Study on Impedance Matching and Maximum Wireless Power Transfer Efficiency of Circuits with Resonant Coupling Based on Simplified S-matrix

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Abstract In this manuscript, the simplified S-matrix of the pair of RLC circuits with resonant coupling at the resonant frequency is revealed. All of the elements of the S-matrix (S-parameters) are expressed by using essential quantities, which are the port-impedance / resistance ratios and the kQ-product. The matching condition and the maximum power transfer efficiency are analytically derived from the elements of the simplified S-matrix.

key words: S-matrix, impedance matching, kQ-product, maximum efficiency
Classification: Electromagnetic theory

1. Introduction

Wireless power transmission has a long history of over 50 years and potential for various industrial applications [1–3]. Since the research team from MIT published a paper in Science [4], the wireless power transfer (WPT) quickly became a hot research area, including energy harvesting technologies [5, 6]. Its technology, which is very attractive to the electric vehicle (EV) charging applications, is developing rapidly in recent years [7]. The condition for maximum power transfer efficiency of magnetic coupled resonance WPT system is studied [8–10] and also seen in the technical manuscript [11]. A numerical analysis has been carried out on its efficiencies focusing on optimal loads and source impedances [12], and the load impedance dependence of the efficiency is measured [13, 14]. An optimization method for output power and transmission efficiency is proposed [15]. The interesting considerations based on kQ and Q are also given to the magnetic coupled resonance WPT system [16–18]. In this manuscript, the impedance matching and the maximum power transfer efficiency of a pair of RLC circuits with resonant coupling are analytically derived based on the simplified S-matrix.

2. S-matrix of coupled RLC circuits at the resonant frequency

Figure 1 shows a pair of RLC series circuits with magnetic coupling. The RLC series circuit in the left side is connected to Port 1 of which impedance is Z₁, and the RLC series circuit in the right side is connected to Port 2 of which impedance is Z₂. The circuit in the side of Port 1 consists of a resistor of R₁, an inductor of L₁ and a capacitor of C₁, and the circuit in the side of Port 2 consists of a resistor of R₂, an inductor of L₂ and a capacitor of C₂. Both circuits include a mutual inductance of Lₘ=k√L₁L₂ between the two coils, where k is the coupling coefficient. The coefficient is essentially discussed in [19]. The impedance matrix of the entire circuit in the Fig.1 is

\[
Z = \begin{pmatrix}
R_1 + j\omega L_1 + \frac{1}{j\omega C_1} & j\omega k\sqrt{L_1L_2} \\
\frac{j\omega k\sqrt{L_1L_2}}{R_2 + j\omega L_2 + \frac{1}{j\omega C_2}} & R_2 + j\omega L_2 + \frac{1}{j\omega C_2}
\end{pmatrix},
\]

(1)

where \(\omega\) is the angular velocity of the signal. In particular, at the resonant frequency of \(f_0=\frac{2\pi}{\omega}\), the impedance matrix is

\[
Z_0 = \begin{pmatrix}
R_1 & j\omega_0 k\sqrt{L_1L_2} \\
j\omega_0 k\sqrt{L_1L_2} & R_2
\end{pmatrix},
\]

(2)

where \(\omega_0\) is the resonant angular velocity. The resonant impedance matrix \(Z_0\) implies Eqs.(2) and (3) in [20]. In general, S-matrices are derived from impedance matrices [21, 22]. By using \(Z_0\), the following S-matrix of the whole

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circuit in Fig.1 at the resonant frequency are obtained.

\[
\begin{pmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{pmatrix} = 
\begin{pmatrix}
\frac{k_2 \omega_0 L_1 L_2 (Z_1 - R_1) (Z_2 + R_2)}{k_1 \omega_0 L_1 L_2 (Z_1 + R_1) (Z_2 - R_2)} & 2 j k_n \eta \sqrt{Q_1 Q_2} \sqrt{Z_1 Z_2} \\
\frac{k_1 \omega_0 L_1 L_2 (Z_1 + R_1) (Z_2 - R_2)}{k_2 \omega_0 L_1 L_2 (Z_1 - R_1) (Z_2 + R_2)} & \frac{k_1 \omega_0 L_1 L_2 (Z_1 + R_1) (Z_2 + R_2)}{k_2 \omega_0 L_1 L_2 (Z_1 - R_1) (Z_2 - R_2)}
\end{pmatrix},
\] (3)

where \(S_{11}, S_{12}, S_{21},\) and \(S_{22}\) are called “\(S\)-parameters” at the resonant frequency.

\(\eta_{21} = |S_{21}|^2\) is the power transfer efficiency when the power goes via port 1, the circuit, and port 2 to reach load \(Z_2,\) and \(\eta_{12} = |S_{12}|^2\) is the power transfer efficiency when the power goes via port 2, the circuit, and port 1 to reach load \(Z_1.\) \(S_{21}\) is equal to \(S_{12}\) as seen in the above equation, that is, \(\eta_{21} = \eta_{12}.\)

### 3. Simplified \(S\)-matrix and power transfer efficiency

The \(S\)-matrix of the coupled \(RLC\) circuits in Fig.1 shown in the previous section is expressed as the simplified matrix as follows.

\[
\begin{pmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{pmatrix} = 
\begin{pmatrix}
(k \sqrt{Q_1 Q_2})^2 - \frac{Z_1}{R_1} \left( \frac{Z_2}{R_2} + 1 \right) & \frac{2 j k \sqrt{Q_1 Q_2} \sqrt{Z_1 Z_2}}{(k \sqrt{Q_1 Q_2})^2 + (\frac{Z_2}{R_2} + 1) (\frac{Z_1}{R_1} + 1)} \\
\frac{2 j k \sqrt{Q_1 Q_2} \sqrt{Z_1 Z_2}}{(k \sqrt{Q_1 Q_2})^2 + (\frac{Z_2}{R_2} + 1) (\frac{Z_1}{R_1} + 1)} & \frac{(k \sqrt{Q_1 Q_2})^2 - (\frac{Z_1}{R_1} + 1) (\frac{Z_2}{R_2} + 1)}{(k \sqrt{Q_1 Q_2})^2 + (\frac{Z_2}{R_2} + 1) (\frac{Z_1}{R_1} + 1)}
\end{pmatrix}
\] (4)

In the above matrix, \(Q_1 \equiv \frac{1}{\sqrt{L_1 C_1}} = \frac{\omega_0 L_1}{R_1},\) and \(Q_2 \equiv \frac{1}{\sqrt{L_2 C_2}} = \frac{\omega_0 L_2}{R_2}\) is the \(Q\) factor for \(RLC\) series circuit in the side of Port 1, and \(Q_2 \equiv \frac{1}{\sqrt{L_2 C_2}} = \frac{1}{\omega_0 L_2 R_2}\) is the \(Q\) factor for \(RLC\) series circuit in the side of Port 2. It is found that the elements of the above matrix (\(S\)-parameters) depend on only three essential quantities of \(r_1 \equiv \frac{Z_1}{R_1},\) \(r_2 \equiv \frac{Z_2}{R_2},\) and \(k_q \equiv k \sqrt{Q_1 Q_2},\) and that the power transfer efficiency \(\eta\) also depends on the three quantities as follows.

\[
\eta = \eta_{21} = |S_{21}|^2 = |S_{12}|^2 = \eta_{12} = \left( \frac{2 k \sqrt{Q_1 Q_2} \sqrt{Z_1 Z_2}}{(k \sqrt{Q_1 Q_2})^2 + (\frac{Z_2}{R_2} + 1) (\frac{Z_1}{R_1} + 1)} \right)^2
\] (5)

\(k_q \equiv k \sqrt{Q_1 Q_2},\) which is called “\(Q\)-product”, in the above equation is equal to \(\frac{k \omega_0 L_1}{\sqrt{R_1 C_1}} = \frac{k \omega_0 L_2}{\sqrt{R_2 C_2}}\), and \(k = \frac{1}{\omega_0 \sqrt{L_1 C_1} \sqrt{L_2 C_2}}\) in the circuit. \(k_q \equiv k \sqrt{Q_1 Q_2}\) is equal to \(X_M \sqrt{R_1 R_2}\), where \(X_M = \omega_0 k \sqrt{L_1 L_2} = \omega_0 L_M\) is the mutual impedance at resonant frequency. Similar formula can be seen in [23].

### 4. Impedance matching

Firstly, the reflection coefficients of \(S_{11}\) and \(S_{22}\) are discussed from the point of view of “impedance matching.”

Equation 6, which satisfies \(S_{11} = 0\) in Eq.4, is the impedance matching condition at the resonant frequency at Port1, and Eq.7, which satisfies \(S_{22} = 0\) in Eq.4, is the impedance matching condition at the resonant frequency at Port2.

\[
\left( \frac{Z_1}{R_1} - 1 \right) \left( \frac{Z_2}{R_2} + 1 \right) = (k \sqrt{Q_1 Q_2})^2
\] (6)

\[
\left( \frac{Z_1}{R_1} + 1 \right) \left( \frac{Z_2}{R_2} - 1 \right) = (k \sqrt{Q_1 Q_2})^2
\] (7)

In Fig.3, solid lines and broken lines show the relations by Eq.6 and Eq.7, respectively, when \(k \sqrt{Q_1 Q_2} = 10, 20, 30, 40,\) and \(50.\) In particular, when \(|S_{11}| = |S_{22}| = 0,\)

\[
\frac{Z_{1\text{opt}}}{R_1} = \frac{Z_{2\text{opt}}}{R_2} = \sqrt{(k \sqrt{Q_1 Q_2})^2 + 1},
\] (8)

which is derived from Eq.6 and Eq.7. The optimal values of \(Z_{1\text{opt}}/R_1, Z_{2\text{opt}}/R_2,\) which are obtained from \(k \sqrt{Q_1 Q_2}\) by using the above equation, are on the dashed-dotted line in Fig.3. The optimum impedance Eq.(8) agree with Eq.(12) in [8], Eq.(5) in [11], Eqs.(15) and (16) in [12], and Eq.(11) in [18].

### 5. Maximum power transfer efficiency

Secondly, the transmission coefficients of \(S_{12}\) and \(S_{21}\) are...
discussed from the point of view of “maximum power transfer efficiency.”

As shown bellow, \( \sqrt{\eta} \), \(|S_{12}|\) and \(|S_{21}|\) are two variable functions of \( r_1 \) and \( r_2 \).

\[
\sqrt{\eta} = |S_{12}| = |S_{21}| = \frac{2k_q \sqrt{r_1 r_2}}{k_q^2 + (r_1 + 1)(r_2 + 1)}, \tag{9}
\]

where \( r_1, r_2, \) and \( k_q \) are \( \frac{Z_1}{R_1}, \frac{Z_2}{R_2} \), and \( k \sqrt{Q_1 Q_2} \), respectively, as defined above.

The following mathematical differentiation for \( \sqrt{\eta} \) are derived from Eq.(9).

\[
\frac{\partial \sqrt{\eta}}{\partial r_1} = k_q \sqrt{r_2} \frac{k_q^2 - (r_1 - 1)(r_2 + 1)}{k_q^2 + (r_1 + 1)(r_2 + 1)} \tag{10}
\]

\[
\frac{\partial \sqrt{\eta}}{\partial r_2} = k_q \sqrt{r_1} \frac{k_q^2 - (r_1 + 1)(r_2 - 1)}{k_q^2 + (r_1 + 1)(r_2 + 1)} \tag{11}
\]

From the above differentiation, the following results are obtained.

\[
\frac{\partial \sqrt{\eta}}{\partial r_1} \bigg|_{r_1=r_2=\sqrt{k_q^2+1}} = 0 \tag{12}
\]

\[
\frac{\partial \sqrt{\eta}}{\partial r_2} \bigg|_{r_1=r_2=\sqrt{k_q^2+1}} = 0 \tag{13}
\]

When \((r_1, r_2)=(\sqrt{k_q^2+1}, \sqrt{k_q^2+1})\),

\[
\frac{\partial^2 \sqrt{\eta}}{\partial r_1^2} \bigg|_{r_1=r_2=\sqrt{k_q^2+1}} = -\frac{1}{4} \frac{k_q^2 + 1 - 1}{k_q^2 + 1} < 0, \tag{14}
\]

\[
\frac{\partial^2 \sqrt{\eta}}{\partial r_2^2} \bigg|_{r_1=r_2=\sqrt{k_q^2+1}} = -\frac{1}{4} \frac{k_q^2 + 1 - 1}{k_q^2 + 1} < 0, \tag{15}
\]

and

\[
\frac{\partial^2 \sqrt{\eta}}{\partial r_1^2} \bigg|_{r_1=r_2=\sqrt{k_q^2+1}} = \frac{1}{4} \frac{(k_q^2 + 1 - 1)^2}{k_q^2 + 1} > 0. \tag{16}
\]

Eq.(12), Eq.(13), Eq.(14), Eq.(15), and Eq.(16) mean that \( \sqrt{\eta} \) is maximum (maximal) when \( r_1 = r_2 = \sqrt{k_q^2 + 1} \).

The point of \( r_1 = r_2 = \sqrt{k_q^2 + 1} \) and \( r_1 = r_2 = \sqrt{k_q^2 + 1} \) are two variable functions of \( r_1 \) and \( r_2 \), respectively.

The point of \( r_1 = r_2 = \sqrt{k_q^2 + 1} \) is maximum, and the point defined by Eq.(8) correspond exactly. By substituting \( \sqrt{k_q^2 + 1} \) for \( r_1 \) and \( r_2 \) in Eq.(5), \( \eta_{max} \) (the maximum value of \( \eta \)) is obtained as follows.

\[
\eta_{max} = \eta \bigg|_{r_1=r_2=\sqrt{k_q^2+1}} \left( \frac{k_q}{\sqrt{k_q^2 + 1}} \right)^2 \tag{17}
\]

That is,

\[
\eta_{max} = \left( \frac{k \sqrt{Q_1 Q_2}}{\sqrt{(k \sqrt{Q_1 Q_2})^2 + 1}} \right)^2 = \left( \frac{1}{k \sqrt{Q_1 Q_2}} \right)^2 + 1 \tag{18}
\]

The maximum efficiency Eq.(18) agree with Eq.(13) in [8], Eq.(6) in [11], Eq.(20) in [12], and Eq.(14) in [18]. In reverse, \( k \sqrt{Q_1 Q_2} \) is obtained from \( \eta_{max} \) by the following equation.

\[
k \sqrt{Q_1 Q_2} = 2 \frac{\sqrt{\eta_{max}}}{1 - \eta_{max}} \tag{19}
\]

Figure 4 shows the \( k \sqrt{Q_1 Q_2} \) dependence of the maximum power transfer efficiency \( \eta_{max} \). This figure is equivalent to Fig.3 in [8].
Fig. 5. A diagram for comprehending $k\sqrt{Q_1/Q_2}$, $Z_{1\text{opt}}/R_1$, $Z_{2\text{opt}}/R_2$, $|S_{12}|_{\text{max}}$, $|S_{21}|_{\text{max}}$, and $n_{\text{max}}$.

6. Conclusion

To conclude, the simplified $S$-matrix of a pair of $RLC$ circuits with resonant coupling at the resonant frequency, which depends on three essential quantities ($Z_1/R_1, Z_2/R_2, k\sqrt{Q_1/Q_2}$), is revealed, and the matching condition of $Z_{1\text{opt}}/R_1 = Z_{2\text{opt}}/R_2 = \sqrt{(k\sqrt{Q_1/Q_2})^2 + 1}$ and the maximum power transfer efficiency of $\left(\frac{k\sqrt{Q_1/Q_2}}{\sqrt{(k\sqrt{Q_1/Q_2})^2 + 1} + 1}\right)^2$ are analytically derived from the $S$-matrix.

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