conclusion

We have derived the novel time-domain asymptotic solution for the transient WG mode radiation from the aperture plane of a cylindrically curved concave conducting boundary. The transient WG mode is excited by the Gaussian-type modulated pulse source. We have newly derived the group velocity of the transient WG mode radiation field. Comparisons with the reference solution calculated numerically reveal the validity of the proposed time-domain asymptotic solution for the transient WG mode radiation. It is clarified that the asymptotic solution proposed here is easy to understand the physical phenomena of the radiation field of the transient WG mode.