Relationship between $Q$ factor and complex resonant frequency: investigations using RLC series circuit

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Abstract: It is well known that $Q$ factor can be calculated from complex resonant frequency of a non-excitation problem. However, two definitions are used to obtain $Q$ factor from complex resonant frequency. One definition uses the real part of the complex frequency while the other uses an absolute value from the numerator of the $Q$ factor. The meaning and difference of the two definitions is investigated using an RLC series circuit, and the findings are presented in this article.

Keywords: Complex resonant frequency, non-excitation, $Q$ factor

Classification: Electromagnetic theory

References


1 Introduction

Q factor [1] is an important value for evaluating the quality of inductors or capacitors, designing filters [2], wireless power transfer systems [3][4], and so on. There are various expressions to define the Q factor [5][6]. One expression is based on complex resonant frequency, wherein there are two types of definitions. This study investigates the difference between the two definitions using a simple RLC series circuit model.

2 Various definitions of Q

The Q factor for a well-known RLC series circuit shown in Fig. 1(a) [2] can be obtained as

\[
Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{1}{R \sqrt{LC}}.
\]

where \( \omega_0 = 1/\sqrt{LC} \) is the resonant angular frequency. At \( \omega_0 \), the ratio \( |I/V| = |1/Z| \) holds peak value, where \( I \) and \( V \) are the current and voltage, respectively. In addition,

\[
Z = R + j\omega L + \frac{1}{j\omega C}.
\]

The second definition for the Q factor is obtained using bandwidth \( \Delta \omega = \omega - \omega_0 \), where \( \omega \) is the angular frequency such that \( |I/V| = |1/Z| \) becomes \( 1/\sqrt{2} \) against the peak value.

\[
Q = \frac{\omega_0}{\Delta \omega}.
\]

The Q factor, according to [2][3], can be defined as

\[
Q = \frac{\omega_0}{2} \left| \frac{1}{Z} \frac{dZ}{d\omega} \right|.
\]

The definition of Q factor using energy [2][7][8], which can be applied generally to any cavities, is expressed as
\[ Q = \omega_0 \frac{U}{\frac{dU}{dt}} \]  

(5)

where \( U \) is the electromagnetic energy stored in the resonator. We now introduce the complex resonant frequency,

\[ \omega_c = \omega_r + j \omega_i \]  

(6)

to take energy dissipation into account in time harmonic scenario \( (e^{j\omega t} = e^{j\omega t} e^{-\alpha t}) \). The complex resonant frequency is normally introduced in eigenmode analysis for electromagnetic cavities. Therefore, the Q factor can be expressed [2], [7] as

\[ Q = \frac{\omega_c}{2\omega_i} \]  

(7)

However, in [9][10], a different expression with Eq.(6) was also proposed.

\[ Q = \frac{\left| \omega_c \right|}{2\omega_i} \]  

(8)

The reason for the definition of Eq.(8) does not seem to be clear. The values of Eq.(7) and Eq.(8) are obviously different. The difference between Eq.(7) and Eq.(8) is discussed in the following two sections using the RLC series circuit model shown in Fig. 1.

3 Definitions of resonance in mathematical style

Fig. 1(a) shows the excitation scenario, wherein the circuit is excited by voltage source \( V \). The Q factor for this circuit can be calculated using (1), (3), and (4). Fig. 1(b) illustrates the non-excitation problem and the equation for this circuit is

\[ ZI = 0 \]  

(9)

in mathematical style. Eq.(9) is an indeterminate equation. \( Z \) must be zero so that \( I \)
exist without the voltage source. By solving $Z = 0$, $\omega_c$ can be obtained.

It is emphasized that the definition of resonance is different between Eq.(9) and Eq.(1). In (1), the real valued angular frequency $\omega_0$ is searched to maximize $|I/V|$. 

### 4 Q factor evaluation by complex resonant frequency for RLC series circuit

Substituting Eq.(2) in $Z = 0$, and solving for the complex resonant angular frequency, we get

$$\omega_c = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} + j \frac{R}{2L}$$

(10)

by choosing the appropriate solution such that $\text{Re}[\omega_c] > 0$. It is clear that $\omega_c = \omega_0 = 1/\sqrt{LC}$ when $R = 0$. Substituting Eq.(10) into Eq.(7) yields

$$Q = \frac{1}{R} \sqrt{\frac{L}{C} - \left(\frac{R}{2}\right)^2}$$

(11)

Substituting Eq.(10) into Eq.(8) yields

$$Q = \frac{1}{R} \sqrt{\frac{L}{C}}$$

(12)

which is unexpectedly identical with Eq.(1). Moreover, $|\omega_c| = 1/\sqrt{LC} = \omega_0$ using Eq.(10) can also be derived.

Complex angular frequency ($\omega_c$) and Q factor with $R$ are shown in Fig. 2 by fixing $L = 1$ H and $C = 1$ F ($\omega_0 = 1$ rad/s). In Fig. 2(a), $\omega_c$ is overlapped with $\omega_0$ when the Q factor $> 10$. In Fig. 2(b), the Q factor defined by Eq.(7) also overlaps the Q factor defined by Eq.(1), Eq.(3), Eq.(4), and Eq.(8). The difference of (real part of) resonant frequencies and Q factors are both 0.125% when Q factor by Eq.(8) is 10. The resonators are normally designed with a Q factor $> 10$, so the very small difference (0.125%) illustrated in the graphs is negligible.
5 Conclusion

The two definitions used to obtain Q factor from complex resonant frequency are presented, and the difference between the two definitions were investigated using an RLC series circuit model. It was found that the resonant angular frequency corresponds to $\omega_0 = 1/\sqrt{LC}$ incidentally when Q factor is defined using the absolute value of complex resonant frequency in the numerator. However, it was also found that the difference between the two definitions is negligible (0.125% when $Q = 10$) when Q factor is greater than 10.

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