Rotational invariants for Tchebichef moments

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Abstract: Image moments that are invariants to distortions such as translation, scale and rotation are an important tool in pattern recognition. In this paper, derivation of invariants for Tchebichef moments with respect to rotation will be presented. The rotational invariants are achieved neither by tempering with the image nor transforming the coordinates from rectangular to polar. They are derived using moment normalization method, which attempts to map the distorted moments with the undistorted ones. Experimental results show that the derivation is correct and it poses as a viable solution to test whether one image is a rotationally distorted version of another.

Keywords: pattern recognition, Tchebichef moments, moment invariants

Classification: Science and engineering for electronics

References

1 Introduction

Image moments are often used as descriptors for images because they are able to collect statistical information from images. An invariant version of such moments is an important tool in the field of pattern recognition because they remain unchanged during distortions such as translation, scale and/or rotation. Moreover, moments that are discrete and orthogonal are often preferable because they do not have spatial approximations and they are more noise tolerant. Tchebichef moments [1] are an example of discrete and orthogonal moments.

Although the translation and scale invariants for Tchebichef moments have been proposed using its distorted moments as the basis function [2], the rotational invariants have not. Unlike translation and scale invariants, rotational invariants are more difficult to derive because both x and y axes must be evaluated together. Radial Tchebichef [3, 4] offers a solution. It uses the principles of rectangular to polar space translation and Zernike moments to derive the invariants.

Without transformation of coordinate space, this paper will apply moment normalization method to derive rotational invariants for Tchebichef moments using its distorted moments as the basis function. It will first present how the rotational distortion parameters are found and how they are applied to map the moments before and after distortion to gain the invariants property. Finally, this paper will test the performance of the proposed invariants.

2 Retrieving distortion parameter

If an image is distorted, there exists a mapping function which relates the new coordinate $x'$, $y'$ and the original coordinate $x$, $y$. In the case of rotationally distorted images, it can be expressed as [5]:

$$x' = a(x - x_0) - b(y - y_0) \quad \text{and} \quad y' = b(x - x_0) + a(y - y_0),$$

where the centroids, $x_0$ and $y_0$, are defined as $\sum \sum f(x,y)$ and $\sum \sum f(x,y)$, respectively. The distortion parameters, $a$ and $b$, can actually be retrieved by matching the central moments of images that are before distortion, $\mu_{nm}$, and after distortion, $\mu'_{nm}$. The distortion parameters, $a$ and $b$, can be solved as follows [5]:

$$a = 3 \sqrt{\frac{PQ + QS}{P^2 + Q^2} |J|} \quad \text{and} \quad b = 3 \sqrt{\frac{QR + PS}{P^2 + Q^2} |J|},$$

where

$$|J| = \left| \frac{\mu'_{00}}{\mu_{00}} \right|$$

$$P = \mu_{21}(\mu_{30}\mu_{30} - \mu_{03}\mu_{03}) + 2\mu_{03}(\mu_{12}\mu_{12} + \mu_{21}\mu_{21}) - 2\mu_{12}\mu_{03}\mu_{30} + 3\mu_{21}(\mu_{21}\mu_{21} + \mu_{12}\mu_{12})$$

$$Q = \mu_{12}(\mu_{03}\mu_{03} - \mu_{30}\mu_{30}) + 2\mu_{30}(\mu_{21}\mu_{21} + \mu_{12}\mu_{12}) - 2\mu_{21}\mu_{03}\mu_{30} + 3\mu_{12}(\mu_{12}\mu_{12} + \mu_{21}\mu_{21})$$
\[
R = 3\mu'_{21}(\mu_{21}\mu_{21} + \mu_{12}\mu_{12}) + \mu'_{30}(\mu_{30}\mu_{21} - \mu_{03}\mu_{12}) \\
+ \mu'_{03}(\mu_{30}\mu_{12} - \mu_{03}\mu_{21}) + 2\mu'_{03}(\mu_{21}\mu_{21} + \mu_{12}\mu_{12})
\]
\[
S = 3\mu'_{12}(\mu_{21}\mu_{21} + \mu_{12}\mu_{12}) + \mu'_{03}(\mu_{30}\mu_{12} - \mu_{03}\mu_{21}) \\
+ \mu'_{30}(\mu_{03}\mu_{21} - \mu_{30}\mu_{12}) + 2\mu'_{30}(\mu_{21}\mu_{21} + \mu_{12}\mu_{12})
\]

For images that are scaled with a factor \(r\), and rotated with an angle \(\theta\), the distortion parameters, \(a\) and \(b\), can be expressed as \(r \cos \theta\) and \(r \sin \theta\) respectively. The scale value, \(r\), can be found by \(\sqrt{a^2 + b^2}\), and the rotation angle, \(\theta\), can be found by \(\arctan(b/a)\). The distortion parameters are crucial for the derivation of rotational invariants as they will be used in the mapping between moments.

3 Tchebichef moments

One of the most important property of Tchebichef polynomials is that it is discrete and orthogonal. The discrete orthogonality relation for Tchebichef polynomials is given by [1]:
\[
\sum_{x=0}^{N-1} p_{n,N}(x)p_{m,M}(x) = \delta_{nm},
\]
where \(\delta_{nm}\) is the Kronecker delta and, \(p_{n,N}\) and \(p_{m,M}\) are the normalized and weighted Tchebichef polynomials. The polynomials are selected because they do not need a separate weight and normalise functions. Because of the orthogonality relationship in Eq. (4), Tchebichef polynomials are often used as a basis function for moments’ calculation. The resulted moments can be used to perfectly reconstruct the original image. This important property shows that all image features are effectively captured by the moments.

The Tchebichef polynomials \(p_{n,N}(x)\) can be expressed by grouping the coefficients as [1]:
\[
p_{n,N}(x) = \sum_{i=0}^{n} \psi_{n,N}(i)x^i.
\]
The coefficient \(\psi_{n,N}(i)\) is defined as [1]:
\[
\psi_{n,N}(i) = \frac{\sum_{k=0}^{n} \sum_{j=0}^{m} \psi_{n,N}^{(i)}(j) \psi_{m,M}^{(i)} \left(\frac{N-1-k}{n}\right) \left(\frac{N-1}{n}\right) \left(\frac{s_k}{n}\right)^i \sqrt{\frac{N}{2n+1}}(1 - \frac{2}{N^2})(1 - \frac{2}{N^2}) \cdots (1 - \frac{n^2}{N^2})}.
\]
where \(s_k^{(i)}\) are the Stirling numbers of the first kind, which satisfies \(\frac{x^i}{(x-k)!} = \sum_{k=0}^{n} s_k^{(i)} x^k\).

Central moments, \(\mu_{nm}\), are helpful for invariants derivation because they are invariants to translation and the centroids can be used as a reference point for distortions such as skew and rotation. Central Tchebichef, \(\rho_{nm}\), are the Tchebichef version of such moments. They can be defined as follows [6]:
\[
\rho_{nm} = \sum_{i=0}^{n} \sum_{j=0}^{m} \psi_{n,N}(i) \psi_{m,M}(j) \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} \bar{x}^i \bar{y}^m f(x, y)
\]
where $\bar{x} = x - x_0$ and $\bar{y} = y - y_0$. The size of image is $N \times M$ pixels and the intensity function of the image is $f(x, y)$. The Tchebichef moment, $P_{n,m}$, and the coefficient $A_{i,n,N}(x_0)$ are defined as:

$$P_{n,m} = \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} p_{n,m}(x)p_{n,m}(y) f(x, y),$$

$$A_{i,n,N}(x_0) = \sum_{p=0}^{i} \lambda_{i+p,N}(i) \sum_{q=0}^{i+p} \psi_{n,N}(i + p + q) \left( \frac{i + p + q}{q} \right) (-x_0)^q$$

with

$$\lambda_{n,N}(i) = -\sum_{k=1}^{n-i} \psi_{n,N}(n-k) \lambda_{n-k,N}(i).$$  \hfill (8)

### 4 Invariants derivation

Rotational invariants for Tchebichef moments can be derived using these four steps. **Step 1:** Based on Eq. (7), define the Rotational Tchebichef moments, $\rho'_{n,m}$, by substituting $\bar{x}$ and $\bar{y}$ with the mapping function, $(a\bar{x} - b\bar{y})$ and $(b\bar{x} + a\bar{y})$, from Eq. (1). **Step 2:** Expand $(a\bar{x} - b\bar{y})^n$ and $(b\bar{x} + a\bar{y})^m$ and separate out the distortion parameters, $a$ and $b$, with $\bar{x}$ and $\bar{y}$. **Step 3:** Group and simplify the coefficient of the polynomial, $\psi$ with the coefficient $B$. **Step 4:** Finally, substitute the summation of $\bar{x}$ and $\bar{y}$ using Central Tchebichef, $\rho_{n,m}$, in Eq. (7). These steps can also be shown as the following:

$$\rho'_{n,m} = |J| \sum_{i=0}^{n} \sum_{j=0}^{m} \psi_{n,N}(i) \psi_{m,M}(j) \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} (a\bar{x} - b\bar{y})^n (b\bar{x} + a\bar{y})^m f(x, y)$$

$$= |J| \sum_{i=0}^{n} \sum_{j=0}^{m} \psi_{n,N}(i) \psi_{m,M}(j) \sum_{p=0}^{n+m} B_{n+m,p} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} (\bar{x})^p (\bar{y})^{n+m-p} f(x, y)$$

$$= |J| \sum_{i=0}^{n} \sum_{j=0}^{m} C_{n,m,i,j} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} (\bar{x})^i (\bar{y})^{n+m-i-j} f(x, y)$$

$$= |J| \sum_{i=0}^{n} \sum_{j=0}^{m} D_{n,m,i,j} \rho_{i,j}$$  \hfill (9)

The coefficients are grouped as follows:

$$B_{n+m,p} = \sum_{q=0}^{\min(p, n, b_2+q)} \left( \begin{array}{c} n+m \left( \begin{array}{c} m \left( \begin{array}{c} b_1+q \n 1 \end{array} \right) \left( \begin{array}{c} b_2+q \n b_1+q \end{array} \right) \right) b_1+2q \n b_1+n-p \end{array} \right)$$

$$C_{n,m,i,j} = \sum_{k_1+q \leq n} \sum_{k_2+q \leq m} \psi_{n,N}(k_1) \psi_{m,M}(k_2) B_{k_1+k_2,i,j}$$

$$D_{n,m,i,j} = \sum_{p=0}^{\min(p, n, b_2+q)} \lambda_{i+p,N}(i) \sum_{q=0}^{\min(q, n, b_2+q)} \lambda_{n+m-i-p-q,M}(j) C_{n,m,i+p,q}$$  \hfill (10)

The Eq. (9) shows that the moments, $\rho'_{n,m}$, which are rotationally invariants, can be composed of Central Tchebichef, which are rotationally variants.
However, if the distortion parameters do not match the image, the mapping process shown in Eq. (9) would fail to produce invariant moments.

Radial Tchebichef [3, 4] uses a different approach to tackle rotational invariants. It changes the domain from rectangular $x$ and $y$ to polar $r$ and $k$. Radial Tchebichef is defined as:

$$
\eta_{p,q} = \left( S_{p,q}^{(c)} \right)^2 + \left( S_{p,q}^{(s)} \right)^2
$$

(11)

where

$$
S_{p,q}^{(c)} = \sum_{k=0}^{n-1} \cos(q\theta_k) \sum_{r=0}^{m-1} p_{p,m}(r) f(r, \theta_k), \quad \text{and}
$$

$$
S_{p,q}^{(s)} = \sum_{k=0}^{n-1} \sin(q\theta_k) \sum_{r=0}^{m-1} p_{p,m}(r) f(r, \theta_k).
$$

(12)

The angle $\theta_k$ is $2\pi k/n$. The mapping of $r, k$ into $x$ and $y$ are done as follows:

$$
x = \frac{N}{2} \left( \frac{r}{m-1} \cos(\theta_k) + 1 \right) \quad \text{and}
$$

$$
y = \frac{N}{2} \left( \frac{r}{m-1} \sin(\theta_k) + 1 \right).
$$

(13)

5 Results and discussion

Table I shows the images that are used for the experiments. Some are only rotated, while others are rotationally scaled as well. The proposed method can actually work on images that are rotated from any point since it is using Central Tchebichef as its basis function. However, since Radial Tchebichef requires the image to be rotated right at the center of the image, all test images are rotated at the center for the purpose of comparison.

### Table I. Test images

<table>
<thead>
<tr>
<th>Degree, Scale</th>
<th>0°, 1</th>
<th>30°, 1</th>
<th>60°, 1</th>
<th>90°, 1</th>
<th>120°, 1</th>
<th>150°, 1</th>
<th>180°, 1</th>
<th>210°, 1</th>
<th>240°, 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree, Scale</td>
<td>250°, 1</td>
<td>300°, 1</td>
<td>350°, 0.7</td>
<td>50°, 0.9</td>
<td>100°, 1.2</td>
<td>150°, 1.4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table II shows the invariants for Radial Tchebichef and the proposed Rotational Tchebichef moments. These invariants are generated from Eq. (11) and Eq. (9). The results show that although the formulation is more complex, the generated results of proposed method are more accurate. This is because there aren’t any rectangular to polar transformation as shown in Eq. (13). Unlike the Radial Tchebichef, the proposed approach is able to work on rotationally scaled images as well. This will be shown next.
Table II. Radial Tchebichef and Rotational Tchebichef

<table>
<thead>
<tr>
<th>Degree</th>
<th>Central Tchebichef $\eta_{\nu,\rho}$</th>
<th>Rotational Tchebichef $\rho'_{\nu,\rho}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\eta_{1,1}$</td>
<td>$\eta_{2,2}$</td>
</tr>
<tr>
<td>0°</td>
<td>0.017086</td>
<td>0.024975</td>
</tr>
<tr>
<td>30°</td>
<td>0.011810</td>
<td>0.026228</td>
</tr>
<tr>
<td>80°</td>
<td>0.012835</td>
<td>0.027253</td>
</tr>
<tr>
<td>120°</td>
<td>0.010364</td>
<td>0.024690</td>
</tr>
<tr>
<td>170°</td>
<td>0.009852</td>
<td>0.024717</td>
</tr>
<tr>
<td>220°</td>
<td>0.007841</td>
<td>0.023421</td>
</tr>
<tr>
<td>270°</td>
<td>0.010304</td>
<td>0.022418</td>
</tr>
</tbody>
</table>

Table III shows the Central Tchebichef and Rotational Tchebichef moments for rotated and scaled images. Central Tchebichef, which is the basis for Rotational Tchebichef calculation, is invariant to translation only; therefore the values are significantly different for images with different rotation and scale distortions.

Table III. Central Tchebichef and Rotational Tchebichef

<table>
<thead>
<tr>
<th>Degree</th>
<th>Central Tchebichef $\rho_{\nu,\rho}$</th>
<th>Rotational Tchebichef $\rho'_{\nu,\rho}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\rho_{1,1}$</td>
<td>$\rho_{5,3}$</td>
</tr>
<tr>
<td>0°</td>
<td>−0.4706</td>
<td>2.3175</td>
</tr>
<tr>
<td>50°</td>
<td>0.0759</td>
<td>−0.4533</td>
</tr>
<tr>
<td>150°</td>
<td>0.3271</td>
<td>−2.0308</td>
</tr>
<tr>
<td>210°</td>
<td>−0.6256</td>
<td>2.6208</td>
</tr>
<tr>
<td>290°</td>
<td>0.7784</td>
<td>3.1838</td>
</tr>
</tbody>
</table>

From the perspective of image properties, the Rotational Tchebichef moments are no different compared to the original moments because no information is cancelled off or nullify. Only the moments have been mapped to have the same rotational parameters.

6 Conclusion

This paper derived the rotational invariants for Tchebichef moments using its distorted moments as basis function. The resulted invariants are more accurate and robust. This is because there was no discretization of pixel coordinates involved and it can work on both scaled and translated images as well. Besides derivation of rotational invariants, the presented method can also be applied for the derivation of skew invariants or any other type of invariants by selecting a proper mapping function and following the derivation steps as shown in the paper.

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