A novel uniform asymptotic solution for reflection of a Gaussian beam at a dielectric interface

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Abstract: We have derived a novel uniform asymptotic solution for reflection of the Gaussian beam that is incident on a dielectric interface from a denser medium to a rarer one. The validity of the novel uniform asymptotic solution has been confirmed by comparing with the reference solution. It has been shown that the transition beam plays an important role to connect two different solutions smoothly through the transition region. It becomes apparent that, near the critical angle of the total reflection, the Gaussian beam is displaced to either direction by the interference between the geometrically reflected beam and the lateral beam.

Keywords: asymptotic solution, reflection, lateral beam, Gaussian beam, transition beam, dielectric interface

Classification: Electromagnetic theory

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1 Introduction

Past studies on the reflection of a Gaussian beam that is incident from a denser dielectric medium to a rarer one have shown that the Gaussian beam reflected by a single medium interface do not exactly follow the path of geometrical-optics [1, 2, 3, 4, 5, 6, 7, 8]. Many theoretical studies are concerned with the investigation of the nonspecular effects in the Gaussian beam reflection on the dielectric interface [1, 2, 3, 4, 5, 6, 8].

The studies on the asymptotic analysis methods for the reflected Gaussian beam fields applicable uniformly through the transition region are not enough until today [2, 9, 10, 11, 12]. Horowitz and Tamir [2] have investigated the basic wave mechanism, including both the Goos-Hänchen shift and the weak radiance that accompanies the reflected beam. The integral representation and its asymptotic solution for the Gaussian beam reflection by Horowitz and Tamir [2] can be applied in the limited case when the Gaussian beam is incident on the dielectric interface nearly from the normal direction to the interface.

In this paper, we shall investigate the reflection of a Gaussian beam that is incident on the interface from a denser dielectric medium to a rarer one at an arbitrary angle [9, 10, 11, 12]. We will derive a novel uniform asymptotic solution for reflection of the Gaussian beam consisting of the geometrically reflected beam, the lateral beam, and the transition beam. The validity and the physical phenomena of the novel uniform asymptotic solution for the Gaussian beam reflection are confirmed by comparing with the reference

solution calculated numerically.

2 Uniform asymptotic solution for Gaussian beam reflection

2.1 Formulation and integral representations

Fig. 1 shows the schematic figure for reflection of a Gaussian beam incident on a plane dielectric interface from a denser dielectric medium (ε_1, μ_0) to a rarer one (ε_2, μ_0), the Cartesian coordinate system (x, y, z) , the incident beam coordinate system (x_i, z_i) , and the reflected beam coordinate system (x_r, z_r) . The relative refractive index n defined by $n = \sqrt{\varepsilon_2/\varepsilon_1}$ satisfies $n < 1$.

Fig. 1. Reflection of Gaussian beam by a plane dielectric interface and coordinate systems.

We assume that the y -component of the incident Gaussian beam $U_y^i(x_i, 0)$ along the aperture plane $z_i = 0$ is given by [2, 3, 11]

$$U_y^i(x_i, 0) = \frac{1}{\sqrt{\pi}W} \exp \{ -(x_i/W)^2 \} \quad (1)$$

in the incident beam coordinate system (x_i, z_i) . Here $2W$ denotes the beam width, and $U_y^i(x_i, 0) = H_y^i(x_i, 0)$ for the TM case and $U_y^i(x_i, 0) = E_y^i(x_i, 0)$ for the TE case. Then one may obtain the reflected Gaussian beam representation observed at $P(x, z)$ in (x, z) coordinate system as follows [2, 11, 12]:

$$U_y^r(x, z) = \frac{k_1}{2\pi} \int_{P_\theta} \Gamma(\theta) B(\theta, \theta_b) e^{ik_1 R_1 q(\theta)} d\theta \quad (2)$$

$$B(\theta, \theta_b) = e^{-\{k_1 W \sin(\theta - \theta_b)/2\}^2} \cos(\theta - \theta_b) \quad (3)$$

$$\Gamma(\theta) = \frac{m \cos \theta - \sqrt{n^2 - \sin^2 \theta}}{m \cos \theta + \sqrt{n^2 - \sin^2 \theta}}, \quad q(\theta) = \cos(\theta - \theta_s) \quad (4)$$

where $m = n^2$ in the TM case and $m = 1$ in the TE case. The notations k_1 , R_1 ($= \sqrt{x^2 + (h+z)^2}$), θ_b , and θ_s denote, respectively, the wavenumber in the upper medium 1, the propagation distance along $Q_i(0,0) \rightarrow A \rightarrow$

$P(x, z)$, the incident angle of the beam axis, and the incident angle of the geometrically reflected beam (see Fig. 1). In Fig. 2 (a), the integration path P_θ and the branch point singularities $\pm\delta$ ($\delta = \sin^{-1} n$) associated with the branch cuts are shown in the complex θ -plane.

To obtain a novel uniform asymptotic solution, one may express $\Gamma(\theta)$ in (4) as follows:

$$\Gamma(\theta) = \Gamma(\theta_s) + \Gamma^s(\theta), \quad \Gamma^s(\theta) = \bar{\Gamma}^s(\theta) + \Gamma^{(2)}(\theta)\sqrt{\sin(\delta - \theta)} \quad (5)$$

$$\bar{\Gamma}^s(\theta) = \Gamma^{(1)}(\theta) - \Gamma^{(1)}(\theta_s) - \Gamma^{(2)}(\theta_s)\sqrt{\sin(\delta - \theta_s)} \quad (5a)$$

$$\Gamma^{(1)}(\theta) = \frac{m^2 \cos^2 \theta - (\sin^2 \theta - n^2)}{m^2 \cos^2 \theta + (\sin^2 \theta - n^2)} \quad (5b)$$

$$\Gamma^{(2)}(\theta) = \frac{-2m \cos \theta}{m^2 \cos^2 \theta + (\sin^2 \theta - n^2)} \sqrt{\sin(\delta + \theta)} \quad (5c)$$

where θ_s used in the above equations is the saddle point of the integrand in (2) determined from $(d/d(\theta))q(\theta) = 0$. Now substituting (5) associated with (5a) ~ (5c) into (2) and suppressing the subscript “ y ” and the coordinate system “ (x, z) ” in the notations, one may obtain

$$U^r = U^{go} + U^s \quad (6)$$

$$U^{go} = \frac{k_1}{2\pi} \Gamma(\theta_s) \int_{P_\theta} B(\theta, \theta_b) e^{ik_1 R_1 q(\theta)} d\theta \quad (7)$$

where the scattered beam field U^s in (6) is expressed by the summation of the scattered beam integrals U^{s1} and U^{s2} as follows.

$$U^s = U^{s1} + U^{s2} \quad (8)$$

$$U^{s1} = \frac{k_1}{2\pi} \int_{P_\theta} \bar{\Gamma}^s(\theta) B(\theta, \theta_b) e^{ik_1 R_1 q(\theta)} d\theta \quad (9)$$

$$U^{s2} = \frac{k_1}{2\pi} \int_{P_\theta} \Gamma^{(2)}(\theta) \sqrt{\sin(\delta - \theta)} B(\theta, \theta_b) e^{ik_1 R_1 q(\theta)} d\theta \quad (10)$$

(a) Original integration path P_θ and branch points $\pm\delta$ associated with branch cuts.

(b) Steepest descent path SDP and integration path C_B around branch cut.

Fig. 2. Integration paths in complex θ -plane.

2.2 Novel asymptotic solutions for reflected Gaussian beam

The geometrically reflected beam integral U^{go} in (7) and the scattered beam integral U^{s1} in (9) can be evaluated asymptotically by applying the isolated saddle point technique [11, 13] for large $k_1 R_1$. While the integral U^{s2} in (10) is evaluated by deforming the integration path P_θ into the steepest descent path SDP passing through the saddle point θ_s (see Fig. 2 (b)). The branch cut contribution U^B arises when the saddle point θ_s is larger than the branch point δ . Thus U^{s2} can be represented by [11]

$$U^{s2} = U^{SDP} + u(\theta_s - \delta)U^B \quad (11)$$

The integrals for U^{SDP} and U^B are obtained from (10) by replacing the integration path P_θ by SDP and C_B , respectively, where C_B denotes the integration path around the branch cut (see Fig. 2 (b)). In (11), $u(\theta_s - \delta)$ is 1 for $\theta_s > \delta$ and 0 for $\theta_s < \delta$.

When the geometrically reflected beam solution for U^{go} in (7) and the scattered beam solution for U^s in (8) associated with the asymptotic solutions for U^{s1} in (9) and U^{s2} in (11) are substituted into U^r in (6) and the solutions U^{s1} and U^{SDP} are combined to produce the transition beam U^{tran} , one may obtain a novel uniform asymptotic solution as follows.

$$U^r = U^{go} + u(\theta_s - \delta)U^B + U^{tran} \quad (12)$$

The geometrically reflected beam U^{go} , the lateral beam U^B , and the transition beam U^{tran} are given, respectively, by

$$U^{go} = \Gamma(\theta_s) \sqrt{\frac{k_1}{2\pi R_1}} B(\theta_s, \theta_b) e^{ik_1 R_1 - i\pi/4} \quad (13)$$

$$U^B = \frac{k_1}{m} B(\delta, \theta_b) \frac{D_{-3/2}(\eta + i\eta) e^{i5\pi/8}}{\{k_1 \cos(\theta_s - \delta)\}^{3/4} e^{-i\eta^2/2}} \sqrt{\frac{2n}{\pi \cos \delta}} e^{ik_1(L_0+L) + ik_2 L_1} \quad (14)$$

$$U^{tran} = i\Gamma^{(2)}(\theta_s) \sqrt{\sin(\theta_s - \delta)} B(\theta_s, \theta_b) \left\{ \frac{D_{1/2}(\eta + i\eta)}{2^{1/4} \eta^{1/2} e^{-i\pi/8} e^{i\eta^2/2}} - 1 \right\} \cdot \sqrt{\frac{k_1}{2\pi R_1}} e^{ik_1 R_1 - i\pi/4}, \quad \eta = \sqrt{2k_1 R_1} \sin \frac{\theta_s - \delta}{2} \quad (15)$$

where $D_{-3/2}(z)$ and $D_{1/2}(z)$ are the parabolic cylinder functions [2, 9, 10, 11, 14], and k_2 is the wavenumber in the medium 2. The geometrically reflected beam U^{go} in (13) and the lateral beam U^B in (14) are shown in Fig. 1.

When the observation point $P(x, z)$ is located in the region satisfying $\theta_s < \delta$, the transition beam U^{tran} in (12) for $\theta_s < \delta$ can be obtained directly from (15) by substituting $-i\sqrt{\sin(\delta - \theta_s)}$ for $\sqrt{\sin(\theta_s - \delta)}$ and ζ for η in (15). Here ζ is given by $\zeta = \sqrt{2k_1 R_1} \sin\{(\delta - \theta_s)/2\}$.

3 Numerical results and discussions

In Fig. 3 (a) and Fig. 3 (b), we have calculated the TM type (i.e., $m = n^2$) Gaussian beams reflected on the dielectric interface for two incident beam

angles $\theta_b = 60^\circ$, which is equal to the critical angle δ of the total reflection, and $\theta_b = 60.7^\circ$, respectively. The geometrically reflected beam solution $U^{go} (= H^{go})$, the lateral beam solution $U^B (= H^B)$, and the transition beam solution $U^{tran} (= H^{tran})$ are shown, respectively, by the blue curve (—), the green curve (—), and the brown curve (—). The total Gaussian beam solution H^r calculated from (12) is shown by the red curve (—) and is compared with the reference solution ($\circ \circ \circ$: open circles) calculated numerically from the integral representation in (2).

It is clarified that the total Gaussian beam solution (—) calculated from (12) agrees excellently with the reference solution ($\circ \circ \circ$) in all regions including the transition region shown by the sign $\leftrightarrow$ in Fig. 3(a) and 3(b). Note that, in (12), the solution in the near region where $\theta_s < \delta$ is different from the solution in the far region where $\theta_s > \delta$. It is shown that the transition beam (—) plays an important role to connect two different solutions in the near region and in the far region smoothly through the transition region (see Fig. 3(a), (b) and also Fig. 1).

It is very interesting to observe that the point where the total beam field (—) takes the maximum value is shifted to the right direction (to the left direction) from the beam axis where $x_r = 0$ in Fig. 3(a) (in Fig. 3(b)). It becomes apparent that, near the critical angle of the total reflection, the Gaussian beam is displaced to either direction due to the interference between the geometrically reflected beam and the lateral beam [11].

(a) $\theta_b = 60^\circ$ (critical angle)

(b) $\theta_b = 60.7^\circ$

Fig. 3. Numerical calculations of Gaussian beams reflection for TM case ($m = n^2$). Numerical parameters: $h = 2000\lambda$, $z_b = 2000\lambda$, $W = 10\lambda$, $\varepsilon_1 = 1.333\varepsilon_0$, $\varepsilon_2 = 1.0\varepsilon_0$, $n = 0.866$, and $\delta = 60^\circ$.

4 Conclusion

We have derived the novel uniform asymptotic solution for the reflection of the Gaussian beam that is incident from a denser dielectric medium. The validity of the novel uniform asymptotic solution has been confirmed by comparing with the reference solution obtained numerically. It has been shown

that the transition beam plays an important role in the transition region. It becomes apparent that the Gaussian beam is shifted to either direction near the critical angle of the total reflection by the interference of the geometrically reflected beam and the lateral beam.

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