Algebraic Independence of Painlevé First Transcendents

By

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Abstract. It will be proved that Painlevé first transcendents and their first derivatives are algebraically independent over the rational function field with complex coefficients, by the use of the irreducibility. A particular case indicates that the group of differential automorphisms of the differential field generated by a Painlevé first transcendent over the rational function field is trivial.

Key Words and Phrases. Painlevé first transcendent, Differential automorphism, Irreducibility.

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1. Introduction

Let $K$ be an ordinary differential field of characteristic 0 with the field of constants $C$, the complex number field, containing an element $x$ with $x' = 1$ and $U$ be a universal differential field extension of $K$. An element $y \in U$ is called a Painlevé first transcendent, abbr. PI, if it satisfies the differential equation over the rational function field $C(x)$

$$y'' = 6y^2 + x.$$ 

The “irreducibility” property of PI’s is well-known (cf. [N], [U]): If $y$ satisfies an algebraic differential equation of the first order over $K$, then it is algebraic over $K$. We here use the term “irreducibility” in this meaning. The proof of this is essentially due to investigation of a weight function $w$ of $K[y, y']$ defined by

$$w(y) = 2, \quad w(y') = 3, \quad w(a) = 0 \quad \text{(for all } a \in K \backslash \{0\}).$$

together with a $K$-derivation $X$ of $K[y, y']$ defined by

$$X = y' \frac{\partial}{\partial y} + 6y^2 \frac{\partial}{\partial y'},$$

and a distinguished polynomial $\gamma = y'^2 - 4y^3 \in K[y, y']$. This method applies to the proof of the following.
Lemma. If \( y \) is a PI and transcendental over \( K \) then there exists no \( K \)-differential automorphism of the differential field extension \( K(y, y') \) of \( K \) other than the unit element.

From this results the following.

Theorem 1. If \( y_i \) \( (1 \leq i \leq n) \) are distinct PIs and each of them is transcendental over \( K \), then \( y_i, y'_i \) \( (1 \leq i \leq n) \) are algebraically independent over \( K \).

For example if as \( K \) we take a differential field extension of \( \mathbb{C}(x) \) generated with solutions of linear differential equations over \( \mathbb{C}(x) \), every PI is transcendental over \( K \) (cf. [N], [U]).

As for the relation between Painlevé’s first and second equations, the following indicates their independence.

Theorem 2. Let \( y \) be a PI over \( K \) and \( z \) satisfy the Painlevé’s second equation \( z'' = 2z^3 + xz + \alpha \) \( (\alpha \in \mathbb{C}) \) over \( K \). Suppose that \( y \) and \( z \) are transcendental over \( K \) and no irreducible polynomial in \( K[y, y'] \) or \( K[z, z'] \) divides its derivative. Then \( y, y', z, z' \) are algebraically independent over \( K \).

We here wish to note most equalities were calculated by the mathematical software Mathematica.

Section 2 is devoted to describe some properties of the differential algebra associated with a PI over \( K \), which will be used in the proofs of Theorem 1 and 2, and verify the proof of the “irreducibility” of PIs. In sections 3 and 4 the proof of Lemma will be given. Section 5 and 6 will conclude the proofs of Theorems 1 and 2 respectively.

2. Differential algebra \( K[y, y'] \)

Let \( y \) be a PI which is transcendental over \( K \). The polynomial algebra with coefficients from \( K, K[y, y'] \), is also treated as a differential algebra.

The weight function \( w \) of \( K[y, y'] \) over \( K \) which we frequently use satisfies the properties.

1) \( w(y) = 2, \ w(y') = 3 \).
2) \( w(u + v) \leq \max\{w(u), w(v)\} \), where the equality holds if \( w(u) \neq w(v) \).
3) \( w(uv) = w(u) + w(v) \).
4) \( w(u') \leq w(u) + 1 \).

For any nonnegative integer \( n \) we denote by \( V_n \) the \( K \)-vector space generated with power products \( y^i y'^j \) \( (2i + 3j = n) \). Clearly \( V_0 = K, \ V_1 = 0 \).

Each \( u \in K[y, y'] \) is described as a unique sum of polynomials in \( V_n \) \( (n \geq 0) \): \( u = \sum_{n \geq 0} u_n \), \( u_n \) will be called the \( n \)-component of \( u \). For nonzero \( u \) the \((w(u) + 1)\)-component of \( u' \) agrees with \( Xu_{w(u)} \), where \( X = y'(\partial/\partial y) + \)
$6y^2(\partial/\partial y') \in \text{Der}(K[y, y']/K)$ (the set of all derivations of $K[y, y']$ to itself over $K$), which satisfies $XV_n \subset V_{n+1}$.

**Proposition 1.** Suppose $n \geq 2$ and $u \in V_n$, $u(1, 2) \neq 0$. Then for any $s > 0$ we have

$$X^s u \in V_{n+s}, X^s u(1, 2) = n(n + 1) \ldots (n + s - 1)u(1, 2) \neq 0.$$  

**Proof.** For power product $y^i y^j$, $2i + 3j = n$, $Xu = iy^{i-1}y^{j+1} + 6y^{i+2}y^j - 1$ and $Xu(1, 2) = i2^{i+1} + 6j2^{j-1} = nu(1, 2)$. Therefore for $u \in V_n$, $Xu(1, 2) = nu(1, 2)$. In particular $Xu \in V_{n+1}$. Assume the assertion holds in case of $s - 1$ ($s > 1$). Then $X^{s-1}u \in V_{n+s-1}$ and $X^{s-1}u(1, 2) = n(n + 1) \ldots (n + s - 2) \cdot u(1, 2) \neq 0$. Applying this for $Xu$,

$$X^s u(1, 2) = (n + 1) \ldots (n + s - 1) Xu(1, 2) = n(n + 1) \ldots (n + s - 1) u(1, 2).$$

A polynomial $\gamma = y^2 - 4y^3$ occupies a specific position in the theory of PI’s. Clearly it satisfies $\gamma(1, 2) = 0$ and $X\gamma = 0$. If $u \in V_n$ satisfies $u(1, 2) = 0$, then $\gamma$ divides $u$. For we may describe $u = f\gamma + gy^e + h$, for some $f \in K[y, y']$, $g, h \in K[y]$. Since $u \in V_n$, $g$ or $h$ must be 0. Evaluating at $(1, 2)$ we have $2g(1) + h(1) = 0$ therefore $g = h = 0$.

**Proposition 2.** For $u \in V_m$ the following hold.

1) If $Xu = a\gamma^n$ or $a\gamma^n (a \in K, n \geq 0)$, then $u = b\gamma^r$ ($b \in K, r \geq 0$) and $a = 0$. In particular $w(u) \equiv 0 \mod 6$.

2) If $X^2 u = 12y + a\gamma^n$ or $12y + a\gamma^n (a \in K, n \geq 0)$, then $u = cy^r\gamma$ or $c\gamma^r$ ($c \in K, r \geq 0$). In particular $w(u) \equiv 0, 3 \mod 6$.

3) If $X^2 u = a\gamma^n (a \in K, n \geq 0)$, then $u = b\gamma^r$ ($b \in K, r \geq 0$) and $a = 0$. In particular $w(u) \equiv 0 \mod 6$.

**Proof.** 1) Assume $u = b\gamma^r$ ($r \geq 0, b(1, 2) \neq 0$) and $b \notin K$. Then, $Xb = a\gamma^{n-r}$. Since if $n > r$, $Xb(1, 2) = 0$, which is impossible by Proposition 1, it is derived that $n = r$, $Xb = a \in K$, and hence that $b \in K, a = 0$. The proof in case $Xu = a\gamma^n$ is similar.

2) Assume $u$ has the same form as in 1). In the first case, $X^2 b = 12yb + a\gamma^{n-r}$. If $n > r$ or $a = 0$, $X^2 b(1, 2) = 12b(1, 2)$ and hence $w(b)(w(b) + 1) = 12$, $w(b) = 3$. This implies $a = 0$. Since we also have $a = 0$ even if $n = r$, $X^2 b = 12yb$, $w(b) = 3$. In the second case, $X^2 b = 12yb + a\gamma^{n-r}$. If $n > r$, we have $a = 0$, and the same result. If $n = r$, $X^2 b = 12yb + ay$ implies $12b + a = 0$.

3) Let $u = b\gamma^r$ as above. Then $X^2 b = a\gamma^n\gamma^r$. If $n > r$ or $a = 0$, $X^2 b(1, 2) = 0$, therefore $b \in K$. Assume $n = r$ and $a \neq 0$. Then $X^2 b = ay^r$, $w(b) = 1$, which is absurd.

As an application of Proposition 2 we include the proof of the fact mentioned in the introduction.
Proposition 3. If a nonzero polynomial \( u \in K[y, y'] \) divides its derivative \( u' \), then \( u \in K \).

Proof. Let \( n = w(u) \) and \( u_n \) be the \( n \)-component of \( u \). Since \( w(u'/u) \leq 1 \), \( c = u'/u \in K \). Looking at the \((n+1)\)-components of the both sides of \( u' = cu \), we find \( Xu_n = 0 \). Hence \( u_n = a v^s \) \((s \geq 0, a \in K \setminus \{0\})\) and \( n = 6s \). Put \( v = u - av^s \). Then

\[
v' = cv + (ac - a')v^s - 2saxy'y^{s-1}.
\]
Let \( m = w(v) \) and \( v_m \) the \( m \)-component of \( v \). If \( m = n - 1 > 4 \)

\[
Xv_m = (ac - a')v^s;
\]
hence, \( m \) is divided by \( 6 \), which is absurd. If \( 6s - 5 < m < 6s - 1 \) then \( a' = ac \), \( m = 6s - 4 \), \( Xv_m = -2saxy'y^{s-1} \).

It follows \( v_m = -2saxy'y^{s-1} \), from which

\[
v' = cv - 2saxy'y^{s-1}, \quad v'_m = cv_m - 2sa(xy' + y)y^{s-1} - 4s(s - 1)ax^2yy'y^{s-2},
\]
and hence

\[
Xv_{m-1} = 2saxy'y^{s-1}.
\]
We have \( a = 0 \), a contradiction.

3. Differential automorphism

We here suppose that \( y \) is a PI and transcendental over \( K \) and consider the differential field \( K(y, y') \). This and next sections are devoted to the proof of Lemma.

We denote by \( O_p \) the local ring at the prime ideal generated by an irreducible polynomial \( p \in K[y, y'] \setminus K \) and \( M_p = pO_p \) its maximal ideal. \( O_p \) is a differential \( K \)-algebra. Every nonzero \( u \in K(y, y') \) can be written in a unique shape \( p^s v \) with \( s \) integer and \( v \in O_p \setminus pO_p \). Defining \( v_p(u) = s \), we have a discrete valuation \( v_p \) of \( K(y, y') \). Since \( p' \) is not divisible by \( p \), \( v_p(u') = v_p(u) - 1 \), provided \( v_p(u) \neq 0 \). In fact, if we let \( u = p^s v \) \((s \neq 0, v \in O_p \setminus M_p)\), we have \( u' = ps^{-1}(sp'v + pv') \) and \( sp'v + pv' \in O_p \setminus M_p \).

Now suppose the converse of Lemma, namely, there exists a \( K \)-differential automorphism \( \sigma \) distinct from the unit element, and let \( z = \sigma y \).

Assume that \( z \in K[y, y'] \). Examining the weights of the both sides of \( z'' = 6z^2 + x \), we have \( w(z) \leq 2 \), which implies \( z = y \), a contradiction. Thus \( v_p(z) < 0 \) for some irreducible polynomial \( p \). Since \( v_p(z'') = v_p(z) - 2 \), \( v_p(z) = -2 \). We
may therefore write as $z = fg^{-2}$, where $f, g$ have no common divisor and $g$ has no multiple factor. The substitution implies
\[
g(-4f'g' + gf'' - 2fg'' - xg^3) = 6f(f - g'^2).
\]
We know that $f - g'^2$ is divisible by $g$, namely, $f = g'^2 + gh$ for some $h \in K[y, y']$. Calculating $g^3(z'' - 6z^2 - x) = 0$, we obtain
\[
g(-xg^2 - 6h^2 - 2gh' - h^g + 2g'' + gh'' + 2g'g^{(3)}) = 10g'^2(h + g'').
\]
It is found that $g, g'$ have no common divisor. In fact, otherwise let $p$ be a common divisor of $g, g'$ and $g = pu$ for $u \in K[y, y']$ with $u$ indivisible by $p$. Then $g' = p'u + pu'$ and hence $p'u$ is divisible by $p$, which contradicts that $p'$ is indivisible by $p$. It follows $h = -g'' + gk$ for some $k \in K[y, y']$, and so
\[
z = -t' + k, \quad t = g'g^{-1}.
\]
To investigate $k$ we extend the weight function $w$ to $K(y, y')$ in a usual manner, namely,
\[
w(uv^{-1}) = w(u) - w(v), \quad (u, v \in K[y, y'], v \neq 0).
\]
The same properties 1)–4) in section 2 are valid. For $z$ it follows $w(z) \leq 2$ since if $w(z) > 0$ then $w(z) + 2 \geq w(z'' - 2w(z) + 2 = 2w(z)$. Hence $w(k) \leq \max\{w(z), w(t')\} \leq 2$. $k \in K[y, y']$ derives a description $k = ay + b$ $(a, b \in K)$.

Letting newly $t = (azg)'(azg)^{-1}$ with $(z'z^{-1})' = -b$ and adopting $K(z, z')$ as new $K$, we may assume $b = 0$. This is guaranteed because $y, y'$ still remain algebraically independent over $K(z, z')$ (cf. [N], [U]). Now we know
\[
z = -t' + ay \quad (t = g'g^{-1}, a \in K, g \in K[y, y'] \setminus K).
\]

**Proposition 4.** Suppose $u, v \in K(y, y')$, $a, b \in K$, $v \neq 0$ satisfy $u' + avv^{-1} + by = 0$. Then $a = b = 0$, $u \in C$.

**Proof.** Let $p$ be an irreducible polynomial in $K[y, y']$ with $v_p(v) \neq 0$. Since $v_p(u') \leq -2$ or $v_p(u') \geq 0$, and $v_p(v'v^{-1}) = -1$, it follows $a = 0$ and hence $u' + by = 0$, which implies $u \in K[y, y']$. If $b \neq 0$, then $w(u) \leq 1$, therefore $u \in K$, a contradiction. Thus $b = 0$ and $u' = 0$. It is known that the latter shows $u \in C$.

By changing the roles of $y$ and $z$, we have $y = -t'_1 + a_1z$, where $t_1 = g'_1g^{-1}_1$, $g_1 \in K[z, z']$ and $a_1 \in K$. This derives
\[
y = -t'_1 + a_1(-t' + ay) = -(t_1 + a_1t)' + a'_1g'g^{-1} + a_1by;
\]
hence $a'_1 = 0, aa_1 = 1$, $t_1 + a_1t \in C$ by Proposition 3. Considering an irreduc-
ible factor $p$ of $g$ and coefficients of $p'p^{-1}$ in $t$ and $t_1$, we see $a_1$, similarly $a$, is an integer, hence $a = \pm 1$. We conclude here for the present

$$z = -t' \pm y.$$

4. Two cases

Case $z = -t' + y$.

Let $S(g) = g^2((-t' + y)' - 6(-t' + y)^2 - x)$. Then

$$S(g) = -3g''^2 + 4g'g^{(3)} - gg^{(4)} - 12g'^2y + 12gg''y,$$

which must vanish. Note $w(S(\zeta)) \leq 2w(\zeta) + 4$ for $\zeta \in K[y, y']$, $\neq 0$. Define

$$B(\zeta, \eta) = S(\zeta + \eta) - S(\zeta) - S(\eta) (\zeta, \eta \in K[y, y']),$$

namely

$$B(\zeta, \eta) = -\zeta(\eta(4) - 12\eta''y) + 4\zeta''y(3) - 6\zeta''\eta''$$

$$+ 4(\zeta''(3) - 6y\zeta''y) + (-\zeta(4) + 12y\zeta''y).$$

Since $w(X^4u - 12yX^2u) = w(u) + 4$ for $u \in V_m$, $m > 0$ by Proposition 2,

$$B(\zeta, \eta) = -\zeta(\eta(4) - 12\eta''y) + \text{(terms of weight lower than } w(\zeta) + w(\eta) + 4)$$

if $w(\zeta) \geq w(\zeta')$, $w(\eta) > 0$.

Let $n = w(g)$ and $g_n$ be the $n$-component of $g$. $g_n$ satisfies the following.

$$-3(X^2g_n)^2 + 4X(g_n)X^3(g_n) - g_n X^4(g_n) - 12(Xg_n)^2y + 12g_nX^2(g_n)y = 0.$$

Set $g_n = a\gamma^r$ with $a \in V_m$, $a(1, 2) \neq 0$ as usual. Then $a$ satisfies the same equation as $g_n$. If $m > 0$, evaluating at $(1, 2)$, we have $-6m(m - 1)a(1, 2)^2 = 0$, which is impossible. Thus $a \in K$ and $n = 6r$.

Substitution

$$g = a\gamma^r + h, \quad h \in K[y, y'], \quad w(h) < 6r,$$

implies

$$S(a^r\gamma^r) = (-12a'^2y + 12a''y - 3a''y - 4a' a(3) - 3a(4) + 2r + \cdots,$$

$$B(a^r\gamma^r, h) = a(-h(4) + 12h''y) + \cdots.$$

Let $m = w(h)$ and $h_m$ the $m$-component of $h$. We have readily $m \leq n - 2$.

Assume $m = n - 2$. Looking at the $(2n + 2)$-component of $S(g)$,

$$X^4h_m = 12yX^2h_m - 12(a'^2 - a''a)a^{-1}y\gamma^r.$$

It follows $m$ is 0, 3 modulo 6, which is a contradiction.

By $w(B(a^r\gamma^r)) = m + n + 4$ we obtain $m < n - 3$. 
Let us examine the case where \( m = n - 4 \). In this case we see \(-a^2 + a a'' = 0\), \( c = a' a^{-1} \in \mathbb{C} \), and
\[
a(-X^4 h + 12 X^2 h y) \gamma^r - 24 a^2 r x y^2 \gamma^{2r-1} = 0.
\]
Suppose \( r = 1 \) and \( m = 2 \). We find \( h = a x + b \) (\( x, b \in K \)). For \( h_2 \)
\[
X^4 h_2 - 12 y X^2 h_2 + 24 a x y^2 = 0.
\]
From \( h_2 = a x \) it follows \( x = -2 a x \). By \( g = a y' - 2 a x y + b \),
\[
S(g) = 12 a (b'' - 2 b' c + b c^2) y y'^2 + a(-10 a - b(4) + 4 b(3) c + 6 b'' c^2 + 4 b' c^3 - b c^4) y'^2
\]
+ polynomial linear in \( y' \).

Therefore
\[
b'' - 2 b' c + b c^2 = 0, \quad -10 a - b(4) + 4 b(3) c - 6 b'' c^2 + 4 b' c^3 - b c^4 = 0.
\]
The first equality reduces the second one into \(-10 a = 0\), which is absurd.

Thus \( r > 1 \), then
\[
X^4 h_m = 12 y X^2 h_m - 24 a r x y^2 \gamma^{r-1}.
\]
If we put \( h_m = e \gamma^s \) (\( s \geq 0, e(1, 2) \neq 0 \)), \( X^4 e = 12 y X^2 e - 24 a r x y^2 \gamma^{r-s-1} \). Assume \( r - s > 1 \). Then \( e \in K \) as before, which is absurd. Hence \( r - s = 1 \) and \( X^4 e = 12 y X^2 e - 24 a r x y^2 \), thereby \( e = -2 a r x y \).

By substitution \( g = a \gamma' - 2 a r x y \gamma^{r-1} + h \), we have
\[
S(a \gamma' - 2 a r x y \gamma^{r-1}, h) = 384 a^2 (r(1) - 1) \gamma^s (r - 2) x y^5 (-y'^2 + 2 y^3) \gamma^{2r-3} + \cdots.
\]
\[
B(a \gamma' - 2 a r x y \gamma^{r-1}, h) = a(-h(4) + 12 y h'') y' + \cdots.
\]
If \( r > 3 \), \( X^4 h_m - 12 y X^2 h_m \) is divided by \( \gamma \), where \( h_m \) denotes the \( m(= w(h)) \)-component of \( h \). But this is impossible. If \( r = 3 \), \( X^4 h_m - 12 y X^2 h_m = 2304 a x y^5 (-y'^2 + 2 y^3) \), which has no solution. Hence \( r = 2 \).

Then
\[
S(a \gamma^2 - 4 a x y \gamma + h) = a(-h(4) + 12 h'' y) y^2 + 96 a^2 x^2 y (4 y'^2 + 9 y^3) y^2 + \cdots,
\]
which yields \( w(h) = 6 \). But there is no solution \( u \in V_6 \) with \( X^4 u - 12 y X^2 u = 96 a x^2 y (4 y'^2 + 9 y^3) \), which completes the proof in case \( z = -t' - y \).

**Case** \( z = -t' - y \).

Let again \( S = g^2 ((-t' - y)'' - 6(-t' - y)^2) - x) \). Then
\[
S = -3 g''^2 + 4 g' g(3) - 6 g''(4) + 12 g'^2 y - 12 g'' y - 2 g^2 x - 12 g^2 y^2.
\]
Let \( n = w(g) \) and \( g_n \) be the \( n \)-component of \( g \). \( g_n \) satisfies the following.

\[
-3(X^2g_n)^2 + 4X(g_n)X^3(g_n) - g_nX^4(g_n) + 12(Xg_n)^2y
- 12X(g_n)X^2(g_n)y - 12g_n^2y^2 = 0.
\]

Set \( g_n = a_j^{r_j} \) with \( a \in V_m, a(1,2) \neq 0 \) as usual. We have \(-6(m+1)(m+2) \cdot a(1,2)^2 = 0\), this is impossible, completing the proof of Lemma.

5. Proof of Theorem 1

The Theorem for \( n = 1 \) is clearly valid. Suppose that the Theorem holds for \( n - 1 \). Let \( y_i \) (\( 1 \leq i \leq n \)) be distinct PI’s and transcendental over \( K \). We shall deduce a contradiction under the assumption that \( y_i, y_i' \) (\( 1 \leq i \leq n \)) are algebraically dependent over \( K \). Let \( L \) denote the algebraic closure of differential extension field \( K(y_1, y_1', \ldots, y_{n-2}, y_{n-2}') \) in \( U \) and set \( y = y_{n-1}, z = y_n \) for simplicity. Then, by our assumption, \( y, y', z, z' \) depend algebraically over \( L \), which indicates \( z \) is algebraic over \( L(y, y') \). We shall show \( z \in L(y, y') \).

Regarding \( z \) as algebraic over \( L_1(y) \), where \( L_1 \) denotes the algebraic closure of \( L(y') \) in \( U \), we have expansions in a local parameter \( t \) at \( z \in L_1 \)

\[
y = x + t^\epsilon, \quad z = \sum_{i=r}^{\infty} a_it^i \quad (a_i \in L_0, a_r \neq 0)
\]

with \( \epsilon \) the ramification exponent. Differentiation of the first yields

\[
et^{-1}t' = y' - x^* - x_{y'}(6x^2 + x) = y' - x^* - x_{y'}(6x^2 + x) - 12x_{y'}t^\epsilon - 6x_{y'}t^{2\epsilon}.
\]

Here "*" indicates the extension of the derivation of \( L[y'] \) defined by

\[
(\sum c_i y_i')^* = \sum c_i' y_i',
\]

and \( x_{y'} = \partial / \partial y' \) the derivation with respect to \( y' \).

Assume that

\[
\beta = y' - x^* - x_{y'}(6x^2 + x) = 0.
\]

Let \( F \in L[y, y'] \) be an irreducible polynomial with \( F(x, y') = 0 \). Then

\[
F^*(x, y') + x^*F_y(x, y') = 0 \quad \text{and} \quad F_{y'}(x, y') + x_{y'}F_y(x, y') = 0.
\]

Accordingly

\[
y'F_y(x, y') + F^*(x, y') + (6x^2 + x)F_{y'}(x, y') = 0,
\]

which implies \( F'(x, y') = 0 \), and hence, \( F \) divides \( F' \), noting \( y, y' \) are algebraically independent over \( L \). This is absurd, therefore we have \( \beta \neq 0 \) and so

\[
t' = e^{-1}\beta t^{1-\epsilon} + \cdots.
\]

If \( r < 0 \), then \( r = -2\epsilon, a_r = \beta^2 \). Now, let \( j \) be the least index indivisible by \( \epsilon \) with \( a_j \neq 0 \). Equating the coefficients of \( t^{j-2\epsilon} \) in \( z'' \) and \( 6z^2 + x \).
6. Proof of Theorem 2

Suppose conversely that \( y, y', z, z' \) are algebraically dependent over \( K \). We assume further \( K \) is algebraically closed. Then \( y, y' \) is algebraically dependent over \( K(z, z') \) and \( z, z' \) over \( K(y, y') \). By the irreducibility \( y \) is algebraic over \( K(y, y') \), and \( z \) over \( K(z, z') \). Using the same argument as in the preceding section, we have \( K(y, y') = K(z, z') \). As usual we adopt the weight function \( w \) of \( K(y, y') \) defined by \( w(y) = 2, w(y') = 3, w(a) = 0 \) for \( a \in K \). Investigating the weight in the equation of PII, we know \( w(z) \leq 1 \). In fact if \( w(z) \geq 2 \),

\[
w(z) + 2 \leq w(z'') = w(2z^3 + xz + x) = 3w(z),
\]

which is absurd. Let \( p \in K[y, y'] \) be any irreducible divisor of \( z \). Since \( p' \) is not divisable by \( p \), it follows \( v_p(p') = 0 \) and \( v_p(f') = v_p(f) - 1 \) holds for \( f \in K(y, y') \) with \( v_p(f) < 0 \). Let \( v_p(z) < 0 \). Then

\[
v_p(z) - 2 = v_p(z'') = v_p(2z^3 + xz + x) = 3v_p(z),
\]

which derives \( v_p(z) = -1 \). We can describe as \( z = fp^{-1} \) with \( f \in K(y, y') \). Putting it into equation of PII,

\[
f''p^{-1} - 2f'p'p^{-2} + f(-p''p^{-2} + 2p'^2p^{-3}) = 2f^3p^{-3} + xfp^{-1} + x.
\]

From this

\[
f''p^2 - 2f''p'p - fp''p + 2fp'^2 = 2f^3 + xfp^2 + xp^3,
\]

and hence \( p \) divides \( 2f(f^2 - p'^2) \), namely, \( v_p(f - \varepsilon(p)p') > 0 \), where \( \varepsilon(p) \) indicates \( \pm 1 \) depending on \( p \). Thus \( v_p(z - \varepsilon(p)p'p^{-1}) \geq 0 \). On letting \( h = IIp^{\varepsilon(p)} \), we find

\[
v_p(z - h'h^{-1}) \geq 0,
\]

and hence \( z - h'h^{-1} \in K[y, y'] \). Viewing \( w(z - h'h^{-1}) \leq \max\{w(z), w(h'h^{-1})\} \)
\[ \leq 1, \text{ we see } z - h'h^{-1} = a \in K. \text{ The } h \text{ as an element of } K(z, z') \text{ satisfies } h' = (z - a)h, \text{ which contradicts, however, the irreducibility of } PII. \]

References


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