Local Solvability of Operators with Principal Symbol $\xi_1^2 + \cdots + \xi_{n-1}^2 + x_n^2 \xi_n^2$ in the Spaces of Distributions and Ultradistributions

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Abstract. In this paper we shall illustrate how to study local solvability in the spaces of distributions and ultradistributions through the studies on operators of a special form. We shall give necessary conditions and sufficient conditions for these operators.

Key Words and Phrases. Local solvability, Solvability in the spaces of distributions and ultradistributions.

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1. Introduction

Local solvability has been studied by many authors in the spaces of distributions and ultradistributions. In the framework of distributions Hörmander [6] gave a necessary condition of local solvability, that is, for a differential operator P he proved that the transposed operator tP of P satisfies some estimates if P is locally solvable. Conversely, Treves [18] and Yoshikawa [21] proved that the same type of estimates implies that P is locally solvable. In the frameworks of ultradistributions and hyperfunctions the corresponding treatment is possible (see [4], [1], [3] and [19]).

In this paper we shall investigate local solvability of pseudodifferential operators with principal symbol $\xi_1^2 + \cdots + \xi_{n-1}^2 + x_n^2 \xi_n^2$ in the spaces of distributions and ultradistributions. In [19] we proved that these operators are locally solvable in the space of hyperfunctions (see, also, Funakoshi [5]). Our purpose is to illustrate, with these examples, how to study local solvability in the spaces of distributions and ultradistributions.

Let us first define Gevrey classes and symbol classes. Let K be a regular compact set in $\mathbf{R}^n$, and let $\kappa > 1$ and $h > 0$. We denote by $\mathcal{E}^{\{\kappa\},h}(K)$ the space of all $f \in C^\infty(K)$ satisfying, with some $C > 0$,

$$(1.1) \quad |D^\alpha f(x)| \leq Ch^{|\alpha|} |\alpha|!^\kappa \quad \text{for any } x \in K \text{ and } \alpha \in (\mathbf{Z}_+)^n,$$

where $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, $D = i^{-1} \partial = i^{-1} (\partial / \partial x_1, \dots, \partial / \partial x_n)$, $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$

and $|\alpha| = \sum_{j=1}^n \alpha_j$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbf{Z}_+)^n$. We also denote by $\mathcal{D}_K^{\{\kappa\}, h}$ the space of all $f \in C^\infty(\mathbf{R}^n)$ with $\text{supp } f \subset K$ satisfying (1.1). $\mathcal{E}^{\{\kappa\}, h}(K)$ and $\mathcal{D}_K^{\{\kappa\}, h}$ are Banach spaces under the norm defined by

$$|f; \mathcal{E}^{\{\kappa\}, h}(K)| = \sup_{x \in K, \alpha \in (\mathbf{Z}_+)^n} |D^\alpha f(x)| / (h^{|\alpha|} |\alpha|!^\kappa).$$

Let Ω be an open subset of $\mathbf{R}^n$. We introduce the following locally convex spaces (Gevrey classes):

$$\begin{aligned} \mathcal{E}^{(\kappa)}(\Omega) &= \varprojlim_{K \Subset \Omega} \mathcal{E}^{(\kappa)}(K), & \mathcal{E}^{(\kappa)}(K) &= \varprojlim_{h \rightarrow 0} \mathcal{E}^{\{\kappa\}, h}(K), \\ \mathcal{E}^{\{\kappa\}}(\Omega) &= \varprojlim_{K \Subset \Omega} \mathcal{E}^{\{\kappa\}}(K), & \mathcal{E}^{\{\kappa\}}(K) &= \varprojlim_{h \rightarrow \infty} \mathcal{E}^{\{\kappa\}, h}(K), \\ \mathcal{D}^{(\kappa)}(\Omega) &= \varinjlim_{K \Subset \Omega} \mathcal{D}_K^{(\kappa)}, & \mathcal{D}_K^{(\kappa)} &= \varinjlim_{h \rightarrow 0} \mathcal{D}_K^{\{\kappa\}, h}, \\ \mathcal{D}^{\{\kappa\}}(\Omega) &= \varinjlim_{K \Subset \Omega} \mathcal{D}_K^{\{\kappa\}}, & \mathcal{D}_K^{\{\kappa\}} &= \varinjlim_{h \rightarrow \infty} \mathcal{D}_K^{\{\kappa\}, h}, \end{aligned}$$

where $A \Subset B$ means that the closure $\bar{A}$ of A is compact and included in the interior $\overset{\circ}{B}$ of B . We denote by $\mathcal{D}^{*f}(\Omega)$ and $\mathcal{E}^{*f}(\Omega)$ the strong dual spaces of $\mathcal{D}^*(\Omega)$ and $\mathcal{E}^*(\Omega)$, respectively, where $*$ denotes (κ) or $\{\kappa\}$. Elements of these spaces are called ultradistributions (see, *e.g.*, [12]). We also write $\mathcal{E}^*, \dots$, instead of $\mathcal{E}^*(\mathbf{R}^n), \dots$. Let us define symbol classes $S_*^{m; \delta}$, where $m, \delta \in \mathbf{R}$. We say that a symbol $p(x, \xi)$ belongs to $S_{(\kappa)}^{m; \delta}$ (resp. $S_{\{\kappa\}}^{m; \delta}$) if $p(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and for any A there is $C \equiv C_A > 0$ (resp. there are $A > 0$ and $C > 0$) such that

$$(1.2) \quad |p_{(\beta)}^{(\alpha)}(x, \xi)| \leq C A^{|\alpha|+|\beta|} (|\alpha| + |\beta|)!^\kappa \langle \xi \rangle^{m-|\alpha|} e^{\delta \langle \xi \rangle}$$

for any $x, \xi \in \mathbf{R}^n$ and $\alpha, \beta \in (\mathbf{Z}_+)^n$, where $p_{(\beta)}^{(\alpha)}(x, \xi) = \partial_\xi^\alpha \partial_x^\beta p(x, \xi)$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. We define

$$S_{(\kappa)}^{0; \infty} = \bigcup_{\delta > 0} S_{(\kappa)}^{0; \delta}, \quad S_{\{\kappa\}}^+ = \bigcap_{\delta > 0} S_{\{\kappa\}}^{0; \delta}.$$

We also use the usual symbol classes $S_{\rho, \delta}^m$, where $0 \leq \rho, \delta \leq 1$ and $m \in \mathbf{R}$. We say that $p(x, \xi) \in S_{\rho, \delta}^m$ if $p(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and there are positive constants $C_{\alpha, \beta}$ ($\alpha, \beta \in (\mathbf{Z}_+)^n$) such that

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|} \quad \text{for any } x, \xi \in \mathbf{R}^n \text{ and } \alpha, \beta \in (\mathbf{Z}_+)^n.$$

Next we shall define the Fourier transformation and pseudodifferential operators in the space of ultradistributions. Let $\kappa > 1$ and $\varepsilon \in \mathbf{R}$, and define

$$\hat{\mathcal{S}}_{\kappa, \varepsilon} = \{v(\xi) \in C^\infty(\mathbf{R}^n); \exp[\varepsilon \langle \xi \rangle^{1/\kappa}] v(\xi) \in \mathcal{S}\},$$

where $\mathcal{S}$ denotes the Schwartz space. We introduce the topology in $\hat{\mathcal{S}}_{\kappa,\varepsilon}$ so that the mapping $\hat{\mathcal{S}}_{\kappa,\varepsilon} \ni v(\xi) \mapsto \exp[\varepsilon\langle\xi\rangle^{1/\kappa}]v(\xi) \in \mathcal{S}$ is a homeomorphism. Since $\mathcal{D} (= C_0^\infty(\mathbf{R}^n))$ is dense in $\hat{\mathcal{S}}_{\kappa,\varepsilon}$, the dual space $\hat{\mathcal{S}}'_{\kappa,\varepsilon}$ of $\hat{\mathcal{S}}_{\kappa,\varepsilon}$ is identified with $\{\exp[\varepsilon\langle\xi\rangle^{1/\kappa}]v(\xi) \in \mathcal{D}'; v \in \mathcal{S}'\}$. We denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transformation and the inverse Fourier transformation on $\mathcal{S}$ (or $\mathcal{S}'$), respectively. For $u \in \mathcal{S}$ the Fourier transform $\mathcal{F}[u](\xi) (\equiv \hat{u}(\xi))$ of u is defined by $\hat{u}(\xi) = \int e^{-ix \cdot \xi} u(x) dx$, where $x \cdot \xi = \sum_{j=1}^n x_j \xi_j$ for $x = (x_1, \dots, x_n)$ and $\xi = (\xi_1, \dots, \xi_n)$. Let $\varepsilon \geq 0$, and define

$$\mathcal{S}_{\kappa,\varepsilon} = \mathcal{F}^{-1}[\hat{\mathcal{S}}_{\kappa,\varepsilon}] (= \mathcal{F}[\hat{\mathcal{S}}_{\kappa,\varepsilon}] = \{u \in \mathcal{S}; \exp[\varepsilon\langle\xi\rangle^{1/\kappa}]\hat{u}(\xi) \in \mathcal{S}\}).$$

We introduce the topology in $\mathcal{S}_{\kappa,\varepsilon}$ so that $\mathcal{F} : \hat{\mathcal{S}}_{\kappa,\varepsilon} \rightarrow \mathcal{S}_{\kappa,\varepsilon}$ is a homeomorphism. Denote by $\mathcal{S}'_{\kappa,\varepsilon}$ the dual space of $\mathcal{S}_{\kappa,\varepsilon}$. Then we can define the transposed operators ${}^t\mathcal{F}$ and ${}^t\mathcal{F}^{-1}$ of $\mathcal{F}$ and $\mathcal{F}^{-1}$ which map $\mathcal{S}'_{\kappa,\varepsilon}$ and $\hat{\mathcal{S}}'_{\kappa,\varepsilon}$ onto $\hat{\mathcal{S}}'_{\kappa,\varepsilon}$ and $\mathcal{S}'_{\kappa,\varepsilon}$, respectively. Since $\hat{\mathcal{S}}_{\kappa,-\varepsilon} \subset \hat{\mathcal{S}}'_{\kappa,\varepsilon} (= \mathcal{D}')$, we can define $\mathcal{S}_{\kappa,-\varepsilon} = {}^t\mathcal{F}^{-1}[\hat{\mathcal{S}}_{\kappa,-\varepsilon}]$, and introduce the topology so that ${}^t\mathcal{F}^{-1} : \hat{\mathcal{S}}_{\kappa,-\varepsilon} \rightarrow \mathcal{S}_{\kappa,-\varepsilon}$ is a homeomorphism. $\mathcal{S}'_{\kappa,-\varepsilon}$ denotes the dual space of $\mathcal{S}_{\kappa,-\varepsilon}$. Then we have $\mathcal{S}'_{\kappa,-\varepsilon} = \mathcal{F}[\hat{\mathcal{S}}'_{\kappa,-\varepsilon}]$. We also define

$$L^2_{\kappa,\varepsilon} = \{u \in \mathcal{S}'_{\kappa,-\varepsilon}; \exp[\varepsilon\langle\xi\rangle^{1/\kappa}]{}^t\mathcal{F}[u](\xi) \in L^2\} \quad \text{for } \varepsilon \in \mathbf{R},$$

$$\mathcal{S}_{\kappa,\infty} = \bigcap_{\varepsilon>0} \mathcal{S}_{\kappa,\varepsilon}, \quad \mathcal{S}_{\kappa,+} = \bigcup_{\varepsilon>0} \mathcal{S}_{\kappa,\varepsilon},$$

$$\mathcal{F}_{(\kappa)} = \bigcup_{\varepsilon>0} \mathcal{S}'_{\kappa,\varepsilon}, \quad \mathcal{F}_{\{\kappa\}} = \bigcap_{\varepsilon>0} \mathcal{S}'_{\kappa,\varepsilon},$$

$$\mathcal{E}_{\kappa,\infty} = \bigcup_{\varepsilon>0} \mathcal{S}_{\kappa,-\varepsilon}, \quad \mathcal{E}_{\kappa,0} = \bigcap_{\varepsilon>0} \mathcal{S}_{\kappa,-\varepsilon}.$$

From the definitions it follows that (i) $\hat{\mathcal{S}}'_{\kappa,-\varepsilon} \subset \mathcal{S}' \subset \hat{\mathcal{S}}'_{\kappa,\varepsilon}$ and $\mathcal{S}'_{\kappa,-\varepsilon} \subset \mathcal{S}' \subset \mathcal{S}'_{\kappa,\varepsilon}$ for $\varepsilon \geq 0$, (ii) $\mathcal{F} = {}^t\mathcal{F}$ on $\mathcal{S}'$, (iii) $\mathcal{D}^{(\kappa)}$ is a dense subspace of $\mathcal{S}_{\kappa,\varepsilon}$, (iv) $\mathcal{D}_{\{\kappa\}} \subset \mathcal{S}_{\kappa,+}$ and $\mathcal{F}_{\{\kappa\}} \subset \mathcal{D}_{\{\kappa\}}'$, (v) $\mathcal{E}^{(\kappa)'} \subset \mathcal{E}_{\kappa,\infty}$ and $\mathcal{E}_{\{\kappa\}}' \subset \mathcal{E}_{\kappa,0}$, and (vi) $\mathcal{D}^{(\kappa)} \subset \mathcal{S}_{\kappa,\varepsilon} \subset \mathcal{S}_{\kappa,\varepsilon'} \subset L^2_{\kappa,\varepsilon'} \subset L^2_{\kappa,\varepsilon''} \subset \mathcal{S}'_{\kappa,-\varepsilon''} \subset \mathcal{F}_{(\kappa)} \subset \mathcal{D}_{(\kappa)}'$ if $\varepsilon \geq \varepsilon' \geq \varepsilon''$ (see, e.g., [11]). So we write ${}^t\mathcal{F}$ as $\mathcal{F}$. Let $p(\xi, y, \eta)$ be a symbol satisfying

$$|\partial_\xi^\alpha D_y^\beta \partial_\eta^\gamma p(\xi, y, \eta)| \leq C_{\alpha,\gamma} A^{|\beta|} |\beta|!^\kappa \exp[\delta_1 \langle \xi \rangle^{1/\kappa} + \delta_2 \langle \eta \rangle^{1/\kappa}]$$

for any $(\xi, y, \eta) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n$ and $\alpha, \beta, \gamma \in (\mathbf{Z}_+)^n$, where $A > 0$, $\delta_1, \delta_2 \in \mathbf{R}$ and the positive constants $C_{\alpha,\gamma}$ are independent of β . Throughout this paper we denote by $C_{a,b,\dots}$ and $C_{a,b,\dots}(A, B, \dots)$ constants depending on $a, b, \dots$ and $a, b, \dots, A, B, \dots$, respectively. Define

$$p(D_x, y, D_y)u(x) = (2\pi)^{-n} \mathcal{F}_\xi^{-1} \left[\int e^{-iy \cdot \xi} \left(\int e^{iy \cdot \eta} p(\xi, y, \eta) \hat{u}(\eta) d\eta \right) dy \right] (x)$$

for $u \in \mathcal{S}_{\kappa,\infty}$.

Proposition 1.1 (Proposition 2.3 of [11]). $p(D_x, y, D_y)$ maps continuously $\mathcal{S}_{\kappa, \varepsilon_2}$ to $\mathcal{S}_{\kappa, \varepsilon_1}$ and $\mathcal{S}'_{\kappa, -\varepsilon_2}$ to $\mathcal{S}'_{\kappa, -\varepsilon_1}$ if $\delta_2 - \kappa(nA)^{-1/\kappa} < \varepsilon_2$, $\varepsilon_1 \leq \varepsilon_2 - \delta_1 - \delta_2$ and $\varepsilon_1 < \kappa(nA)^{-1/\kappa} - \delta_1$.

Let $p(x, \xi) \in S_{(\kappa)}^{0; \infty}$. From Proposition 1.1 we can define $p(x, D)$ and ${}^t p(x, D)$ by

$$(1.3) \quad p(x, D) = p(D_x, y, D_y), \quad {}^t p(x, D) = q(D_x, y, D_y),$$

where $p(\xi, y, \eta) = p(y, \eta)$ and $q(\xi, y, \eta) = p(y, -\xi)$. It follows from Proposition 1.1 that $p(x, D)$ and ${}^t p(x, D)$ map continuously $\mathcal{S}_{\kappa, \varepsilon}$ to $\mathcal{S}_{\kappa, \varepsilon - \delta}$ and $\mathcal{S}'_{\kappa, \varepsilon}$ to $\mathcal{S}'_{\kappa, \varepsilon + \delta}$ for any $\varepsilon \in \mathbf{R}$ if $p(x, \xi) \in S_{(\kappa)}^{0; \delta}$, and that $p(x, D)$ and ${}^t p(x, D)$ map $\mathcal{S}_{\kappa, \infty}$ to $\mathcal{S}_{\kappa, \infty}$ and $\mathcal{F}_{(\kappa)}$ to $\mathcal{F}_{(\kappa)}$. Let $p(x, \xi) \in S_{\{\kappa\}}^+$. Similarly, we can define $p(x, D)$ and ${}^t p(x, D)$ by (1.3), which map $\mathcal{S}_{\kappa, +}$ to $\mathcal{S}_{\kappa, +}$, $\mathcal{E}_{\kappa, 0}$ to $\mathcal{E}_{\kappa, 0}$ and $\mathcal{F}_{\{\kappa\}}$ to $\mathcal{F}_{\{\kappa\}}$. In order to state our main results we give definitions of local solvability adopted here.

Definition 1.2. Let $x^0 \in \mathbf{R}^n$. (i) For $p(x, \xi) \in S_{(\kappa)}^{0; \infty}$ (resp. $S_{\{\kappa\}}^+$) we say that $p(x, D)$ is locally solvable at x^0 in $\mathcal{D}^{*'} if there is an open neighborhood U of x^0 such that for any $f \in \mathcal{D}^{*'}$ there is $u \in \mathcal{F}_*$ satisfying $p(x, D)u = f$ in U (in $\mathcal{D}^{*'}(U)$), where $*$ = (κ) (resp. $*$ = $\{\kappa\}$). Moreover, we say that $p(x, D)$ is locally solvable at x^0 in $\mathcal{D}^{*'}$ in a germ sense if for any $f \in \mathcal{D}^{*'}$ there are an open neighborhood U of x^0 and $u \in \mathcal{F}_*$ satisfying $p(x, D)u = f$ in U (in $\mathcal{D}^{*'}(U)$). (ii) For $p(x, \xi) \in S_{1,0}^m$ we say that $p(x, D)$ is locally solvable at x^0 in $\mathcal{D}'$ if there is an open neighborhood U of x^0 such that for any $f \in \mathcal{D}'$ there is $u \in \mathcal{S}'$ satisfying $p(x, D)u = f$ in U (in $\mathcal{D}'(U)$). Similarly, we define local solvability at x^0 in $\mathcal{D}'$ in a germ sense.$

Remark. (i) We remark that the above definitions of local solvability are slightly different from usual ones. (ii) In $\mathcal{D}^{\{\kappa\}'}$ “local solvability in a germ sense” implies “local solvability” for properly supported pseudodifferential operators (see Proposition A.2 below).

Let $\kappa > 1$. We denote (κ) or $\{\kappa\}$ by $*$. Let $\alpha(x, \xi) \in S_*^{1;0}$, and let

$$L(x, \xi) = |\xi'|^2 + x_n^2 \xi_n^2 + \alpha(x, \xi),$$

where $\xi' = (\xi_1, \dots, \xi_{n-1})$ for $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$. Then we have the following

Theorem 1.3. (i) If $\kappa \leq 2$ when $*$ = (κ) , and if $\kappa < 2$ when $*$ = $\{\kappa\}$, then $L(x, D)$ is locally solvable at the origin in $\mathcal{D}^{*'}$. (ii) Assume that $\alpha(x, \xi)$ can be written as

$$\alpha(x, \xi) = \sum_{k=1}^{n-1} \alpha_k(x, \xi) \xi_k + x_n \alpha_n(x, \xi) + \alpha_0(x, \xi),$$

where $\alpha_j(x, \xi) \in S_*^{0;0}$ ($0 \leq j \leq n-1$) and $\alpha_n(x, \xi) \in S_*^{1;0}$. Then $L(x, D)$ is locally solvable at the origin in $\mathcal{D}'$.

Remark. It was shown that $L(x, D)$ is locally solvable at the origin in the space of hyperfunctions if $\alpha(x, \xi)$ is an analytic symbol (see [5] and Chapter V of [19]).

Let $P(x, D)$ be a differential operator of the form

$$P(x, D) = D_1^2 + \cdots + D_{n-1}^2 + x_n^2 D_n^2 - x_n a(x) D_n - (1 + 2ix_1 + x_1^2 b(x)) D_n - \sum_{k=1}^{n-1} c_k(x) D_k + d(x).$$

Then we have the following theorem which gives necessary conditions of local solvability.

Theorem 1.4. (i) Assume that $a(x)$, $b(x)$, the $c_k(x)$ and $d(x)$ are analytic near the origin. Then $P(x, D)$ is not locally solvable at the origin in $\mathcal{D}'$ if $\kappa > 2$. (ii) Assume that $a(x), b(x), c_k(x), d(x) \in C^\infty(\mathbf{R}^n)$. Then $P(x, D)$ is not locally solvable at the origin in $\mathcal{D}'$.

Remark. From Hörmander [7] and Olejnik and Radkevich [14] it follows that the operator

$$D_1^2 + \cdots + D_{n-1}^2 + x_n^2 D_n^2 + (i\alpha(x) + x_n a(x)) D_n + \sum_{k=1}^{n-1} b_k(x) D_k + c(x)$$

is (hypoelliptic and) locally solvable at the origin in $\mathcal{D}'$ if $\alpha(x), a(x), b_k(x), c(x) \in C^\infty(\mathbf{R}^n)$, $\alpha(x)$ is real-valued and there is $\gamma \in (\mathbf{Z}_+)^n$ such that $\gamma_n = 0$ and $(D^\gamma \alpha)(0) \neq 0$ (see, also, [20]).

Let A be an operator defined by $Au(x) = (x_n D_n u(x) + D_n(x_n u(x)))/2$, i.e., $A = x_n D_n - i/2$. Moreover, let $Q(x, D) = D_1^m + \sum_{|\alpha| \leq m, \alpha_1 < m} a_\alpha D^{\alpha'} A^{\alpha_n}$, where $m \in \mathbf{N}$, $a_\alpha \in \mathbf{C}$, $\alpha' = (\alpha_1, \dots, \alpha_{n-1})$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbf{Z}_+)^n$ and $D^{\alpha'} = D_1^{\alpha_1} \cdots D_{n-1}^{\alpha_{n-1}}$.

Theorem 1.5. $Q(x, D)$ is locally solvable at the origin in $\mathcal{D}'$.

Remark. By the above theorem the operator

$$P \equiv D_1^2 + \cdots + D_{n-1}^2 + x_n^2 D_n^2 + \sum_{k=1}^{n-1} a_k D_k + a_n x_n D_n + b$$

is locally solvable at the origin in $\mathcal{D}'$, where $a_k, b \in \mathbf{C}$. (ii) In [16] and [17] Tahara studied more general operators and proved local solvability of those

operators in $\mathcal{D}'$ in a germ sense. (iii) The argument used in the proof of Theorem 1.5 gives an alternative proof of local solvability of differential operators with constant coefficients.

In §2 we shall give criteria (abstract necessary conditions and sufficient conditions) for local solvability. Using the results in §2 we shall prove Theorem 1.3 in §3, Theorem 1.4 in §4 and Theorem 1.5 in §5. In Appendix we shall prove that “local solvability in a germ sense” implies “local solvability” in the framework of $\mathcal{D}^{\{\kappa\}'}$, following [3].

2. Criteria for local solvability

We begin with well-known results on local solvability in $\mathcal{D}'$ (see, e.g., [18], [21] and [6]).

Proposition 2.1. *Let $x^0 \in \mathbf{R}^n$ and $p(x, \xi)$ be a symbol in $S_{1,0}^m$, where $m \in \mathbf{R}$. (i) If there is an open neighborhood U of x^0 such that for any $s \geq 0$ there are $\ell \in \mathbf{R}$ and $C > 0$ satisfying*

$$\|\langle D \rangle^s u\| \leq C \{\|\langle D \rangle^{\ell} p(x, D)u\| + \|u\|\}$$

for any $u \in C_0^\infty(U)$, then $p(x, D)$ is locally solvable at x^0 in $\mathcal{D}'$. Here $\|f\|$ denotes the L^2 -norm of f , i.e., $\|f\| = (\int |f(x)|^2 dx)^{1/2}$ for $f \in L^2(\mathbf{R}^n)$. (ii) If $p(x, D)$ is locally solvable at x^0 in $\mathcal{D}'$, then there is an open neighborhood U of x^0 such that for any $s \geq 0$ there are $\ell \in \mathbf{R}$ and $C > 0$ satisfying

$$\|\langle D \rangle^s u\| \leq C \|\langle D \rangle^{\ell} p(x, D)u\| \quad \text{for any } u \in C_0^\infty(U).$$

Repeating the same argument as in the proof of Proposition 2.1 we shall prove Theorems 2.4 and 2.5 below which give criteria for local solvability in $\mathcal{D}^{*'}$. In doing so, we need the following

Lemma 2.2 [Lemma 5.1.8 in [19]]. *Let $f(t)$ be a continuous functions on $[0, \infty)$ such that $f(t) \geq 0$ ($t \in [0, \infty)$) and $\lim_{t \rightarrow \infty} f(t)/t = 0$. Then there is an analytic function $F(t)$ defined in $\mathbf{C} \setminus (-\infty, 0]$ satisfying the following:* (i) $F(t) \geq \max_{0 \leq s \leq t} f(s)$ for $t \geq 0$. (ii) $\lim_{t \rightarrow +\infty} F(t)/t = 0$. (iii) $\lim_{t \rightarrow +\infty} t/(F(t)(1 + \log t)) = 0$. (iv) $0 < F'(t) \leq F(t)/t$ for $t > 0$. (v) There is $C > 0$ such that $F(t)/t \leq CF'(t)$ for $t \geq C$. (vi) $F''(t) < 0$ for $t > 0$. (vii) $\lim_{t \rightarrow +\infty} t^2 F''(t)/F(t) = 0$. (viii) There is $C > 0$ such that

$$|(d/dt)^k F(t)| \leq C(2/t)^k k! F(t) \quad \text{for } t > 0 \text{ and } k \in \mathbf{Z}_+.$$

Definition 2.3. (i) We say that a symbol $\omega(\xi) \in C^\infty(\mathbf{R}^n)$ belongs to $\mathcal{W}_{(\kappa)}$ if there is $\varepsilon \geq 1$ such that $\omega(\xi) = \varepsilon \langle \xi \rangle^{1/\kappa}$. (ii) We say that a symbol $\omega(\xi) \in C^\infty(\mathbf{R}^n)$ belongs to $\mathcal{W}_{\{\kappa\}}$ if there is a real analytic function $F(t)$ defined

near $[1, \infty)$ satisfying the following conditions: (0) $\omega(\xi) = F(\langle \xi \rangle^{1/\kappa})$. (i) $F(t) \geq t/(1 + \log t)$ for $t \geq 1$. (ii) $\lim_{t \rightarrow +\infty} F(t)/t = 0$. (iii) $0 < F'(t) \leq F(t)/t$ for $t \geq 1$. (iv) There is $C > 1$ such that $F(t)/t \leq CF'(t)$ for $t \geq C$. (v) $F''(t) < 0$ for $t \geq 1$. (vi) $\lim_{t \rightarrow +\infty} t^2 F''(t)/F(t) = 0$. (vii) There is $C > 0$ such that $|(d/dt)^k F(t)| \leq C(2/t)^k k! F(t)$ for $t \geq 1$ and $k \in \mathbf{Z}_+$.

Applying the same argument as in the proof of Lemma 5.1.10 of [19], one can prove that $\omega(\xi + \eta) \leq \omega(\xi) + \omega(\eta)$ and

$$(2.1) \quad |\partial^\alpha \omega(\xi)| \leq c_0 10^{|\alpha|} |\alpha|! \langle \xi \rangle^{-|\alpha|} \omega(\xi)$$

if $\omega(\xi) \in \mathcal{W}_*$ and $\xi, \eta \in \mathbf{R}^n$, where $c_0 > 0$. Assume that $p(x, \xi) \in S_{(\kappa)}^{0; \infty}$ if $* = (\kappa)$, and $p(x, \xi) \in S_{\{\kappa\}}^+$ if $* = \{\kappa\}$. For a real-valued symbol $\mu(\xi) \in C^\infty(\mathbf{R}^n)$ satisfying $|\mu(\xi)| \leq C \langle \xi \rangle^{1/\kappa}$ with some $C > 0$ we define

$$L^2(\mu) = \{f \in \mathcal{F}_{(\kappa)}; e^{\mu(\xi)} \hat{f}(\xi) \in L^2(\mathbf{R}^n)\},$$

which is a Hilbert space with the norm defined by $\|e^{\mu(\xi)} \hat{f}(\xi)\|$ ($f \in L^2(\mu)$). Note that $L^2(\mu) = L_{\kappa, \varepsilon}^2$ if $\mu(\xi) = \varepsilon \langle \xi \rangle^{1/\kappa}$.

Theorem 2.4. *Let $x^0 \in \mathbf{R}^n$, and let Ω be an open neighborhood of x^0 . Assume that for any $\omega(\xi) \in \mathcal{W}_*$ there are $\mu(\xi) \in \mathcal{W}_*$ and $C > 0$ such that*

$$\|e^{\omega(D)} v\| \leq C \{\|e^{\mu(D)} {}^t p(x, D) v\| + \|v\|\}$$

for any $v \in \mathcal{D}^{(\kappa)}(\Omega)$. Then $p(x, D)$ is locally solvable at x^0 in $\mathcal{D}^{*f}$.

Proof. Let $\omega_0(\xi) \in \mathcal{W}_*$. Then there is $C > 0$ such that $\|\langle D \rangle v\| \leq C \|\exp[\omega_0(D)] v\|$ for $v \in \mathcal{D}^{(\kappa)}$. By assumption there are $\mu_0(\xi) \in \mathcal{W}_*$ and $C_0 > 0$ such that

$$\|\langle D \rangle v\| \leq C_0 \{\|\exp[\mu_0(D)] {}^t p(x, D) v\| + \|v\|\}$$

for $v \in \mathcal{D}^{(\kappa)}(\Omega)$. Hence it follows from Poincaré's inequality that there is $a_0 > 0$ such that

$$\|v\| \leq C_0 \|\exp[\mu_0(D)] {}^t p(x, D) v\| \quad \text{for } v \in \mathcal{D}^{(\kappa)}(\Omega_0)$$

if Ω_0 is an open subset of Ω and $\text{diam}(\Omega_0) \leq a_0$, where $\text{diam}(\Omega_0)$ denotes the diameter of Ω_0 . Choose an open neighborhood Ω_0 of x^0 so that $\Omega_0 \subset \Omega$ and $\text{diam}(\Omega_0) \leq a_0$. It is obvious that for any $\lambda(\xi) \in \mathcal{W}_*$ there is $\mu(\xi) \in \mathcal{W}_*$ satisfying $\mu(\xi) \geq \max\{\lambda(\xi), \mu_0(\xi)\}$. Thus for any $\omega(\xi) \in \mathcal{W}_*$ there are $\mu(\xi) \in \mathcal{W}_*$ and $C > 0$ such that

$$(2.2) \quad \|e^{\omega(D)} v\| \leq C \|e^{\mu(D)} {}^t p(x, D) v\| \quad \text{for } v \in \mathcal{D}^{(\kappa)}(\Omega_0).$$

Let $f \in \mathcal{E}^{*f}(\Omega_0)$. Then there is $\omega(\xi) \in \mathcal{W}_*$ such that $f \in L^2(-\omega)$ (see, e.g., [12]). By (2.2) there are $\mu(\xi) \in \mathcal{W}_*$ and $C > 0$ such that

$$|\langle f, v \rangle| \leq C \|e^{-\omega(D)} f\| \|e^{\mu(D)} {}^t p(x, D) v\|$$

for $v \in \mathcal{D}^{(\kappa)}(\Omega_0)$, where $\langle \cdot, \cdot \rangle$ denotes the duality between $L^2(-\omega)$ and $L^2(\omega)$ (or the duality between $\mathcal{D}^{(\kappa)'}(\Omega_0)$ and $\mathcal{D}^{(\kappa)}(\Omega_0)$). Put

$$\mathcal{X} = \{{}^t p(x, D) v; v \in \mathcal{D}^{(\kappa)}(\Omega_0)\}.$$

Then $\mathcal{X}$ is a subspace of $L^2(\mu)$ and it follows from the Hahn-Banach theorem that there is $u \in L^2(-\mu)$ ($\subset \mathcal{F}_* \subset \mathcal{D}^{*'}(\Omega_0)$) such that

$$\langle f, v \rangle = \langle u, {}^t p(x, D) v \rangle_{L^2(-\mu), L^2(\mu)} \quad \text{for } v \in \mathcal{D}^{(\kappa)}(\Omega_0),$$

where $\langle \cdot, \cdot \rangle_{X', X}$ denotes the duality between X' and X . This gives $p(x, D)u = f$ in $\mathcal{D}^{*'}(\Omega_0)$. $\square$

We note that there is $\varepsilon_0 > 0$ satisfying ${}^t p(x, D)v \in L^2_{\kappa, \varepsilon_0}$ for every $v \in \mathcal{D}^{(\kappa)}$ if $* = \{\kappa\}$, i.e., if $p(x, \xi) \in \mathcal{S}^+_{\{\kappa\}}$, which is a simple consequence of Proposition 1.1.

Theorem 2.5. *Let $x^0 \in \mathbf{R}^n$. (i) Let $* = (\kappa)$, and assume that $p(x, D)$ is locally solvable at x^0 in $\mathcal{D}^{(\kappa)'}(\Omega)$. Then there is an open neighborhood U of x^0 such that for any $\varepsilon > 0$ there are $\delta > 0$ and $C > 0$ satisfying*

$$(2.3) \quad \|e^{\varepsilon \langle D \rangle^{1/\kappa}} v\| \leq C \|e^{\delta \langle D \rangle^{1/\kappa}} {}^t p(x, D) v\| \quad \text{for any } v \in \mathcal{D}^{(\kappa)}(U).$$

(ii) *Let $* = \{\kappa\}$, and assume that $p(x, D)$ is locally solvable at x^0 in $\mathcal{D}^{\{\kappa\}'}(\Omega)$. Then there is an open neighborhood U of x^0 such that for any $\delta > 0$ with $\delta < \varepsilon_0$ there are $\varepsilon > 0$ and $C > 0$ satisfying*

$$(2.4) \quad \|e^{\varepsilon \langle D \rangle^{1/\kappa}} v\| \leq C \{ \|e^{\delta \langle D \rangle^{1/\kappa}} {}^t p(x, D) v\| + \|v\| \} \quad \text{for any } v \in \mathcal{D}^{(\kappa)}(U),$$

where ε_0 is a positive constant determined by $p(x, \xi)$. If $p(x, D)$ is properly supported, then one can drop the term $\|v\|$ on the right-hand side of (2.4).

Proof. (i) By assumption there is an open neighborhood Ω of x^0 such that for any $\varepsilon > 0$ and $f \in L^2_{\kappa, -\varepsilon} \cap \mathcal{E}^{(\kappa)'}(\Omega)$ there is $u \in \mathcal{F}_{(\kappa)}$ satisfying $p(x, D)u = f$ in $\mathcal{D}^{(\kappa)'}(\Omega)$. Let U and U_1 be open neighborhoods of x^0 satisfying $U \Subset U_1 \Subset \Omega$. Let $\varepsilon > 0$, and denote by $\mathcal{X}$ the closure of $\mathcal{D}^{(\kappa)}(U_1)$ in $L^2_{\kappa, -\varepsilon}$. Note that $\mathcal{X}$ is a Hilbert space and that $\mathcal{X} \subset L^2_{\kappa, -\varepsilon} \cap \mathcal{E}^{(\kappa)'}(\Omega)$. We also define $\mathcal{Y} = \mathcal{D}^{(\kappa)}(U_1)$, whose topology is defined by the family of seminorms $\|e^{j \langle D \rangle^{1/\kappa}} {}^t p(x, D) v\| + \|v\|$ ($v \in \mathcal{Y}, j \in \mathbf{N}$). Then $\mathcal{Y}$ is a metrizable locally convex space. We define a bilinear form $\varphi: \mathcal{X} \times \mathcal{Y} \rightarrow \mathbf{C}$ by $\varphi(f, v) = \langle f, v \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the duality between $\mathcal{D}^{(\kappa)'}(\Omega)$ and $\mathcal{D}^{(\kappa)}(\Omega)$. It is obvious that for a fixed $v \in \mathcal{Y}$ $\varphi(f, v)$ is continuous in $f \in \mathcal{X}$. Fix $f \in \mathcal{X}$. Then there is $u \in \mathcal{F}_{(\kappa)}$ such that $p(x, D)u = f$ in $\mathcal{D}^{(\kappa)'}(\Omega)$. Choose $\chi_j(x) \in \mathcal{D}^{(\kappa)}$ ($j = 1, 2$) so that $\chi_1(x) = 1$ in Ω and $\chi_2(x) = 1$ near $\text{supp } \chi_1$. It follows from Lemma 2.15 of [11] with $p(x, \xi, y, \eta) = (1 - \chi_2(x))p(y, -\xi)\chi_1(y)$ that $(1 - \chi_2){}^t p(x, D)\chi_1$ maps continuously $\mathcal{S}_{\kappa, -\varepsilon_1}$ to $\mathcal{S}_{\kappa, \varepsilon_2}$ for any $\varepsilon_1, \varepsilon_2 \geq 0$. Hence we have

$$g(x) = \chi_1(x)p(x, D)((1 - \chi_2(x))u(x)) \in \mathcal{S}'_{\kappa, -\varepsilon} \quad \text{for any } \varepsilon > 0.$$

Since $g(x) = (2\pi)^{-n} \langle e^{\varepsilon \langle \xi \rangle^{1/\kappa}} \hat{g}(\xi), e^{ix \cdot \xi - \varepsilon \langle \xi \rangle^{1/\kappa}} \rangle_{\mathcal{S}', \mathcal{S}}$ for any $\varepsilon > 0$ and, therefore, $g(x)$ is continuous, we have $g(x) \in L^2(\mathbf{R}^n)$. Noting that $\chi_2 u \in \mathcal{E}^{(\kappa)'}_{\kappa, -\varepsilon'}$ and that there is $\varepsilon' > 0$ such that $\chi_2 u \in L^2_{\kappa, -\varepsilon'}$, we have

$$\begin{aligned} |\langle f, v \rangle| &\leq |\langle p(x, D)(\chi_2(x)u(x)), v \rangle| + |\langle g, v \rangle| = |\langle \chi_2 u, {}^t p(x, D)v \rangle| + |\langle g, v \rangle| \\ &\leq \|e^{-\varepsilon' \langle D \rangle^{1/\kappa}} (\chi_2 u)\| \|e^{\varepsilon' \langle D \rangle^{1/\kappa}} {}^t p(x, D)v\| + \|g\| \|v\| \end{aligned}$$

for $v \in \mathcal{Y}$. This implies that $\varphi(f, v)$ is continuous in $v \in \mathcal{Y}$ and that φ is separately continuous. Therefore, φ is continuous (see, e.g., [15]), i.e., there are $j \in \mathbf{N}$ and $C > 0$ such that

$$(2.5) \quad |\langle f, v \rangle| \leq C \|e^{-\varepsilon \langle D \rangle^{1/\kappa}} f\| (\|e^{j \langle D \rangle^{1/\kappa}} {}^t p(x, D)v\| + \|v\|)$$

for $f \in \mathcal{X}$ and $v \in \mathcal{D}^{(\kappa)}(U_1)$. Choose $\psi(x) \in \mathcal{D}^{(\kappa)}(U_1)$ so that $\psi(x) = 1$ in U . It follows from the proof of Proposition 2.12 of [11] that $q(x, \xi) \in S^0_{1-1/\kappa, 0}$ if $q(x, D) = e^{-\varepsilon \langle D \rangle^{1/\kappa}} \psi(x) e^{\varepsilon \langle D \rangle^{1/\kappa}}$. Putting $f = \psi(x) e^{2\varepsilon \langle D \rangle^{1/\kappa}} \overline{v(x)}$ in (2.5) for $v \in \mathcal{D}^{(\kappa)}(U)$, we have

$$(2.6) \quad \|e^{\varepsilon \langle D \rangle^{1/\kappa}} v\| \leq C' (\|e^{j \langle D \rangle^{1/\kappa}} {}^t p(x, D)v\| + \|v\|).$$

Poincaré's inequality and (2.6) give, with some $C_0 > 0$,

$$\|v\| \leq C_0 \|e^{\langle D \rangle^{1/\kappa}} {}^t p(x, D)v\|$$

for $v \in \mathcal{D}^{(\kappa)}(U)$, shrinking U if necessary. Therefore, this, together with (2.6), yields (2.3). (ii) By assumption there is an open neighborhood Ω of x^0 such that for any $f \in \mathcal{E}^{\{\kappa\}'}(\Omega)$ there is $u \in \mathcal{F}_{\{\kappa\}}$ satisfying $p(x, D)u = f$ in $\mathcal{D}^{\{\kappa\}'}(\Omega)$. Let U and U_1 be open neighborhoods of x^0 satisfying $U \subseteq U_1 \subseteq \Omega$, and fix $\delta > 0$ so that $\delta < \varepsilon_0$. We put

$$\mathcal{X} = \{f \in \mathcal{E}^{\{\kappa\}'}; \text{supp } f \subset \overline{U}_1\}, \quad \mathcal{Y} = \mathcal{D}^{(\kappa)}(U_1).$$

We introduce the topology in $\mathcal{X}$ by the family of seminorms $\|e^{-\langle D \rangle^{1/\kappa}/j} f\|$ ($f \in \mathcal{X}, j \in \mathbf{N}$). It is easy to see that $\mathcal{X}$ is a Fréchet space, since $\bigcap_{\eta > 0} L^2_{\kappa, -\eta} \subset \mathcal{D}^{\{\kappa\}'}_{\kappa, -\eta}$ and $\mathcal{X} = \left\{ f \in \bigcap_{\eta > 0} L^2_{\kappa, -\eta}; \text{supp } f \subset \overline{U}_1 \right\}$. $\mathcal{Y}$ is a normed space under the norm defined by $\|e^{\delta \langle D \rangle^{1/\kappa}} {}^t p(x, D)v\| + \|v\|$ ($v \in \mathcal{Y}$). We define a bilinear form $\varphi : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbf{C}$ by $\varphi(f, v) = \langle f, v \rangle$. It is obvious that $\varphi(f, v)$ is continuous in $f \in \mathcal{X}$ for a fixed $v \in \mathcal{Y}$. Let $f \in \mathcal{X}$. Then there is $u \in \mathcal{F}_{\{\kappa\}}$ such that $p(x, D)u = f$ in $\mathcal{D}^{\{\kappa\}'}(\Omega)$. Let $\chi_j(x) \in \mathcal{D}^{(\kappa)}$ be as in the proof of the assertion (i). Similarly, it follows from Lemma 2.15 of [11] that $(1 - \chi_2){}^t p(x, D)\chi_1$ maps continuously $\mathcal{S}_{\kappa, -\varepsilon_1}$ to $\mathcal{S}_{\kappa, \varepsilon_2}$ for any $\varepsilon_1, \varepsilon_2 \geq 0$ with $\varepsilon_1, \varepsilon_2 \leq \varepsilon_0$, modifying ε_0 if necessary. Hence we have

$$g(x) = \chi_1(x)p(x, D)((1 - \chi_2(x))u(x)) \in \mathcal{S}'_{\kappa, -\varepsilon_0}.$$

Therefore, we have $g(x) \in L^2(\mathbf{R}^n)$. Noting that $\chi_2 u \in L^2_{\kappa, -\delta}$, we have

$$\begin{aligned} |\langle f, v \rangle| &\leq |\langle \chi_2 u, {}^t p(x, D)v \rangle| + |\langle g, v \rangle| \\ &\leq \|e^{-\delta \langle D \rangle^{1/\kappa}}(\chi_2 u)\| \|e^{\delta \langle D \rangle^{1/\kappa}} {}^t p(x, D)v\| + \|g\| \|v\| \end{aligned}$$

for $v \in \mathcal{Y}$. This implies that $\varphi(f, v)$ is continuous in $v \in \mathcal{Y}$, and that φ is separately continuous. Therefore, φ is continuous and there are $j \in \mathbf{N}$ and $C > 0$ such that

$$|\langle f, v \rangle| \leq C \|e^{-\langle D \rangle^{1/\kappa}/j} f\| (\|e^{\delta \langle D \rangle^{1/\kappa}} {}^t p(x, D)v\| + \|v\|)$$

for $f \in \mathcal{X}$ and $v \in \mathcal{D}^{(\kappa)}(U_1)$. Repeating the same argument as in the proof of the assertion (i), we obtain (2.4). Now assume that $p(x, D)$ is properly supported. Then for any $f \in \mathcal{E}^{\{\kappa\}'}(\Omega)$ there is $u \in \mathcal{E}^{\{\kappa\}'}$ satisfying $p(x, D)u = f$ in $\mathcal{D}^{\{\kappa\}'}(\Omega)$. Let $v \in \mathcal{Y}$ satisfy $\|e^{\delta \langle D \rangle^{1/\kappa}} {}^t p(x, D)v\| = 0$. For any $f \in \mathcal{D}^{(\kappa)}(U_1)$ there is $u \in \mathcal{E}^{\{\kappa\}'}$ satisfying $p(x, D)u = f$ in $\mathcal{D}^{\{\kappa\}'}(\Omega)$. Then we have

$$0 = \langle u, {}^t p(x, D)v \rangle = \langle f, v \rangle,$$

which gives $v = 0$. So $\|e^{\delta \langle D \rangle^{1/\kappa}} {}^t p(x, D)v\|$ for $v \in \mathcal{Y}$ is a norm of $\mathcal{Y}$. We replace the norm of $\mathcal{Y}$ by $\|e^{\delta \langle D \rangle^{1/\kappa}} {}^t p(x, D)v\|$ ($v \in \mathcal{Y}$). Repeating the same argument, we can prove that $\varphi(f, v)$ is continuous since for $f \in \mathcal{X}$ there is $u \in L^2_{\kappa, -\delta}$ satisfying $p(x, D)u = f$ in $\mathcal{D}^{\{\kappa\}}(\Omega)$, and we can take $\chi_2(x) \equiv 1$. This completes the proof. $\square$

In the rest of this section we assume that $p(x, \xi) \in S_*^{m;0}$, where $m \in \mathbf{R}$. Let $\omega(\xi) \in \mathcal{W}_*$, and put

$$p_\omega(\xi, y, \eta) = e^{-\omega(\xi)} p(y, \eta) e^{\omega(\eta)}.$$

Let us apply Proposition 2.13 of [11] to $p_\omega(\xi, y, \eta)$. By induction on $|\alpha|$ and (2.1) we have

$$\begin{aligned} &|\partial^\beta \{e^{\mp \omega(\xi)} \partial^\alpha e^{\pm \omega(\xi)}\}| \\ &\leq 20^{|\beta|} A_1(\rho)^{|\alpha|} (|\alpha| + |\beta|)! \langle \xi \rangle^{-|\alpha| - |\beta|} \sum_{k=0}^{|\alpha|} \rho^k \omega(\xi)^k / k! \end{aligned}$$

if $\rho > 0$ and $A_1(\rho) = \max\{80c_0/\rho, 40\}$. Here we have also used the identity

$$\begin{aligned} &\partial^\beta \{e^{\mp \omega(\xi)} \partial^{\alpha+e} e^{\pm \omega(\xi)}\} \\ &= \partial^{\beta+e} \{e^{\mp \omega(\xi)} \partial^\alpha e^{\pm \omega(\xi)}\} \pm \partial^\beta \{(\partial^e \omega(\xi)) e^{\mp \omega(\xi)} \partial^\alpha e^{\pm \omega(\xi)}\}, \end{aligned}$$

where $|e| = 1$. In particular, we have

$$|\partial^\alpha e^{\pm\omega(\xi)}| \leq A_1^{|\alpha|} |\alpha|! \langle \xi \rangle^{-|\alpha|} e^{(1 \pm 1)\omega(\xi)},$$

where $A_1 = A_1(1)$. Put

$$p_{\omega,j}(x, \xi) = \sum_{|\alpha|=j} e^{\omega(\xi)} (\partial^\alpha e^{-\omega(\xi)}) p_{(\alpha)}(x, \xi) / \alpha!$$

($j \in \mathbf{Z}_+$). Since $\sum_{k=0}^{\infty} k!(1-\kappa)t^k < \infty$ for $t \geq 0$, we have

$$\begin{aligned} |p_{\omega,j(\beta)}^{(\alpha)}(x, \xi)| &\leq C_\kappa(p, A)(2^\kappa A)^{|\alpha|+|\beta|} (2^{1+\kappa} n A_1 A)^j \\ &\quad \times (|\alpha| + |\beta|)!^\kappa j!^\kappa \langle \xi \rangle^{m-|\alpha|-j} \sum_{k=0}^j \omega(\xi)^k / k!, \end{aligned}$$

where A is a positive constant satisfying

$$(2.7) \quad |p_{(\beta)}^{(\alpha)}(x, \xi)| \leq C(p) A^{|\alpha|+|\beta|} (|\alpha| + |\beta|)!^\kappa \langle \xi \rangle^{m-|\alpha|}.$$

Recall that one can take A to be arbitrary positive constant in (2.7) if $* = (\kappa)$. If $* = (\kappa)$, then we can apply Proposition 2.13 of [11] to $p(x, D)$. If $* = \{\kappa\}$, then for any $\delta > 0$ there is $C_\delta > 0$ such that $e^{2\omega(\xi)} \leq C_\delta e^{\delta \langle \xi \rangle^{1/\kappa}}$. Therefore, we can modify the proof of Proposition 2.13 of [11] and apply Proposition 2.13 of [11] with $C_0 = \lambda_0 = \lambda_1 = \delta = 0$ to $p(x, D)$ when $* = \{\kappa\}$. Thus there are symbols $\tilde{p}_\omega(x, \xi)$ and $r_\omega(x, \xi)$ and $c > 0$ such that

$$\begin{aligned} p_\omega(D_x, y, D_y) &= \tilde{p}_\omega(x, D) + r_\omega(x, D), \\ \left| \partial_\xi^\alpha D_x^\beta \left\{ \tilde{p}_\omega(x, \xi) - \sum_{j=0}^{N-1} p_{\omega,j}(x, \xi) \right\} \right| \\ &\leq C_\kappa(p, A, N)(2^\kappa A)^{|\alpha|+|\beta|} (|\alpha| + |\beta|)!^\kappa \langle \xi \rangle^{m-|\alpha|-(1-1/\kappa)N} \quad (N \in \mathbf{Z}_+), \\ |r_{\omega(\beta)}^{(\alpha)}(x, \xi)| &\leq C'_\kappa(p, A, |\alpha|)(2^{3\kappa} A)^{|\beta|} |\beta|!^\kappa e^{-cA^{1/\kappa} \langle \xi \rangle^{1/\kappa}}, \end{aligned}$$

where A is a positive constant satisfying (2.7) and $C_\kappa(p, A, N), C'_\kappa(p, A, |\alpha|) > 0$. Let $\rho > 0$, and let $p_\omega^\rho(x, \xi)$ be a symbol in $S_{1,0}^m$ satisfying

$$\tilde{p}_\omega(x, \xi) \equiv p_\omega^\rho(x, \xi) \pmod{S_{1,0}^{m-\rho}}.$$

Theorem 2.6. *Let $x^0 \in \mathbf{R}^n$, and let Ω be an open neighborhood of x^0 . Assume that for any $\omega(\xi) \in \mathcal{W}_*$ and $a > 0$ there is $C > 0$ such that*

$$(2.8) \quad \|{}^t p_\omega^\rho(x, D)u\| \geq a \|\langle D \rangle^{m-\rho} u\| - C \|\langle D \rangle^{m-\rho-1} u\|$$

for $u \in C_0^\infty(\Omega)$. Then $p(x, D)$ is locally solvable at x^0 in $\mathcal{D}^{*f}$.

Proof. Let $\omega(\xi) \in \mathcal{W}_*$. Then it follows from (2.8) that there is $C > 0$ satisfying

$$(2.9) \quad \| {}^t p_{2\omega}(D_x, y, D_y) u \| \geq \| \langle D \rangle^{m-\rho} u \| - C \| e^{-\langle D \rangle} u \|$$

for $u \in C_0^\infty(\Omega)$. Let U and U_1 be open neighborhoods of x^0 such that $U \Subset U_1 \Subset \Omega$, and choose $\psi \in \mathcal{D}^{(\kappa)}(\Omega)$ so that $\psi(x) = 1$ in U_1 . For $v \in \mathcal{D}^{(\kappa)}(U)$ we take $u(x) = \psi(x)e^{2\omega(D)}v(x)$ in (2.9). From Propositions 2.8 and 2.12 of [11] and Proposition 1.1 there are $\varepsilon > 0$ and $C > 0$ such that

$$\| e^{2\omega(D)} {}^t p(x, D) e^{-2\omega(D)} ((1 - \psi(x))e^{2\omega(D)}v) \| \leq C \| v \|,$$

$$\| \langle D \rangle^{m-\rho} ((1 - \psi(x))e^{2\omega(D)}v) \| \leq C \| v \|,$$

$$\| e^{-\langle D \rangle} (\psi(x)e^{2\omega(D)}v) \| \leq C \| v \|$$

for $v \in \mathcal{D}^{(\kappa)}(U)$. Therefore, there are positive constants C and C' such that

$$\| e^{\omega(D)} v \| \leq C \| \langle D \rangle^{m-\rho} e^{2\omega(D)} v \| \leq C' \{ \| e^{2\omega(D)} {}^t p(x, D) v \| + \| v \| \}$$

for $v \in \mathcal{D}^{(\kappa)}(U)$. So, by Theorem 2.4 $p(x, D)$ is locally solvable at x^0 in $\mathcal{D}^{*}$. $\square$

3. Proof of Theorem 1.3

Let $\lambda(t)$ be a function in $\mathcal{E}^{(\kappa)}(\mathbf{R})$ such that $|\lambda(t)| \leq 1/5$, $\lambda(t) = t$ if $|t| \leq 1/10$, and $\lambda(t) = \pm 1/5$ if $\pm t \geq 1$, and put $X_n = \lambda(x_n)$, $\Omega = \{x \in \mathbf{R}^n; |x| < 1/20\}$ and $\Omega_1 = \{x \in \mathbf{R}^n; |x| < 1/10\}$. We modify $L(x, \xi)$ as follows:

$$\tilde{L}(x, \xi) = |\xi'|^2 + X_n^2 \xi_n^2 + \tilde{\alpha}(x, \xi),$$

where $\tilde{\alpha}(x, \xi) = \alpha(x, \xi)$ when we prove the first part of Theorem 1.3, and

$$\tilde{\alpha}(x, \xi) = \sum_{k=1}^{n-1} \alpha_k(x, \xi) \xi_k + X_n \alpha_n(x, \xi) + \alpha_0(x, \xi)$$

when we prove the second part of Theorem 1.3. Then we have $L(x, D)u = \tilde{L}(x, D)u$ in Ω . Therefore, it suffices to prove Theorem 1.3 with $L(x, D)$ replaced by $\tilde{L}(x, D)$. Let $\omega(\xi) \in \mathcal{W}_*$ and $\omega(\xi) = F(\langle \xi \rangle^{1/\kappa})$, and put

$$\tilde{L}_\omega(x, D) = e^{-\omega(D)} \tilde{L}(x, D) e^{\omega(D)}.$$

Then we have

$$\tilde{L}_\omega(x, \xi) \equiv \tilde{L}(x, \xi; i\nabla\omega(\xi)) - \sum_{|\alpha|=2} \tilde{L}_{(\alpha)}(x, \xi; i\nabla\omega(\xi)) \partial^\alpha \omega(\xi) / \alpha! \pmod{S_{1,0}^0},$$

where $\tilde{L}(x, \xi; \zeta) = \sum_{|\beta| < 2\kappa/(\kappa-1)} (i\zeta)^\beta \tilde{L}_{(\beta)}(x, \xi)/\beta!$ and $\tilde{L}_{(\beta)}^{(\alpha)}(x, \xi; \zeta) = \partial_\xi^\alpha D_x^\beta \tilde{L}(x, \xi; \zeta)$ (see Proposition A.1 below). Put

$$q_\omega(x, \xi) = \ell_\omega(x, \xi) + s_\omega(x, \xi),$$

where

$$\begin{aligned} (3.1) \quad \ell_\omega(x, \xi) &= |\xi'|^2 + (X_n - i\partial_n \omega(\xi))^2 \xi_n^2, \\ s_\omega(x, \xi) &= \tilde{\alpha}(x, -\xi; -i\nabla \omega(\xi)) - 2i(X_n - i\partial_n \omega(\xi))\xi_n, \\ \tilde{\alpha}(x, \xi; \zeta) &= \sum_{|\beta| < \kappa/(\kappa-1)} (i\zeta)^\beta \tilde{\alpha}_{(\beta)}(x, \xi)/\beta!. \end{aligned}$$

Then we have

$$\begin{aligned} q_\omega(x, \xi) &\equiv \tilde{L}(x, -\xi; -i\nabla \omega(\xi)) + (i/2) \sum_{j=1}^n (\partial^2 \tilde{L} / \partial x_j \partial \xi_j)(x, -\xi; -i\nabla \omega(\xi)), \\ &\exp[-\partial_\xi \cdot D_x / 2] \tilde{L}_\omega(x, \xi) \equiv q_\omega(x, -\xi) \\ &(\text{mod } S_{1,0}^0 \text{ in } \Omega_1), \text{ since} \\ &-(1/2) \sum_{0 < |\gamma| < 2\kappa/(\kappa-1)} \sum_{j=1}^n \tilde{L}_{(\gamma+e_j)}(x, \xi) \partial_j (-\nabla \omega(\xi))^\gamma / \gamma! \\ &= (1/2) \sum_{j,k=1}^n \sum_{0 < |\gamma| < 2\kappa/(\kappa-1)} \gamma_k \tilde{L}_{(\gamma-e_k+e_j+e_k)}(x, \xi) (\partial_j \partial_k \omega(\xi)) (-\nabla \omega(\xi))^{\gamma-e_k} / \gamma! \\ &= \sum_{|\beta|=2} \sum_{|\gamma| < 2\kappa/(\kappa-1)-1} \tilde{L}_{(\gamma+\beta)}(x, \xi) (-\nabla \omega(\xi))^\gamma (\partial^\beta \omega(\xi)) / (\gamma! \beta!), \\ &(-\partial \cdot D_x) \tilde{L}(x, \xi; i\nabla \omega(\xi)) / 2 \\ &\equiv (i/2) \sum_{j=1}^n (\partial^2 \tilde{L} / \partial x_j \partial \xi_j)(x, \xi; i\nabla \omega(\xi)) \\ &+ \sum_{|\beta|=2} \tilde{L}_{(\beta)}(x, \xi; i\nabla \omega(\xi)) \partial^\beta \omega(\xi) / \beta! \quad (\text{mod } S_{1,0}^0 \text{ in } \Omega_1), \end{aligned}$$

where “ $a(x, \xi) \equiv b(x, \xi) \pmod{S_{1,0}^k \text{ in } \Omega_1}$ ” means that $a(x, \xi) - b(x, \xi) \in S_{1,0}^k$ in Ω_1 and e_j denotes the multi-index in $(\mathbb{Z}_+)^n$ whose k -component is equal to $\delta_{j,k}$ ($1 \leq k \leq n$) (see, e.g., [8]). This gives $\sigma({}^t \tilde{L}_\omega(x, D) - q_\omega^w(x, D))(x, \xi) \in S_{1,0}^0$ in Ω_1 , i.e., $\varphi(x) \sigma({}^t \tilde{L}_\omega(x, D) - q_\omega^w(x, D))(x, \xi) \in S_{1,0}^0$ if $\varphi(x) \in C_0^\infty(\Omega_1)$, where $\sigma(a(x, D))(x, \xi) = a(x, \xi)$ and $q_\omega^w(x, D)$ denotes the pseudodifferential operator with Weyl symbol $q_\omega(x, \xi)$ defined by

$$q_{\omega}^w(x, D)u(x) = (2\pi)^{-n} \int \left(\int e^{i(x-y)\cdot\xi} q_{\omega}((x+y)/2, \xi) u(y) dy \right) d\xi$$

for $u \in \mathcal{S}$. Let $E_{\omega}(x, \xi)$ be a symbol in $S_{1,0}^4$ such that

$$\overline{q_{\omega}(x, \xi)} \# q_{\omega}(x, \xi) \equiv E_{\omega}(x, \xi) \pmod{S_{1,0}^0},$$

where $a \# b$ denotes the Weyl symbol of $a^w(x, D)b^w(x, D)$. First we assume that $(1 <) \kappa < 3/2$. Put

$$\begin{aligned} I_{\omega}(x, \xi) &= |q_{\omega}(x, \xi)|^2 + 4\kappa^{-1}\xi_n^4 \langle \xi \rangle^{1/\kappa-2} (\kappa^{-2} F'(\langle \xi \rangle^{1/\kappa})^2 \langle \xi \rangle^{2/\kappa-4} \xi_n^2 + X_n^2) \\ &\quad \times [F'(\langle \xi \rangle^{1/\kappa}) (2 - (2 - 1/\kappa) \langle \xi \rangle^{-2} \xi_n^2) + \kappa^{-1} F''(\langle \xi \rangle^{1/\kappa}) \langle \xi \rangle^{1/\kappa-2} \xi_n^2] \\ &\quad \text{for } x_1 \in \Omega_1 \end{aligned}$$

Then a simple calculation yields

$$I_{\omega}(x, \xi) = |q_{\omega}(x, \xi)|^2 + (i/2) \{ \ell_{\omega}(x, \xi), \overline{\ell_{\omega}(x, \xi)} \} \quad \text{for } x \in \Omega_1,$$

$$E_{\omega}(x, \xi) \equiv I_{\omega}(x, \xi) \pmod{S_{1,0}^2 \text{ in } \Omega_1},$$

where $\{f(x, \xi), g(x, \xi)\} = \sum_{j=1}^n ((\partial f / \partial \xi_j)(x, \xi)(\partial g / \partial x_j)(x, \xi) - (\partial f / \partial x_j)(x, \xi) \cdot (\partial g / \partial \xi_j)(x, \xi))$. Recall that $F''(t) \equiv 0$ if $* = (\kappa)$, and that $F''(t) < 0 < F'(t)$ for $t \geq 1$ if $* = \{\kappa\}$. If $|\xi_n| \geq |\xi'|$ and $|\xi| \geq 1$, then we have $|\xi_n| \geq \langle \xi \rangle / 2$ and

$$I_{\omega}(x, \xi) \geq 2^{-4} \kappa^{-4} \langle \xi \rangle^{3/\kappa} F'(\langle \xi \rangle^{1/\kappa})^2 (F'(\langle \xi \rangle^{1/\kappa}) + F''(\langle \xi \rangle^{1/\kappa}) \langle \xi \rangle^{1/\kappa}).$$

Noting that there is $\varepsilon > 0$ satisfying $F(t) = \varepsilon t$ if $* = (\kappa)$, and that there are $R_{\omega} > 0$ and $c_{\omega} > 0$ satisfying

$$F'(\langle \xi \rangle^{1/\kappa}) + F''(\langle \xi \rangle^{1/\kappa}) \langle \xi \rangle^{1/\kappa} \geq F'(\langle \xi \rangle^{1/\kappa}) / 2 \geq c_{\omega} / (1 + \log \langle \xi \rangle)$$

if $|\xi| \geq R_{\omega}$ and $* = \{\kappa\}$, we see that there are positive constants δ , R'_{ω} and c'_{ω} such that

$$I_{\omega}(x, \xi) \geq c'_{\omega} \langle \xi \rangle^{2+\delta} \quad \text{if } |\xi| \geq R'_{\omega}.$$

Here we have also used the inequalities $|\xi'| \geq \langle \xi \rangle / 2$ and $I_{\omega}(x, \xi) \geq |q_{\omega}(x, \xi)|^2 \geq 2^{-6} \langle \xi \rangle^4$ if $|\xi_n| \leq |\xi'|$ and $|\xi| \gg 1$. The Fefferman-Phong inequality gives, with some $C_j > 0$ ($j = 1, 2$),

$$\begin{aligned} (3.2) \quad \|q_{\omega}^w(x, D)u\|^2 &\geq (I_{\omega}^w(x, D)u, u)_{L^2} + ((E_{\omega}^w(x, D) - I_{\omega}^w(x, D))u, u)_{L^2} - C_1 \|u\|^2 \\ &\geq c'_{\omega} \|\langle D \rangle^{1+\delta/2} u\|^2 - C_2 \|\langle D \rangle u\|^2 \end{aligned}$$

for $u \in C_0^{\infty}(\Omega)$, where $(f, g)_{L^2} = \int f(x) \overline{g(x)} dx$ for $f, g \in L^2(\mathbf{R}^n)$ (see, e.g., [8]). Therefore, Theorem 2.6, together with (3.2), proves Theorem 1.3 if $\kappa < 3/2$. Next we shall consider the case where $\kappa \geq 3/2$. It is obvious that

$$q_\omega(x, \xi) = \ell_\omega(x, \xi) + \tilde{\alpha}(x, -\xi) + \sum_{0 < |\beta| \leq 2} (\nabla \omega(\xi))^{\beta} \tilde{\alpha}_{(\beta)}(x, -\xi) / \beta! \\ - 2i(X_n - i\partial_n \omega(\xi)) \xi_n$$

and that $\partial^\alpha \omega(\xi) / \omega(\xi) \in S_{1,0}^{-|\alpha|}$. So we can write

$$(3.3) \quad q_\omega(x, \xi) = \ell_\omega(x, \xi) + a(x, \xi) + \alpha_n(x, \xi) X_n + \sum_{k=1}^{n-1} \alpha_k(x, \xi) \xi_k \\ + \omega(\xi) b(x, \xi) + \alpha_0(x, \xi),$$

where $a(x, \xi), \alpha_n(x, \xi) \in S_{1,0}^1$ and $\alpha_k(x, \xi), b(x, \xi) \in S_{1,0}^0$ ($0 \leq k \leq n-1$) and they also depend on ω . Taking account of the hypotheses of Theorem 1.3, we assume that $a(x, \xi) \equiv 0$ if $\kappa > 2$ or if $\kappa = 2$ and $* = \{\kappa\}$. Now it suffices to prove the estimates 2.8 with $m = \rho = 2$ and ${}^t p_\omega^\rho(x, D)$ replaced by $q_\omega^w(x, D)$ in order to complete the proof of Theorem 1.3. Put

$$q_\omega^1(x, \xi) = \operatorname{Re} q_\omega(x, \xi), \quad q_\omega^2(x, \xi) = \operatorname{Im} q_\omega(x, \xi).$$

Noting that $\{a(x, \xi), \overline{a(x, \xi)}\} \omega(\xi)^{-2} \in S_{1,0}^0$ and $\omega(\xi)^{-1} \in S_{1,0}^0$, a straightforward calculation gives

$$(3.4) \quad E_\omega(x, \xi) \equiv \sum_{j=1}^2 q_\omega^j(x, \xi) \# q_\omega^j(x, \xi) \\ + i(q_\omega^1(x, \xi) \# q_\omega^2(x, \xi) - q_\omega^2(x, \xi) \# q_\omega^1(x, \xi)) \pmod{S_{1,0}^0}, \\ Q_\omega(x, \xi) \equiv (i/2) \{q_\omega(x, \xi), \overline{q_\omega(x, \xi)}\} \\ + i \sum_{|\alpha|+|\beta|=3} (-1)^{|\beta|} (4\alpha! \beta!)^{-1} q_{\omega(\beta)}^{1(\alpha)}(x, \xi) q_{\omega(\alpha)}^{2(\beta)}(x, \xi) \\ \equiv 4\kappa^{-1} w_0(x_n, \xi) \lambda'(x_n) \langle \xi \rangle^{1/\kappa-2} \xi_n^4 [F'(\langle \xi \rangle^{1/\kappa}) (2 - (2 - 1/\kappa) \langle \xi \rangle^{-2} \xi_n^2) \\ + \kappa^{-1} F''(\langle \xi \rangle^{1/\kappa}) \langle \xi \rangle^{1/\kappa-2} \xi_n^2] + \sum_{k=1}^{n-1} 2\xi_k \operatorname{Im}(\partial a / \partial x_k)(x, \xi) \\ - 2\lambda'(x_n) [X_n \xi_n^2 \operatorname{Im}(\partial a / \partial \xi_n)(x, \xi) + (\partial_n \omega(\xi)) \xi_n^2 \operatorname{Re}(\partial a / \partial \xi_n)(x, \xi)] \\ + \sum_{j,k=1}^{n-1} \xi_j \xi_k b_{j,k}(x, \xi) + \sum_{k=1}^{n-1} \xi_k (X_n b_k^1(x, \xi) + \omega(\xi) b_k^0(x, \xi)) \\ + X_n^2 b^2(x, \xi) + X_n \omega(\xi) b^1(x, \xi) + \omega(\xi)^2 b^0(x, \xi) \pmod{S_{1,0}^{1/\kappa}},$$

where

$$Q_\omega(x, \xi) = i(q_\omega^1(x, \xi) \# q_\omega^2(x, \xi) - q_\omega^2(x, \xi) \# q_\omega^1(x, \xi)),$$

$$w_0(x_n, \xi) = X_n^2 + \kappa^{-2} F'(\langle \xi \rangle^{1/\kappa})^2 \langle \xi \rangle^{2/\kappa-4} \xi_n^2,$$

$b_{j,k}(x, \xi), b_k^0(x, \xi), b^0(x, \xi) \in S_{1,0}^0$ ($1 \leq j, k \leq n-1$), $b_k^1(x, \xi), b^1(x, \xi) \in S_{1,0}^1$ ($1 \leq k \leq n-1$) and $b^2(x, \xi) \in S_{1,0}^2$. We choose $\Phi(t) \in C^\infty(\mathbf{R})$ so that $0 \leq \Phi \leq 1$, $\Phi(t) = 1$ for $t \geq 6$ and $\Phi(t) = 0$ for $t \leq 4$, and put

$$(3.5) \quad w(x_n, \xi) = (X_n^2 + \kappa^{-2} F'(\langle \xi \rangle^{1/\kappa})^2 \langle \xi \rangle^{2/\kappa-2}) \langle \xi \rangle^2,$$

$$\chi_\omega(x_n, \xi) = \Phi(|\xi'|^2/w(x_n, \xi)).$$

Moreover, we fix a positive constant κ' so that $\kappa' = \kappa$ if $* = (\kappa)$, $\kappa' > \kappa$ if $* = \{\kappa\}$, and $\kappa < \kappa' < 2$ if $* = \{\kappa\}$ and $(3/2 \leq) \kappa < 2$. Then we have

$$|(D_{x_n}^j X_n^2) \langle \xi \rangle^2|/w(x_n, \xi) \leq \begin{cases} Cw(x_n, \xi)^{-1/2} \langle \xi \rangle \leq C' \langle \xi \rangle^{1-1/\kappa'} & \text{if } j = 1, \\ C_j(w(x_n, \xi)^{-1} \langle \xi \rangle^2 + w(x_n, \xi)^{-1/2} \langle \xi \rangle) \\ \leq C_j' \langle \xi \rangle^{2(1-1/\kappa')} & \text{if } j \geq 2. \end{cases}$$

Therefore, there are positive constants $C_{j,k} \equiv C_{\omega,j,k}$ ($j, k \in \mathbf{Z}_+$) such that

$$|\partial_\xi^\alpha D_{x_n}^j w(x_n, \xi)|/w(x_n, \xi) \leq C_{j,|\alpha|} \langle \xi \rangle^{j(1-1/\kappa')-|\alpha|}$$

for $x_n \in \mathbf{R}$, $\xi \in \mathbf{R}^n$, $j \in \mathbf{Z}_+$ and $\alpha \in (\mathbf{Z}_+)^n$. Moreover, there are positive constants R_ω and $\hat{C}_{j,k}$ ($j, k \in \mathbf{Z}_+$) such that the $\hat{C}_{j,k}$ are independent of ε and

$$|\partial_\xi^\alpha D_{x_n}^j w(x_n, \xi)|/w(x_n, \xi) \leq \hat{C}_{j,|\alpha|} (\kappa/\varepsilon)^j \langle \xi \rangle^{j(1-1/\kappa)-|\alpha|}$$

for $|\xi| \geq R_\omega$ if $\omega(\xi) = \varepsilon \langle \xi \rangle^{1/\kappa}$, since $w(x_n, \xi) = (X_n^2 \langle \xi \rangle^2 + (\varepsilon/\kappa)^2 \langle \xi \rangle^{2/\kappa})$. Therefore, by induction we have

$$|\partial_\xi^\alpha D_{x_n}^j w(x_n, \xi)^{-1}| \leq C_{j,|\alpha|} \langle \xi \rangle^{j(1-1/\kappa')-|\alpha|} w(x_n, \xi)^{-1},$$

for $|\xi| \geq R_\omega$, and

$$|\partial_\xi^\alpha D_{x_n}^j w(x_n, \xi)^{-1}| \leq \hat{C}_{j,|\alpha|} (\kappa/\varepsilon)^j \langle \xi \rangle^{j(1-1/\kappa)-|\alpha|} w(x_n, \xi)^{-1},$$

for $|\xi| \geq R_\omega$ if $\omega(\xi) = \varepsilon \langle \xi \rangle^{1/\kappa}$, modifying the $C_{j,k}$, the $\hat{C}_{j,k}$ and R_ω if necessary (see, e.g., Lemma 3.2 of [11]). Since $|\partial_\xi^\alpha |\xi'|^2|/w(x_n, \xi) \leq Cw(x_n, \xi)^{-|\alpha|/2}$ if $d\chi_\omega(x_n, \xi) \neq 0$, and $|\partial_\xi^\alpha |\xi'|^2|/w(x_n, \xi) = 0$ if $3\alpha_n + |\alpha'| \geq 3$, we have

$$(3.6) \quad |\partial_\xi^\alpha D_{x_n}^j \chi_\omega(x_n, \xi)| \leq C_{j,|\alpha|} \langle \xi \rangle^{j(1-1/\kappa')-|\alpha'|/\kappa'-\alpha_n}$$

for $|\xi| \geq R_\omega$, and

$$|\partial_\xi^\alpha D_{x_n}^j \chi_\omega(x_n, \xi)| \leq \hat{C}_{j,|\alpha|} (\kappa/\varepsilon)^{j+|\alpha'|} \langle \xi \rangle^{j(1-1/\kappa)-|\alpha'|/\kappa-\alpha_n}$$

for $|\xi| \geq R_\omega$ if $\omega(\xi) = \varepsilon \langle \xi \rangle^{1/\kappa}$, modifying the $C_{j,k}$, the $\hat{C}_{j,k}$ and R_ω if necessary. Define the Riemannian metric $g_{\kappa'}$ on $\mathbf{R}^{2n}$ by

$$g_{\kappa'(x,\xi)}(y,\eta) = |y'|^2 + \langle \xi \rangle^{2(1-1/\kappa')} y_n^2 + \langle \xi \rangle^{-2/\kappa'} |\eta'|^2 + \langle \xi \rangle^{-2} \eta_n^2,$$

where $y' = (y_1, \dots, y_{n-1})$ for $y = (y_1, \dots, y_n) \in \mathbf{R}^n$. Then one can easily show that $g_{\kappa'}$ is σ temperate in the sense of Hörmander (see, e.g., [8]). Note that

$$g_{\kappa'(x,\xi)}(y,\eta)/g_{\kappa'(x,\xi)}^\sigma(y,\eta) = \langle \xi \rangle^{-2/\kappa'}.$$

(3.6) implies that $\chi_\omega(x_n, \xi) \in S(1, g_{\kappa'})$. From (2.1), (3.1), (3.3), (3.5) and the assumption on $a(x, \xi)$ there are positive constants $c_\omega^0, R_\omega^0, C_{\alpha,\beta} \equiv C_{\omega,\alpha,\beta}$ ($\alpha, \beta \in (\mathbf{Z}_+)^n$), $A \equiv A_\omega$ and $B \equiv B_\omega$ such that

$$(3.7) \quad |q_\omega(x, \xi)| \geq |q_\omega^1(x, \xi)| \geq |\xi'|^2/2 - |a(x, \xi)| - A|\xi'| - B\langle \xi \rangle^{1/\kappa} \\ \geq (|\xi'|^2 + w(x_n, \xi))/4 \geq c_\omega^0 \langle \xi \rangle^{2/\kappa'},$$

$$|q_{\omega(\beta)}^{(\alpha)}(x, \xi)|/|q_\omega(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{\beta_n(1-1/\kappa') - |\alpha'|/\kappa' - \alpha_n}$$

if $|\xi'|^2 \geq 2w(x_n, \xi)$ and $|\xi| \geq R_\omega^0$, and if $\varepsilon \geq 4\sqrt{2}(\sup_{(y,\eta) \in \mathbf{R}^{2n}} |a(y, \eta)|/\langle \eta \rangle)^{1/2}$ in the case where $\omega(\xi) = \varepsilon \langle \xi \rangle^{1/2}$. Indeed, we have

$$|\ell_{\omega(\beta)}^{(\alpha)}(x, \xi)|/|q_\omega(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{\beta_n(1-1/\kappa') - |\alpha'|/\kappa' - \alpha_n},$$

$$|a(x, \xi)|/|q_\omega(x, \xi)| \leq C,$$

$$|\partial_\xi^\alpha D_x^\beta \{\alpha_n(x, \xi) X_n\}|/|q_\omega(x, \xi)| \leq \begin{cases} C_{\alpha,\beta} \langle \xi \rangle^{-1/\kappa' - |\alpha|} & \text{if } \beta_n = 0, \\ C_{\alpha,\beta} \langle \xi \rangle^{1-2/\kappa' - |\alpha|} & \text{if } \beta_n \geq 1, \end{cases}$$

$$\left| \partial_\xi^\alpha D_x^\beta \sum_{k=1}^{n-1} \alpha_k(x, \xi) \xi_k \right| / |q_\omega(x, \xi)| \leq \begin{cases} C_{\alpha,\beta} \langle \xi \rangle^{-1/\kappa' - \alpha_n} & \text{if } \alpha' = 0, \\ C_{\alpha,\beta} \langle \xi \rangle^{1-2/\kappa' - |\alpha|} & \text{if } |\alpha'| \geq 1, \end{cases}$$

$$|\partial_\xi^\alpha D_x^\beta (\omega(\xi) b(x, \xi))|/|q_\omega(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{-1/\kappa' - |\alpha|}$$

if $|\xi'|^2 \geq 2w(x_n, \xi)$ and $|\xi| \geq R_\omega^0$, which gives (3.7). Assume that $\varepsilon \geq 4\sqrt{2} \times (\sup_{(x,\xi) \in \mathbf{R}^{2n}} |a(x, \xi)|/\langle \xi \rangle)^{1/2}$ in the case where $\omega(\xi) = \varepsilon \langle \xi \rangle^{1/2}$. Therefore, we can construct a parametrix of $q_\omega^w(x, D)$ in $\{(x, \xi); |\xi'| \geq 3w(x_n, \xi)\}$ and prove that for any $s, s' \in \mathbf{R}$ there is $C_{\omega,s,s'} > 0$ such that

$$(3.8) \quad \|\langle D \rangle^s \chi_\omega^w(x_n, D) u\| \leq C_{\omega,s,s'} (\|\langle D \rangle^{s-2/\kappa'} q_\omega^w(x, D) u\| + \|\langle D \rangle^{s'} u\|)$$

for $u \in C_0^\infty(\Omega)$ (see, e.g., Theorem 5.4 of [13]). It is easy to see that $|\xi'|^2 + w(x_n, \xi)$ is σ , $g_{\kappa'}$ temperate weight and $q_\omega(x, \xi) \in S(|\xi'|^2 + w(x_n, \xi), g_{\kappa'})$. We omit the proof as we shall not use this fact in the paper. Write

$$q_\omega^w(x, D)(1 - \chi_\omega^w(x_n, D))u \\ = (1 - \chi_\omega^w(x_n, D))q_\omega^w(x, D)u + q_{\omega,1}^w(x, D)u + q_{\omega,0}^w(x, D)u,$$

where $q_{\omega,1}(x, \xi) \in S(\langle \xi \rangle^{2-1/\kappa'}, g_{\kappa'}), q_{\omega,0}(x, \xi) \in S(1, g_{\kappa'})$ and $\text{supp } q_{\omega,1}(x, \xi) \subset \{(x, \xi); 4w(x_n, \xi) \leq |\xi'|^2 \leq 6w(x_n, \xi)\}$. Since

$$|q_{\omega(\beta+\beta^1)}^{(\alpha+\alpha^1)}(x, \xi) \chi_{\omega(\alpha+\beta^2)}^{(\beta+\alpha^2)}(x_n, \xi)| / |q_\omega(x, \xi)| \\ \leq C_{\alpha, \beta, \alpha^1, \beta^1, \alpha^2, \beta^2} \langle \xi \rangle^{-(|\alpha|+|\beta|)/\kappa' + (\beta_n^1 + \beta_n^2)(1-1/\kappa') - (|\alpha^1| + |\alpha^2|)/\kappa' - \alpha_n^1 - \alpha_n^2},$$

there are $C_{\alpha, \beta} \equiv C_{\omega, \alpha, \beta} > 0$ ($\alpha, \beta \in (\mathbf{Z}_+)^n$) such that

$$|\partial_\xi^\alpha D_x^\beta q_{\omega,1}(x, \xi)| / |q_\omega(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{-1/\kappa' + \beta_n(1-1/\kappa') - |\alpha^1|/\kappa' - \alpha_n}.$$

Therefore, there are $r_{\omega,j}(x, \xi) \in S(\langle \xi \rangle^{-(j+1)/\kappa'}, g_{\kappa'})$ ($1 \leq j < 2\kappa' - 1$) such that

$$\left(q_{\omega,1}(x, \xi) / q_\omega(x, \xi) + \sum_{1 \leq j < 2\kappa' - 1} r_{\omega,j}(x, \xi) \right) \# q_\omega(x, \xi) \equiv q_{\omega,1}(x, \xi)$$

(mod $S(1, g_{\kappa'})$). This gives, with some $C \equiv C_\omega > 0$,

$$(3.9) \quad \|q_\omega^w(x, D)(1 - \chi_\omega^w(x_n, D))u\| \leq C(\|q_\omega^w(x, D)u\| + \|u\|)$$

for $u \in C_0^\infty(\Omega)$. On the other hand, we have

$$(3.10) \quad \|q_\omega^w(x, D)(1 - \chi_\omega^w(x_n, D))u\|^2 \\ \geq (Q_\omega^w(x, D)(1 - \chi_\omega^w(x_n, D))u, (1 - \chi_\omega^w(x_n, D))u)_{L^2}$$

for $u \in C_0^\infty(\Omega)$. Recall that $\text{supp}(1 - \chi_\omega(x_n, \xi)) \subset \{(x, \xi); |\xi'|^2 \leq 6w(x_n, \xi)\}$. We assume that $|\xi'|^2 \leq 10w(x_n, \xi)$. Then there is $R_\omega \geq 2$ such that

$$|\xi'|^2 \leq 2\langle \xi \rangle^2 / 5 + 10\kappa^{-2}\omega(\xi)^2 \leq \langle \xi \rangle^2 / 2 \quad \text{if } |\xi| \geq R_\omega.$$

We also assume that $|\xi| \geq R_\omega$. Then we have $|\xi_n| \geq \langle \xi \rangle / 2$ and $4w_0(x_n, \xi) \geq w(x_n, \xi)\langle \xi \rangle^{-2}$. From (3.4), (3.5), the properties of $F(t)$ and the assumption on $a(x, \xi)$ there are positive constants $c_\omega^0, R_\omega^0, C_{\alpha, \beta} \equiv C_{\omega, \alpha, \beta}$ ($\alpha, \beta \in (\mathbf{Z}_+)^n$), A and $B \equiv B_\omega$ such that

$$(3.11) \quad Q_\omega(x, \xi) \geq 2^{-4}\kappa^{-2}w(x_n, \xi)(F'(\langle \xi \rangle^{1/\kappa})\langle \xi \rangle^{1/\kappa} + F''(\langle \xi \rangle^{1/\kappa})\langle \xi \rangle^{2/\kappa}) \\ - AK(a)w(x_n, \xi)^{1/2}\langle \xi \rangle - Bw(x_n, \xi)$$

$$\geq 2^{-5}\kappa^{-2}w(x_n, \xi)F'(\langle \xi \rangle^{1/\kappa})\langle \xi \rangle^{1/\kappa} - AK(a)w(x_n, \xi)^{1/2}\langle \xi \rangle \\ \geq 2^{-6}\kappa^{-2}w(x_n, \xi)F'(\langle \xi \rangle^{1/\kappa})\langle \xi \rangle^{1/\kappa} \geq c_\omega^0\langle \xi \rangle^{3/\kappa'},$$

$$(3.12) \quad |Q_{\omega(\beta)}^{(\alpha)}(x, \xi)| / Q_\omega(x, \xi) \leq C_{\alpha, \beta} \langle \xi \rangle^{\beta_n(1-1/\kappa') - |\alpha^1|/\kappa' - \alpha_n},$$

where $K(a) = \sup_{(y, \eta) \in \mathbf{R}^{2n}} \{ \sum_{k=1}^{n-1} |\operatorname{Im}(\partial a / \partial x_k)(y, \eta)| / \langle \eta \rangle + |(\partial a / \partial \xi_n)(y, \eta)| \}$, if $x \in \Omega_1$ and $|\xi| \geq R_\omega^0$, and if $\varepsilon \geq 2^4(2AK(a))^{1/2}$ in the case where $\omega(\xi) = \varepsilon \langle \xi \rangle^{1/2}$. Here we have used the fact that $K(a) = 0$ if $\kappa > 2$ or if $* = \{2\}$. Moreover, we can easily prove that

$$\begin{aligned} |\partial_\xi^\alpha D_{x_n}^j \{w_0(x_n, \xi) \langle \xi \rangle^{-2} \xi^4\} / w(x_n, \xi)| &\leq C_{j, |\alpha|} \langle \xi \rangle^{j(1-1/\kappa') - |\alpha|}, \\ |\partial^\alpha \{F^{(k)}(\langle \xi \rangle^{1/\kappa}) \langle \xi \rangle^{k/\kappa}\} / \omega(\xi)| &\leq C_{|\alpha|} \langle \xi \rangle^{-|\alpha|}, \\ |\partial_\xi^\alpha \xi_k / w_n(x_n, \xi)|^{1/2} &\leq C_{|\alpha|} \langle \xi \rangle^{-|\alpha'|/\kappa'} \quad (1 \leq k \leq n-1), \\ |D_{x_n}^j X_n / w(x_n, \xi)|^{1/2} &\leq C_j \langle \xi \rangle^{-1+j(1-1/\kappa')}. \end{aligned}$$

Now assume that $\varepsilon \geq 2^4(2AK(a))^{1/2}$ in the case where $\omega(\xi) = \varepsilon \langle \xi \rangle^{1/2}$. It follows from (3.11), (3.12) and induction that

$$\begin{aligned} Q_\omega^{1/2}(x, \xi) &\geq \sqrt{c_\omega^0} \langle \xi \rangle^{3/(2\kappa')}, \\ |Q_{\omega(\beta)}^{1/2(\alpha)}(x, \xi)| / Q_\omega^{1/2}(x, \xi) &\leq C_{\alpha, \beta} \langle \xi \rangle^{\beta_n(1-1/\kappa') - |\alpha'|/\kappa' - \alpha_n}, \\ |\partial_\xi^\alpha D_x^\beta Q_\omega^{1/2}(x, \xi)^{-1}| &\leq C_{\alpha, \beta} \langle \xi \rangle^{\beta_n(1-1/\kappa') - |\alpha'|/\kappa' - \alpha_n} / Q_\omega^{1/2}(x, \xi) \end{aligned}$$

for $x \in \Omega_1$ and $|\xi| \geq R_\omega^0$, with modifications of $\{C_{\alpha, \beta}\}$ if necessary, where $Q_\omega^{1/2}(x, \xi) = (Q_\omega(x, \xi))^{1/2}$. Let $\psi_j(x)$ ($j=1, 2$) be functions in $C_0^\infty(\Omega_1)$ satisfying $\psi_1(x) = 1$ near $\bar{\Omega}$ and $\psi_2(x) = 1$ near $\operatorname{supp} \psi_1$. Moreover, we choose $\Phi_0(t) \in C^\infty(\mathbf{R})$ so that $0 \leq \Phi_0 \leq 1$, $\Phi_0(t) = 1$ for $t \geq 3/2$ and $\Phi_0(t) = 0$ for $t \leq 1$. Put $\chi_\omega^j(x_n, \xi) = \Phi_0(|\xi'|^2/w(x_n, \xi) - 5 - j)$ and $\Psi_j(x, \xi) = (1 - \chi_\omega^j(x_n, \xi))\psi_j(x)$ for $j=1, 2$. Then $\chi_\omega^j(x_n, \xi) = 1$ if $|\xi'|^2/w(x_n, \xi) \geq 13/2 + j$, and $\chi_\omega^j(x_n, \xi) = 0$ if $|\xi'|^2/w(x_n, \xi) \leq 6 + j$. Moreover, we have $\chi_\omega^j(x_n, \xi) \in S(1, g_{\kappa'})$ ($j=1, 2$) and there is a symbol $s_\omega(x, \xi)$ such that $\operatorname{supp} s_\omega(x, \xi) \subset \mathcal{C}$ and

$$\begin{aligned} |s_{\omega(\beta)}^{(\alpha)}(x, \xi)| &\leq C_{\alpha, \beta} \langle \xi \rangle^{-1/\kappa' + \beta_n(1-1/\kappa') - |\alpha'|/\kappa' - \alpha_n} / Q_\omega^{1/2}(x, \xi) \quad ((x, \xi) \in \mathcal{C}), \\ (3.13) \quad S_\omega(x, \xi) \# \tilde{Q}_\omega^{1/2}(x, \xi) &\equiv \Psi_1(x, \xi) \pmod{S^{-\infty}}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{C} &= \{(x, \xi) \in \Omega_1 \times \mathbf{R}^n; |\xi| \geq R_\omega^0 \text{ and } |\xi'|^2 \leq 10w(x_n, \xi)\}, \\ \tilde{Q}_\omega^{1/2}(x, \xi) &= Q_\omega^{1/2}(x, \xi) \Phi_0(|\xi|/R_\omega^0) \Psi_2(x, \xi), \\ S_\omega(x, \xi) &= Q_\omega^{1/2}(x, \xi)^{-1} \Phi_0(|\xi|/R_\omega^0) \Psi_1(x, \xi) + s_\omega(x, \xi) \in S(\langle \xi \rangle^{-3/(2\kappa')}, g_{\kappa'}). \end{aligned}$$

From the Weyl calculus there is a symbol $h_\omega(x, \xi)$ such that $\operatorname{supp} h_\omega(x, \xi) \subset \mathcal{C}$ and

$$(3.14) \quad |h_{\omega(\beta)}^{(\alpha)}(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{-1/\kappa' + \beta_n(1-1/\kappa') - |\alpha'|/\kappa' - \alpha_n} Q_{\omega}(x, \xi) \quad ((x, \xi) \in \mathcal{C}),$$

$$\tilde{Q}_{\omega}^{1/2}(x, \xi) \# \tilde{Q}_{\omega}^{1/2}(x, \xi) = Q_{\omega}(x, \xi) \Psi_2(x, \xi)^2 + h_{\omega}(x, \xi).$$

We note that $Q_{\omega}^{1/2}(x, \xi)^{\pm 1}$ and $Q_{\omega}(x, \xi)^{\pm 1}$ play roles of weight functions in our calculation. From (3.13) and (3.14) we have

$$\overline{S_{\omega}(x, \xi)} \# Q_{\omega}(x, \xi) \# S_{\omega}(x, \xi) \equiv \Psi_1(x, \xi) \# \Psi_1(x, \xi) \pmod{S(\langle \xi \rangle^{-1/\kappa'}, g_{\kappa'})},$$

since $\overline{S_{\omega}(x, \xi)} - S_{\omega}(x, \xi) = \overline{s_{\omega}(x, \xi)} - s_{\omega}(x, \xi)$ and $\Psi_1(x, \xi) \Psi_2(x, \xi) = \Psi_1(x, \xi)$. Therefore, this gives

$$(3.15) \quad (Q_{\omega}^w(x, D)(1 - \chi_{\omega}^w(x_n, D))u, (1 - \chi_{\omega}^w(x_n, D))u)_{L^2}$$

$$\geq (Q_{\omega}^w(x, D) \Psi_1^w(x, D)(1 - \chi_{\omega}^w(x_n, D))u, \Psi_1^w(x, D)(1 - \chi_{\omega}^w(x_n, D))u)_{L^2}$$

$$- C\|u\|^2$$

$$\geq (S_{\omega}^w(x, D)^* Q_{\omega}^w(x, D) S_{\omega}^w(x, D) \tilde{Q}_{\omega}^{1/2, w}(x, D)(1 - \chi_{\omega}^w(x_n, D))u,$$

$$\tilde{Q}_{\omega}^{1/2, w}(x, D)(1 - \chi_{\omega}^w(x_n, D))u)_{L^2} - C'\|u\|^2$$

$$\geq \|\Psi_1^w(x, D) \tilde{Q}_{\omega}^{1/2, w}(x, D)(1 - \chi_{\omega}^w(x_n, D))u\|^2$$

$$- C_1 \|\langle D \rangle^{-1/(2\kappa')} \tilde{Q}_{\omega}^{1/2, w}(x, D)(1 - \chi_{\omega}^w(x_n, D))u\|^2 - C'\|u\|^2$$

$$\geq \|\tilde{Q}_{\omega}^{1/2, w}(x, D)(1 - \chi_{\omega}^w(x_n, D))u\|^2/2 - C''\|u\|^2$$

for $u \in C_0^{\infty}(\Omega)$, since $\Psi_1(x, \xi)(1 - \chi_{\omega}(x_n, \xi)) = (1 - \chi_{\omega}(x_n, \xi))\psi_1(x)$. On the other hand, by (3.13) we have

$$\|\langle D \rangle^{3/(2\kappa')} (1 - \chi_{\omega}^w(x_n, D))u\|$$

$$\leq \|\langle D \rangle^{3/(2\kappa')} S_{\omega}^w(x, D) \tilde{Q}_{\omega}^{1/2, w}(x, D)(1 - \chi_{\omega}^w(x_n, D))u\| + C\|u\|$$

$$\leq C_0 \|\tilde{Q}_{\omega}^{1/2, w}(x, D)(1 - \chi_{\omega}^w(x_n, D))u\| + C\|u\|$$

for $u \in C_0^{\infty}(\Omega)$. This, together with (3.10) and (3.15), yields

$$(3.16) \quad \|q_{\omega}^w(x, D)(1 - \chi_{\omega}^w(x_n, D))u\|^2$$

$$\geq (2C_0^2)^{-1} \|\langle D \rangle^{3/(2\kappa')} (1 - \chi_{\omega}^w(x_n, D))u\|^2 - C\|u\|$$

for $u \in C_0^{\infty}(\Omega)$. Therefore, it follows from (3.8), (3.9) and (3.16) that

$$\|\langle D \rangle^{3/(2\kappa')} u\| \leq C(\|q_{\omega}^w(x, D)u\| + \|u\|)$$

for $u \in C_0^{\infty}(\Omega)$, where $C \equiv C_{\omega} > 0$. This gives the estimates (2.8) with $m = \rho = 2$ and ${}^t p_{\omega}^{\rho}(x, D)$ replaced by ${}^t \tilde{L}_{\omega}(x, D)$ and completes the proof of Theorem 1.3.

4. Proof of Theorem 1.4

In this section we shall prove Theorem 1.4, applying the same arguments as used in Cardoso and Treves [2], Ivrii and Petkov [10] and Ivrii [9]. First we shall prove the assertion (i) of Theorem 1.4. Let $\kappa > 2$, and assume that $P(x, D)$ is locally solvable at the origin in $\mathcal{D}^{*f}$. Then it follows from Theorem 2.5 and Poincaré's inequality that there are an open neighborhood U of the origin and positive constants δ and C such that

$$|u(0)| \leq C \|e^{\delta \langle D \rangle^{1/\kappa}} {}^tP(x, D)u\|$$

for any $u \in \mathcal{D}^{(\kappa)}(U)$, since we have $|u(0)| + \|\langle D \rangle u\| \leq C \|\langle D \rangle^{(n+1)/2} u\| \leq C_\varepsilon \times \|e^{\varepsilon \langle D \rangle^{1/\kappa}} u\|$ for a fixed $\varepsilon > 0$. Making an asymptotic change of variables $y = (x', \rho x_n)$ ($\rho \geq 1$), we have

$$(4.1) \quad |v(0)| \leq C \rho^{-1/2} \|\exp[\delta \langle (D', \rho D_n) \rangle^{1/\kappa}] {}^tP_\rho(y, D)v\|$$

for every $v(y) \in \mathcal{D}^{(\kappa)}(W)$ and $\rho \geq \rho_0(W)$, where W is an open bounded neighborhood of the origin in $\mathbf{R}^n$, $x' = (x_1, \dots, x_{n-1})$ for $x = (x_1, \dots, x_n)$, $D' = (D_1, \dots, D_{n-1})$, $\{(y', \rho^{-1}y_n); y \in W\} \subset U$ for $\rho \geq \rho_0(W)$ and

$$\begin{aligned} P_\rho(y, \eta) &= |\eta'|^2 + y_n^2 \eta_n^2 - y_n a(y', \rho^{-1}y_n) \eta_n \\ &\quad - \rho(1 + 2iy_1 + y_1^2 b(y', \rho^{-1}y_n)) \eta_n - \sum_{k=1}^{n-1} c_k(y', \rho^{-1}y_n) \eta_k + d(y', \rho^{-1}y_n). \end{aligned}$$

Indeed, for $f \in \mathcal{D}^{(\kappa)}(U)$ we have

$$\begin{aligned} \|e^{\delta \langle D \rangle^{1/\kappa}} f\|^2 &= \rho^{-2} (2\pi)^{-n} \int e^{2\delta \langle \xi \rangle^{1/\kappa}} |\mathcal{F}_y[f(y; \rho)](\xi', \rho^{-1}\xi_n)|^2 d\xi \\ &= \rho^{-1} \|\exp[\delta \langle (D', \rho D_n) \rangle^{1/\kappa}] f(y; \rho)\|^2, \end{aligned}$$

where $f(y; \rho) = f(y', \rho^{-1}y_n)$. This gives (4.1). Putting

$$\tilde{P}_\rho(y, \eta) = \sigma({}^tP_\rho(y, D))(y, \eta),$$

we have

$$\begin{aligned} \tilde{P}_\rho(y, \eta) &= |\eta'|^2 + y_n^2 \eta_n^2 + y_n a_\rho(y) \eta_n \\ &\quad + \rho(1 + 2iy_1 + y_1^2 b(y', 0)) \eta_n + \sum_{k=1}^{n-1} c_k(y', \rho^{-1}y_n) \eta_k \\ &\quad + y_1^2 b_\rho(y) \eta_n + d_\rho(y), \end{aligned}$$

where $a_\rho(y) = a(y', \rho^{-1}y_n) - 4i$, $b_\rho(y) = \rho(b(y', \rho^{-1}y_n) - b(y', 0))$ and $d_\rho(y) =$

$d(y', \rho^{-1}y_n) - 2 - i\partial_n(y_n a(y', \rho^{-1}y_n)) - iy_1^2(\partial_n b)(y', \rho^{-1}y_n) - i\sum_{k=1}^{n-1} \partial_k c_k(y', \rho^{-1}y_n)$.
Consider the Cauchy problem

$$(4.2) \quad \begin{cases} \sum_{k=1}^{n-1} (\partial_k \varphi(y))^2 + y_n^2 (\partial_n \varphi(y))^2 + (1 + 2iy_1 + y_1^2 b(y', 0)) \partial_n \varphi(y) = 0, \\ \varphi(0, y'') = -y_n + i|y''|^2/4, \end{cases}$$

where $y'' = (y_2, \dots, y_n)$ for $y = (y_1, \dots, y_n)$. Putting $\varphi(y) = \sum_{j=0}^{\infty} \varphi_j(y'') y_1^j / j!$ in (4.2), we have

$$\varphi_0(y'') = -y_n + i|y''|^2/4,$$

$$\begin{aligned} 2\varphi_1(y'')\varphi_{j+1}(y'') &= -\sum_{\ell=1}^{j-1} \binom{j}{\ell} \varphi_{\ell+1}(y'')\varphi_{j+1-\ell}(y'') \\ &\quad - \sum_{k=2}^{n-1} \sum_{\ell=0}^j \binom{j}{\ell} (\partial_k \varphi_{\ell})(y'') (\partial_k \varphi_{j-\ell})(y'') \\ &\quad - y_n^2 \sum_{\ell=0}^j \binom{j}{\ell} (\partial_n \varphi_{\ell})(y'') (\partial_n \varphi_{j-\ell})(y'') - (\partial_n \varphi_j)(y'') \\ &\quad - 2ji(\partial_n \varphi_{j-1})(y'') \\ &\quad + \sum_{\ell=0}^{j-2} j(j-1) \binom{j-2}{\ell} (\partial_1^{\ell} b)(0, y_2, \dots, y_{n-1}, 0) (\partial_n \varphi_{j-\ell-2})(y'') \end{aligned}$$

($j \in \mathbb{N}$), formally, where $\binom{j}{\ell} = j! / (\ell!(j-\ell)!)$. In particular, we have

$$\begin{aligned} \varphi_1(y'') &= \left(1 - iy_n/2 + \sum_{k=2}^{n-1} y_k^2/4 - y_n^2 + iy_n^3 + y_n^4/4 \right)^{1/2} \\ &= 1 - iy_n/4 + O(|y''|^2) \quad \text{as } y'' \rightarrow 0. \end{aligned}$$

Therefore, (4.2) has a unique analytic solution $\varphi(y)$ in an open neighborhood W_0 of the origin. We may assume that $\partial_1 \varphi(y) \neq 0$ in W_0 . Since

$$\varphi_2(y'') = 9i/8 + O(|y''|) \quad \text{as } y'' \rightarrow 0,$$

there are $c_0 > 0$ and $C > 0$ such that $\{y; |y| \leq 2c_0\} \subset W_0$ and

$$\varphi(y) = y_1 - y_n + i|y''|^2/4 - iy_1 y_n/4 + 9iy_1^2/16 + O(y_1 |y|^2) \quad \text{as } y \rightarrow 0,$$

$$\operatorname{Im} \varphi(y) \geq 5y_1^2/16 + \sum_{k=2}^{n-1} y_k^2/4 + 3y_n^2/16 - Cy_1 |y|^2 \geq |y|^2/8 \quad \text{if } |y| \leq c_0.$$

Define $T_{\rho, \varphi}(y, D)$ by

$$\tilde{P}_\rho(y, D)(e^{i\rho\varphi(y)}v(y)) = e^{i\rho\varphi(y)}T_{\rho, \varphi}(y, D)v(y).$$

Then $T_{\rho, \varphi}(y, D)$ can be written as

$$T_{\rho, \varphi}(y, \eta) = 2(\partial_1 \varphi)(y) \left[\rho \left(\eta_1 + \sum_{k=2}^n d_k(y) \eta_k + d_0(y) \right) - \sum_{k=1}^n (q_k^0(y; \rho) \eta_k^2 + q_k^1(y; \rho) \eta_k) - q_0(y; \rho) \right],$$

where the $d_k(y)$, the $q_k^0(y; \rho)$, the $q_k^1(y; \rho)$ and $q_0(y; \rho)$ satisfy, with some positive constants C and A ,

$$|D^\alpha d_k(y)| \leq CA^{|\alpha|} |\alpha|! \quad (k = 0, 2, 3, \dots, n),$$

$$\sum_{j=1}^2 |D^\alpha q_k^j(y; \rho)| \leq CA^{|\alpha|} |\alpha|! \quad (1 \leq k \leq n),$$

$$|D^\alpha q_0(y; \rho)| \leq CA^{|\alpha|} |\alpha|!$$

if $|y| \leq 2c_0$ and $\rho \geq \rho_0 \equiv \rho_0(W_0)$. Assume that $\rho \geq \rho_0$, and define inductively $\{v_j(y; \rho)\}_{j \in \mathbf{Z}_+}$ by the Cauchy problems

$$\begin{cases} D_1 v_j(y; \rho) + \sum_{k=2}^n d_k(y) D_k v_j(y; \rho) + d_0(y) v_j(y; \rho) \\ \quad = \sum_{k=1}^n (q_k^0(y; \rho) D_k^2 + q_k^1(y; \rho) D_k) v_{j-1}(y; \rho) + q_0(y; \rho) v_{j-1}(y; \rho), \\ v_j(0, y''; \rho) = \delta_{0,j} \end{cases}$$

($j \in \mathbf{Z}_+$), where $v_{-1}(y; \rho) \equiv 0$. Let us consider the Cauchy problem

$$(4.3) \quad \begin{cases} D_1 w(y) + \sum_{k=2}^n d_k(y) D_k w(y) + d_0(y) w(y) = f(y), \\ w(0, y'') = h_0 \in \mathbf{C}, \end{cases}$$

where $f(y)$ ($\equiv f(y; \rho)$) is a real analytic function near the origin. Put

$$\Phi(\theta) = \sum_{\alpha=(\alpha_1, \alpha'') \in (\mathbf{Z}_+)^n} \theta_1^{\alpha_1} \theta_2^{|\alpha''|} |(\partial^\alpha w)(0)| / \alpha!,$$

$$\Psi(\theta) = \sum_{\alpha} \theta_1^{\alpha_1} \theta_2^{|\alpha''|} |(\partial^\alpha f)(0)| / \alpha!,$$

where $\theta = (\theta_1, \theta_2)$. We say that

$$g(\theta) = \sum_{j,k=0}^{\infty} g_{j,k} \theta_1^j \theta_2^k \ll h(\theta) = \sum_{j,k=0}^{\infty} h_{j,k} \theta_1^j \theta_2^k$$

if $|g_{j,k}| \leq h_{j,k}$ for any $(j, k) \in (\mathbf{Z}_+)^2$. It is easy to see that there are positive constants A , a_0 and b_0 such that

$$(\partial\Phi/\partial\theta_1)(\theta) \ll A(1 - \theta_1/a_0 - \theta_2/b_0)^{-1}(\partial/\partial\theta_2 + 1)\Phi(\theta) + \Psi(\theta),$$

since

$$\begin{aligned} (\partial\Phi/\partial\theta_1)(\theta) &= \sum_{\alpha} \theta_1^{\alpha_1} \theta_2^{|\alpha''|} |(\partial^{\alpha} \partial_1 w)(0)|/\alpha!, \\ (\partial\Phi/\partial\theta_2)(\theta) &= \sum_{k=2}^n \sum_{\alpha} \theta_1^{\alpha_1} \theta_2^{|\alpha''|} |(\partial^{\alpha} \partial_k w)(0)|/\alpha! \end{aligned}$$

(see §2 of [9]).

Lemma 4.1. *Assume that*

$$\Psi(\theta) \ll (jC_1\Theta_{a,b} + C_2M)\Theta_{a,b}^j e^{M\theta_1},$$

$$\Phi^*(\theta) = C_*\Theta_{a,b}^j e^{M\theta_1},$$

where $\Theta_{a,b} = (1 - \theta_1/a - \theta_2/b)^{-1}$, a, b, C_1, C_2, C_* and M are positive constants and $j \in \mathbf{N}$. Fix μ_0 so that $0 < \mu_0 < 1$. Then we have $\Phi(\theta) \ll \Phi^*(\theta)$ if $a \leq \mu_0 a_0$, $b \leq \mu_0 b_0$,

$$(4.4) \quad \begin{cases} a/b \leq (1 - \mu_0)/(2A), & M \geq 2(1 - \mu_0)^{-1}A, \\ C_* \geq \max\{h_0, 2aC_1, 2a(jC_1 + C_2M)/(j + aM)\}. \end{cases}$$

Proof. We shall repeat the same argument as in §2 of [9]. It is obvious that $\Phi(\theta) \ll \Phi^*(\theta)$ if

$$(4.5) \quad (\partial\Phi^*/\partial\theta_1)(\theta) \gg A(1 - \theta_1/a_0 - \theta_2/b_0)^{-1}(\partial/\partial\theta_2 + 1)\Phi^*(\theta) + \Psi(\theta),$$

$$(4.6) \quad \Phi^*(0, \theta_2) \gg \Phi(0, \theta_2) \quad (= h_0).$$

Assume that $a \leq \mu_0 a_0$ and $b \leq \mu_0 b_0$. Then we have

$$(1 - \theta_1/a_0 - \theta_2/b_0)^{-1}\Theta_{a,b} \ll \sum_{j,k=0}^{\infty} \mu_0^j (\theta_1/a + \theta_2/b)^{j+k} \ll (1 - \mu_0)^{-1}\Theta_{a,b}.$$

Therefore, we have

$$\begin{aligned} (\partial\Phi^*/\partial\theta_1)(\theta) &= (jC_*\Theta_{a,b}/a + MC_*)\Theta_{a,b}^j e^{M\theta_1}, \\ A(1 - \theta_1/a_0 - \theta_2/b_0)^{-1}(\partial/\partial\theta_2 + 1)\Phi^*(\theta) + \Psi(\theta) \\ &\ll \{((1 - \mu_0)^{-1}AjC_*/b + jC_1)\Theta_{a,b} + (1 - \mu_0)^{-1}AC_* + C_2M\}\Theta_{a,b}^j e^{M\theta_1}. \end{aligned}$$

Thus (4.5) and (4.6) are valid if (4.4) is satisfied, since

$$(jC_*/a - (1 - \mu_0)^{-1}AjC_*/b - jC_1)\Theta_{a,b} \gg j(C_* - 2aC_1)/(2a).$$

This proves the lemma. □

Put

$$(4.7) \quad \Phi_j(\theta; \rho) = \sum_{\alpha} \theta_1^{\alpha_1} \theta_2^{|\alpha''|} |(\partial^{\alpha} v_j)(0; \rho)| / \alpha!,$$

$$\Psi_j(\theta; \rho) = \sum_{\alpha} \theta_1^{\alpha_1} \theta_2^{|\alpha''|} |(\partial^{\alpha} f_j)(0; \rho)| / \alpha!$$

($j \in \mathbf{Z}_+$), where

$$(4.9) \quad f_j(y; \rho) = \sum_{k=1}^n (q_k^0(y; \rho) D_k^2 + q_k^1(y; \rho) D_k) v_{j-1}(y; \rho) + q_0(y; \rho) v_{j-1}(y; \rho).$$

We fix positive constants a and b so that $a \leq 1/4$, $a \leq a_0/2$, $b \leq b_0/2$ and $a/b \leq 1/(4A)$. We may assume that $A \geq 1$. Let us prove, by induction, that

$$(4.9) \quad \begin{cases} \Psi_j(\theta; \rho) \ll (12A/a^2)^j \sum_{k=0}^{j+1} A^{j+1-k} k! \Theta_{a,b}^{k+1} e^{4A\theta_1}, \\ \Phi_j(\theta; \rho) \ll (12A/a^2)^j \sum_{k=0}^j A^{j-k} k! \Theta_{a,b}^{k+1} e^{4A\theta_1} =: \Phi_j^*(\theta) \end{cases}$$

($j \in \mathbf{Z}_+$), modifying A , a_0 and b_0 if necessary. It follows from (4.8) that

$$(4.10) \quad \Psi_j(\theta; \rho) \ll A(1 - \theta_1/a_0 - \theta_2/b_0)^{-1} \times (\partial^2/\partial\theta_1^2 + \partial^2/\partial\theta_2^2 + \partial/\partial\theta_1 + \partial/\partial\theta_2 + 1) \Phi_{j-1}^*(\theta)$$

for $j \in \mathbf{N}$, with modifications of A , a_0 and b_0 if necessary. Since $\Phi_0(0, \theta_2; \rho) \equiv 1$ and $\Psi_0(\theta; \rho) \equiv 0$, Lemma 4.1 gives

$$\Phi_0(\theta; \rho) \ll \Theta_{a,b} e^{4A\theta_1} = \Phi_0^*(\theta).$$

So (4.9) is valid for $j = 0$. Suppose that (4.9) is valid for $j = \ell$ (≥ 0). Then, by (4.10) we have

$$\begin{aligned} \Psi_{\ell+1}(\theta; \rho) &\ll A(1 - \theta_1/a_0 - \theta_2/b_0)^{-1} (12A/a^2)^{\ell} \sum_{k=0}^{\ell} A^{\ell-k} k! e^{4A\theta_1} \\ &\quad \times \{ (k+1)(k+2)(a^{-2} + b^{-2}) \Theta_{a,b}^{k+3} \\ &\quad + (k+1)(8A/a + a^{-1} + b^{-1}) \Theta_{a,b}^{k+2} + (16A^2 + 4A + 1) \Theta_{a,b}^{k+1} \} \\ &\ll 2A(12A/a^2)^{\ell} \left\{ 21A^{\ell+2} \Theta_{a,b} + A^{\ell-1} (21A^2 + 10A^2/a) \Theta_{a,b}^2 \right. \\ &\quad + \sum_{k=2}^{\ell} A^{\ell-k} k! (21A^2 + 10A^2/a + 2A^2/a^2) \Theta_{a,b}^{k+1} \\ &\quad \left. + (k+1)! (10A/a + 2A/a^2) \Theta_{a,b}^{\ell+2} + (\ell+2)! (2/a^2) \Theta_{a,b}^{\ell+3} \right\} e^{4A\theta_1} \\ &\ll (12A/a^2)^{\ell+1} \sum_{k=0}^{\ell+2} A^{\ell+2-k} k! \Theta_{a,b}^{k+1} e^{4A\theta_1}. \end{aligned}$$

It follows from Lemma 4.1 that

$$\begin{aligned}\Phi_{\ell+1}(\theta; \rho) &\ll (12A/a^2)^{\ell+1} \left\{ A^{\ell+1} \Theta_{a,b}/2 + \sum_{k=1}^{\ell+2} 2aA^{\ell+2-k} (k-1)! \Theta_{a,b}^k \right\} e^{4A\theta_1} \\ &\ll (12A/a^2)^{\ell+1} \sum_{k=0}^{\ell+1} A^{\ell+1-k} k! \Theta_{a,b}^{k+1} e^{4A\theta_1},\end{aligned}$$

which proves that (4.9) is valid for $j = \ell + 1$ and for any $j \in \mathbf{Z}_+$. Let $N \in \mathbf{N}$, and choose $\kappa_1 > 1$ and $\chi(y) \in \mathcal{D}^{(\kappa_1)}$ so that $\kappa_1 < \kappa/2$ and $\chi(y) = 1$ if $|y| \leq c_1/2$, and $\text{supp } \chi \subset \{y \in \mathbf{R}^n; |y| \leq c_1\}$, where c_1 is a positive constant satisfying $c_1 \leq \max\{c_0, a/4\}$. It follows from (4.9) that the $v_j(y; \rho)$ are defined and analytic in $\{y; |y| < a/2\}$. Put

$$\begin{aligned}v_\rho^{(N)}(y) &= e^{i\rho\varphi(y)} \sum_{j=0}^N \rho^{-j} v_j(y; \rho) \chi(y), \\ w_{\rho,1}^{(N)}(y) &= \sum_{j=0}^N \rho^{-j} \chi(y) T_{\rho,\varphi}(y, D) v_j(y; \rho), \\ w_{\rho,2}^{(N)}(y) &= \sum_{j=0}^N \rho^{-j} [T_{\rho,\varphi}(y, D), \chi(y)] v_j(y; \rho),\end{aligned}$$

where $[A, B] = AB - BA$ for operators A and B . Then we have

$$\begin{aligned}\tilde{P}_\rho(y, D) v_\rho^{(N)}(y) &= e^{i\rho\varphi(y)} (w_{\rho,1}^{(N)}(y) + w_{\rho,2}^{(N)}(y)), \\ w_{\rho,1}^{(N)}(y) &= -2\rho^{-N} (\partial_1 \varphi)(y) \chi(y) \\ &\quad \times \left\{ \sum_{k=1}^n (q_k^0(y; \rho) D_k^2 + q_k^1(y; \rho) D_k) + q_0(y; \rho) \right\} v_N(y; \rho).\end{aligned}$$

On the other hand, (4.7) and (4.9) give

$$\begin{aligned}|D^\alpha v_j(y; \rho)| &= \left| \sum_\gamma y^\gamma (\partial^{\alpha+\gamma} v_j)(0; \rho) / \gamma! \right| \leq (\partial^{|\alpha|} \Phi_j / \partial \theta_1^{z_1} \partial \theta_2^{|\alpha''|})(a/4, b/4) \\ &\leq (12A/a^2)^j \sum_{k=0}^j A^{j-k} k! \sum_{\mu=0}^{\alpha_1} \alpha_1! (k+1+\mu+|\alpha''|)! \\ &\quad \times (\mu! (\alpha_1 - \mu)! (k+1)!)^{-1} a^{-\mu} b^{-|\alpha''|} 2^{k+1+\mu+|\alpha''|} (4A)^{\alpha_1-\mu} e^{aA} \\ &\leq 6e^A (72A/a^2)^j (6/a)^{|\alpha|} j! |\alpha|!\end{aligned}$$

if $|y_1| \leq a/4$ and $|y_2| \leq b/4$, since $|\gamma''|!/\gamma'''! \geq |\beta''|!/\beta'''!$ for $\gamma'' \geq \beta''$, $(k+1+\mu+|\alpha''|)! \leq 3^{k+1+\mu+|\alpha''|}(k+1)!\mu!|\alpha''|!$, $\sum_{k=0}^j (k!/j!)(A/6)^{j-k} \leq e^{A/6}$ and $1/6 + 2a/3 + a < 1$. Therefore, for any $\Gamma > 0$ there is $C_\Gamma > 0$ such that

$$|D^\alpha w_{\rho,1}^{(N)}(y)| \leq \rho^{-N} C_\Gamma B_0^N \Gamma^{|\alpha|} N! |\alpha|!^{\kappa_1},$$

where $B_0 = 72A/a^2$. Note that $\text{supp } w_{\rho,2}^{(N)} \subset \{y; c_1/2 \leq |y| \leq c_1\}$ and

$$|e^{i\rho\varphi(y)}| \leq e^{-c_1^2\rho/32} \quad \text{on } \text{supp } w_{\rho,2}^{(N)}.$$

For any $\Gamma > 0$ there is $C_\Gamma > 0$ such that

$$|D^\alpha w_{\rho,2}^{(N)}(y)| \leq C_\Gamma \rho \Gamma^{|\alpha|} |\alpha|!^{\kappa_1} \sum_{j=0}^N j! (B_0/\rho)^j.$$

Now choose $N \equiv N_\rho \in \mathbf{N}$ so that $\rho/(2B_0) - 1 < N \leq \rho/(2B_0)$. Then we have

$$(4.11) \quad |e^{i\rho\varphi(y)} D^\alpha w_\rho^{(N)}(y)| \leq C_\Gamma e^{-c_2\rho} \Gamma^{|\alpha|} |\alpha|!^{\kappa_1}$$

for any $\Gamma > 0$, modifying C_Γ if necessary, where $w_\rho^{(N)}(y) = w_{\rho,1}^{(N)}(y) + w_{\rho,2}^{(N)}(y)$ and $c_2 = \min\{-(2B_0)^{-1}, c_1^2/64\}$, since $N! \leq c e^{-N} N^{N+1/2}$. Noting that ${}^tP_\rho(y, D)v_\rho^{(N)}(y) = e^{i\rho\varphi(y)} w_\rho^{(N)}(y)$, we have

$$(4.12) \quad |\exp[\delta \langle (\eta', \rho\eta_n) \rangle^{1/\kappa}] \mathcal{F}_y[{}^tP_\rho(y, D)v_\rho^{(N)}(y)](\eta)| \\ \leq C e^{-c_2\rho + 2^{1/\kappa}\delta\rho^{2/\kappa}} \leq C_{\kappa,\ell} \rho^{-\ell}$$

if $|\eta| \leq 2\rho$ and $\ell \in \mathbf{Z}_+$, where $C > 0$ and $C_{\kappa,\ell} > 0$. Modifying $c_1 > 0$ if necessary, we may assume that $|\eta - \rho\nabla\varphi(y)| \geq |\eta|/4 \geq \rho/2$ if $|y| \leq c_1$ and $|\eta| \geq 2\rho$. Put

$$K \equiv \sum_{k=1}^n h_k(y; \rho, \eta) D_{y_k} = |\eta - \rho\nabla\varphi(y)|^{-2} \sum_{k=1}^n (\rho \overline{\partial_k \varphi(y)} - \eta_k) D_{y_k}$$

for $|y| \leq c_1$ and $|\eta| \geq 2\rho$. Then, by induction we can see that there is $B_1 > 0$ such that

$$|D_y^\alpha |\eta - \rho\nabla\varphi(y)|^{-2}| \leq |\eta - \rho\nabla\varphi(y)|^{-2} B_1^{|\alpha|} |\alpha|!$$

if $|y| \leq c_1$ and $|\eta| \geq 2\rho$. Moreover, there is $\hat{C} > 0$ such that

$$(4.13) \quad |D_y^\alpha h_k(y; \rho, \eta)| \leq \hat{C} |\eta|^{-1} B_1^{|\alpha|} |\alpha|!$$

for $1 \leq k \leq n$, $|y| \leq c_1$ and $|\eta| \geq 2\rho$, with modifications of B_1 if necessary. It follows from (4.11) and (4.13) that for any $\Gamma > 0$ there is $K_\Gamma > 0$ such that

$$(4.14) \quad |e^{i\rho\varphi(y)} D_y^{\alpha,t} K^j w_\rho^{(N)}(y)| \leq C_\Gamma e^{-c_2\rho} \Gamma^{|\alpha|+j} (K_\Gamma/|\eta|)^j (|\alpha|+j)!^{\kappa_1}$$

if $|\eta| \geq 2\rho$, where c_2 and C_Γ are as in (4.11). Indeed, we can prove (4.14) by induction on j . In doing so, we choose $K_\Gamma > 0$ so that

$$K_\Gamma \geq n\hat{C} \sum_{k=0}^{\infty} k!^{1-\kappa_1} (B_1/\Gamma)^k.$$

By (4.14) we have

$$\begin{aligned} (4.15) \quad |\mathcal{F}_y[{}^tP_\rho(y, D)v_\rho^{(N)}(y)](\eta)| &\leq \int |e^{i\rho\varphi(y)t} K^j w_\rho^{(N)}(y)| dy \\ &\leq C e^{-c_2\rho} (K_1/|\eta|)^j j!^{\kappa_1} \\ &\leq C' \exp[-\kappa|\eta|^{1/\kappa_1}/(2K_1^{1/\kappa_1})] \end{aligned}$$

if $|\eta| \geq 2\rho$ and $(|\eta|/K_1)^{1/\kappa_1} - 1 < j \leq (|\eta|/K_1)^{1/\kappa_1}$. Therefore, (4.12) and (4.15) give

$$\|\exp[\delta\langle\eta', \rho\eta_n\rangle^{1/\kappa}] \mathcal{F}_y[{}^tP_\rho(y, D)v_\rho^{(N)}(y)](\eta)\| \leq C_{\kappa, \ell} \rho^{-\ell}$$

for any $\ell \in \mathbf{Z}_+$, where $C_{\kappa, \ell} > 0$. This contradicts (4.1) and $v_\rho^{(N)}(0) = 1$, which proves the assertion (i) of Theorem 1.4.

Next we shall prove the assertion (ii) of Theorem 1.4. Assume that $P(x, D)$ is locally solvable at the origin in $\mathcal{D}'$. Then it follows from Proposition 2.1 that there are an open neighborhood U of the origin, $\ell \in \mathbf{Z}_+$ and $C > 0$ such that

$$|u(0)| \leq C \sum_{|\alpha| \leq \ell} \|D^{\alpha t} P(x, D)u\| \quad \text{for every } u \in C_0^\infty(U).$$

Making an asymptotic change of variables $y = (\rho x', \rho^4 x_n)$ ($\rho \geq 1$), we have

$$(4.16) \quad |v(0)| \leq C \rho^{-(n+3)/2} \sum_{|\alpha| \leq \ell} \rho^{|\alpha'| + 4\alpha_n} \|D^{\alpha t} Q_\rho(y, D)v\|$$

for every $v \in C_0^\infty(W)$ and $\rho \geq \rho_1$, where $W = \{y \in \mathbf{R}^n; |y| < 1\}$, $\{(\rho^{-1}y', \rho^{-4}y_n); y \in W\} \subset U$ for $\rho \geq \rho_1$ and

$$\begin{aligned} Q_\rho(y, \eta) &= \rho^2 |\eta'|^2 + y_n^2 \eta_n^2 - y_n a(\rho^{-1}y', \rho^{-4}y_n) \eta_n \\ &\quad - (1 + 2i\rho^{-1}y_1 + \rho^{-2}y_1^2 b(\rho^{-1}y', \rho^{-4}y_n)) \rho^4 \eta_n \\ &\quad - \sum_{k=1}^{n-1} \rho c_k(\rho^{-1}y', \rho^{-4}y_n) \eta_k + d(\rho^{-1}y', \rho^{-4}y_n). \end{aligned}$$

Put

$$\tilde{\mathcal{Q}}_\rho(y, \eta) = \sigma({}^t\mathcal{Q}_\rho(y, D))(y, \eta),$$

$$\varphi(y; \rho) = y_1 - y_n + \rho^{-1}\psi(y), \quad \psi(y) = 9iy_1^2/16 - iy_1y_n/4 + i|y''|^2/4.$$

Then $\psi(y)$ satisfies

$$2\partial_1\psi(y) + \partial_n\psi(y) - 2iy_1 = 0, \quad \psi(0, y'') = i|y''|^2/4, \quad \text{Im } \psi(y) \geq |y|^2/8.$$

A simple calculation yields

$$\tilde{\mathcal{Q}}_\rho(y, D)(e^{i\rho^2\varphi(y; \rho)}v(y)) = e^{i\rho^2\varphi(y; \rho)}S_{\rho, \varphi}(y, D)v(y),$$

$$S_{\rho, \varphi} = \sum_{j=0}^{N-1} \rho^{4-j} S_j(y, \eta) + \rho^{4-j} S_N(y, \eta; \rho),$$

where $N \in \mathbb{N}$, the $S_j(y, \eta)$ are polynomials of y and η and satisfy $\deg_\eta S_j(y, \eta) \leq 2$,

$$S_0(y, \eta) = 2D_1 + D_n - b(0)y_1^2 - |y|^2/4 - 33y_1^2/64 + 19y_n^2/16 - 7y_1y_n/16$$

and $S_N(y, \eta; \rho)$ is written in the form

$$S_N(y, \eta; \rho) = \sum_{|\alpha| \leq 2} s_{N, \alpha}(y; \rho) \eta^\alpha$$

and satisfies

$$|\partial_y^\beta s_{N, \alpha}(y; \rho)| \leq C_{N, \beta} \quad \text{for } y \in W, \rho \geq \rho_1 \text{ and } \alpha, \beta \in (\mathbb{Z}_+)^n \text{ with } |\alpha| \leq 2.$$

Assume that $\rho \geq \rho_1$, and define inductively $\{v_j(y)\}_{j=0,1,2,\dots}$ by the Cauchy problems

$$\begin{cases} S_0(y, D)v_j(y) = -\sum_{k=1}^j S_k(y, D)v_{j-k}(\eta), \\ v_j(0, y'') = \delta_{0,j}. \end{cases}$$

We choose $\chi(y) \in C_0^\infty(W)$ so that $\chi(y) = 1$ if $|y| \leq 1/2$, and put

$$v_\rho^{(N)}(y) = e^{i\rho^2\varphi(y; \rho)} \sum_{j=0}^{N-1} \rho^{-j} v_j(y) \chi(y),$$

$$w_{\rho,1}^{(N)}(y) = \sum_{j=0}^{N-1} \rho^{-j} \chi(y) S_{\rho, \varphi}(y, D)v_j(y),$$

$$w_{\rho,2}^{(N)}(y) = \sum_{j=0}^{N-1} \rho^{-j} [S_{\rho, \varphi}(y, D), \chi(y)] v_j(y).$$

Then we have

$$\tilde{Q}_\rho(y, D)v_\rho^{(N)}(y) = e^{ip^2\varphi(y; \rho)}(w_{\rho,1}^{(N)}(y) + w_{\rho,2}^{(N)}(y)),$$

$$w_{\rho,1}^{(N)}(y) = \rho^{4-N}\chi(y) \left[\sum_{k=1}^{N-1} \sum_{j=0}^{k-1} \rho^{-j} S_k(y, D)v_{N-k+j}(y) + \sum_{j=0}^{N-1} \rho^{-j} S_N(y, D;\rho)v_j(y) \right],$$

$$|e^{ip^2\varphi(y; \rho)}| \leq e^{-\rho^2/32} \quad \text{on } \text{supp } w_{\rho,2}^{(N)}.$$

Therefore, we have

$$\|D^{\alpha'} Q_\rho(y, D)v_\rho^{(N)}(y)\| \leq C_{N,k,\alpha} \rho^{4-N+2|\alpha|} \quad \text{for } N \in \mathbf{N} \text{ and } \alpha \in (\mathbf{Z}_+)^n,$$

where $C_{N,k,\alpha}$ (> 0) is independent of ρ . This contradicts (4.16) and $v_\rho^{(N)}(0) = 1$ if $N > 4 + 6\ell - (n+3)/2$ and $\rho \gg 1$, which proves the assertion (ii) of Theorem 1.4.

5. Proof of Theorem 1.5

Let $X = L^2(\mathbf{R}^n) \oplus L^2(\mathbf{R}^n)$. So $X = L^2(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ and X is a Hilbert space with norm $\|(f, g)\|_X$ defined by $\|(f, g)\|_X = (\|f\|^2 + \|g\|^2)^{1/2}$. Let $\mathcal{T} : L^2(\mathbf{R}^n) \rightarrow X$ be a linear operator defined by

$$(\mathcal{T}u)(y) = (e^{y_n/2}u(y'), e^{y_n/2}u(y', -e^{y_n})).$$

It is obvious that $\|u\| = \|\mathcal{T}u\|_X$ for $u \in L^2(\mathbf{R}^n)$. If $(f, g) \in X$ and

$$u(x) = \begin{cases} x_n^{-1/2}f(x', \log x_n) & (x_n > 0), \\ |x_n|^{-1/2}g(x', \log|x_n|) & (x_n < 0), \end{cases}$$

then $\|u\| = \|(f, g)\|_X$ and $\mathcal{T}u = (f, g)$. Therefore, $\mathcal{T}$ is a unitary operator. We note that $\mathcal{T}(C_0^\infty(\mathbf{R}^n)) \subset \mathcal{S}(\mathbf{R}^n) \times \mathcal{S}(\mathbf{R}^n)$, since $|y_n|^\ell e^{y_n/2} \leq C_{\ell,k}$ if $\ell, k \in \mathbf{Z}_+$ and $y_n \leq k$. For $(f, g) \in \mathcal{S}(\mathbf{R}^n) \times \mathcal{S}(\mathbf{R}^n)$ we define

$$a(y)D^\alpha(f, g) = (a(y)D^\alpha f(y), a(y)D^\alpha g(y)).$$

Recall that $A = x_n D_n - i/2$ and $Q(x, D) = D_1^m + \sum_{|\alpha| \leq m, \alpha_1 < m} a_\alpha D^{\alpha'} A^{\alpha_n}$. Since ${}^t A = -A$, we have ${}^t Q(x, D) = (-1)^m D_1^m + \sum_{|\alpha| \leq m, \alpha_1 < m} (-1)^{|\alpha|} a_\alpha D^{\alpha'} A^{\alpha_n}$. Moreover, we have

$$(5.1) \quad (\mathcal{T}(D_k u))(y) = D_k(\mathcal{T}u)(y) \quad (1 \leq k \leq n-1),$$

$$(\mathcal{T}(A u))(y) = D_n(\mathcal{T}u)(y),$$

$$(\mathcal{T}({}^t Q(x, D)u))(y) = (-1)^m \tilde{Q}(D)(\mathcal{T}u)(y)$$

for $u \in \mathcal{S}(\mathbf{R}^n)$, where $\tilde{Q}(\eta) = \eta_1^m + \sum_{|\alpha| \leq m, \alpha_1 < m} (-1)^{m-|\alpha|} a_\alpha \eta^\alpha$. Write

$$(5.2) \quad \tilde{Q}(\eta) = \prod_{j=1}^m (\eta_1 - \lambda_j(\eta'')),$$

where $\{\lambda_j(\eta'')\}$ is enumerated as

$$\operatorname{Re} \lambda_1(\eta'') \leq \operatorname{Re} \lambda_2(\eta'') \leq \cdots \leq \operatorname{Re} \lambda_m(\eta''),$$

$$\operatorname{Im} \lambda_j(\eta'') \leq \operatorname{Im} \lambda_k(\eta'') \quad \text{if } \operatorname{Re} \lambda_j(\eta'') = \operatorname{Re} \lambda_k(\eta'') \text{ and } j < k.$$

It is obvious that $\operatorname{Re} \lambda_j(\eta'')$ is continuous. Let $T > 0$, and let $v \in \mathcal{S}(\mathbf{R}^n)$ satisfy $\operatorname{supp} v \subset \{y \in \mathbf{R}^n; |y_1| \leq T\}$. Then we have

$$(5.3) \quad |\hat{v}(\eta)|^2 = \left| \int_{-T}^T e^{-iy_1 \eta_1} \tilde{v}(y_1, \eta'') dy_1 \right|^2 \leq 2T \|\tilde{v}(y_1, \eta'')\|_{L^2(\mathbf{R}_{y_1})}^2,$$

where $\tilde{v}(y_1, \eta'') = \mathcal{F}_{y''}[v(y_1, y'')](\eta'')$. Let $\varepsilon > 0$, and let A be a Lebesgue measurable set of $\mathbf{R}^n$ such that $\mu(A(\eta'')) \leq \varepsilon$ for a.e. $\eta'' \in \mathbf{R}^{n-1}$, where $A(\eta'') = \{\eta_1 \in \mathbf{R}; (\eta_1, \eta'') \in A\}$ and μ denotes the Lebesgue measure in $\mathbf{R}$. Then (5.3) yields

$$\begin{aligned} \|v\|^2 &= (2\pi)^{-n} \int_A |\hat{v}(\eta)|^2 d\eta + (2\pi)^{-n} \int_{\mathbf{R}^n \setminus A} |\hat{v}(\eta)|^2 d\eta \\ &\leq 2T(2\pi)^{-n} \int_{\mathbf{R}^{n-1}} \left(\int_{A(\eta'')} \|\tilde{v}(y_1, \eta'')\|_{L^2(\mathbf{R}_{y_1})}^2 d\eta_1 \right) d\eta'' \\ &\quad + (2\pi)^{-n} \int_{\mathbf{R}^n \setminus A} |\hat{v}(\eta)|^2 d\eta \\ &\leq (T\varepsilon/\pi)(2\pi)^{-n+1} \int_{\mathbf{R}^{n-1}} \|\tilde{v}(y_1, \eta'')\|_{L^2(\mathbf{R}_{y_1})}^2 d\eta'' + (2\pi)^{-n} \int_{\mathbf{R}^n \setminus A} |\hat{v}(\eta)|^2 d\eta \\ &= (T\varepsilon/\pi)\|v\|^2 + (2\pi)^{-n} \int_{\mathbf{R}^n \setminus A} |\hat{v}(\eta)|^2 d\eta. \end{aligned}$$

Therefore, we have

$$(5.4) \quad \|v\|^2/2 \leq (2\pi)^{-n} \int_{\mathbf{R}^n \setminus A} |\hat{v}(\eta)|^2 d\eta \quad \text{if } T\varepsilon/\pi \leq 1/2.$$

Now we choose

$$(5.5) \quad A = \{\eta \in \mathbf{R}^n; |\eta_1 - \operatorname{Re} \lambda_j(\eta'')| \leq \varepsilon/(2m) \text{ for some } j\}.$$

Then A is a Lebesgue measurable set of $\mathbf{R}^n$ and $\mu(A(\eta'')) \leq \varepsilon$ for each $\eta'' \in \mathbf{R}^{n-1}$, since $\operatorname{Re} \lambda_j(\eta'')$ is continuous. From (5.2), (5.4) and (5.5) we have

$$\begin{aligned}
\|\tilde{Q}(D)v\|^2 &\geq (2\pi)^{-n} \int_{\mathbf{R}^n \setminus A} |\tilde{Q}(\eta)\hat{v}(\eta)|^2 d\eta \\
&\geq (\varepsilon/(2m))^{2m} (2\pi)^{-n} \int_{\mathbf{R}^n \setminus A} |\hat{v}(\eta)|^2 d\eta \geq 2^{-2m-1} (\varepsilon/m)^{2m} \|v\|^2
\end{aligned}$$

if $v \in \mathcal{S}(\mathbf{R}^n)$, $\text{supp } v \subset \{y \in \mathbf{R}^n; |y_1| \leq T\}$ and $2T\varepsilon \leq \pi$. This, together with (5.1), gives

$$\begin{aligned}
(5.6) \quad \|\iota Q(x, D)u\|^2 &= \|\tilde{Q}(D)\mathcal{F}u\|_X^2 \geq 2^{-2m-1} (\varepsilon/m)^{2m} \|\mathcal{F}u\|_X^2 \\
&= 2^{-2m-1} (\varepsilon/m)^{2m} \|u\|^2
\end{aligned}$$

if $u \in C_0^\infty(\mathbf{R}^n)$, $\text{supp } u \subset \{x \in \mathbf{R}^n; |x_1| \leq T\}$ and $2T\varepsilon \leq \pi$. Let $\gamma \in (\mathbf{Z}_+)^n$. Since $AD^\gamma = D^\gamma(A + i\gamma_n)$, we have $D^{\gamma\iota}Q(x, D)u = {}^tQ^\gamma(x, D)D^\gamma u$, where $Q^\gamma(x, D) = D_1^m + \sum_{|z| \leq m, z_1 < m} a_z D^{z'}(A + i\gamma_n)^{z_n}$. (5.6) with $Q(x, D)$ replaced by $Q^\gamma(x, D)$ yields

$$\|D^\gamma u\| \leq 2^{2m+1/2} (mT/\pi)^m \|D^{\gamma\iota}Q(x, D)u\|$$

for $u \in C_0^\infty(\mathbf{R}^n)$ with $\text{supp } u \subset \{x \in \mathbf{R}^n; |x_1| \leq T\}$. Therefore, for any $s \in \mathbf{Z}_+$ there is $C_s > 0$ such that

$$\|\langle D \rangle^s u\| \leq C_s T^m \|\langle D \rangle^{s\iota} Q(x, D)u\|$$

for $u \in C_0^\infty(\mathbf{R}^n)$ with $\text{supp } u \subset \{x \in \mathbf{R}^n; |x_1| \leq T\}$. This, together with Proposition 2.1, proves Theorem 1.5.

Appendix

First we shall give a proof of the fact on the symbol calculus used in §3.

Proposition A.1. *Let $\kappa > 1$, $p(x, \xi) \in S_{1,0}^m$, and let $\omega(\xi)$ be a real-valued symbol in $S_{1,0}^{1/\kappa}$. Then for any $a \in \mathbf{C}$ and $N \in \mathbf{N}$ we have*

$$\begin{aligned}
&\sum_{|z| < N} e^{ia\omega(\xi)} (\partial^\alpha e^{-ia\omega(\xi)}) p_{(z)}(x, \xi) / \alpha! \\
&\equiv \sum_{|z| + (1-1/\kappa)|\beta| < (1-1/\kappa)N} [\hat{\partial}_\eta^\alpha (-iaW(\xi, \eta))^\beta]_{\eta=0} p_{(\alpha+\beta)}(x, \xi) / (\alpha! \beta!)
\end{aligned}$$

(mod $S_{1,0}^{m-(1-1/\kappa)N}$), where $W(\xi, \eta) = \int_0^1 (\nabla \omega)(\xi + \theta\eta) d\theta$.

Proof. First assume that $a \in \mathbf{R}$. Then the symbol calculus of pseudo-differential operators gives

$$\sigma(e^{-ia\omega(D)} p(x, D) e^{ia\omega(D)})(x, \xi) \sim \sum_\alpha e^{ia\omega(\xi)} (\partial^\alpha e^{-ia\omega(\xi)}) p_{(\alpha)}(x, \xi) / \alpha!.$$

Applying the method used in the calculus of Fourier integral operators, we see that

$$\begin{aligned}
 & \sigma(e^{-ia\omega(D)}p(x, D)e^{ia\omega(D)})(x, \xi) \\
 &= (2\pi)^{-n} \text{Os-} \int e^{-iy \cdot \eta} e^{-ia(\omega(\xi+\eta)-\omega(\xi))} p(x+y, \xi) dy d\eta \\
 &= (2\pi)^{-n} \text{Os-} \int e^{-i(y+aW(\xi, \eta)) \cdot \eta} p(x+y, \xi) dy d\eta \\
 &= (2\pi)^{-n} \text{Os-} \int e^{-iy \cdot \eta} \left[\sum_{|\alpha| < N} (iy)^\alpha p_{(\alpha)}(x-aW(\xi, \eta), \xi) / \alpha! \right. \\
 &\quad \left. + \sum_{|\alpha|=N} (iy)^\alpha N \int_0^1 (1-\theta)^{N-1} p_{(\alpha)}(x+\theta y-aW(\xi, \eta), \xi) d\theta / \alpha! \right] dy d\eta \\
 &\equiv \sum_{|\alpha| < N} [\partial_\eta^\alpha p_{(\alpha)}(x-aW(\xi, \eta), \xi)]_{\eta=0} / \alpha! \pmod{S_{1,0}^{m+1/\kappa-N-1}}
 \end{aligned}$$

($N \in \mathbb{N}$), where $\text{Os-} \int \dots dy d\eta$ denotes the oscillatory integral. For $N \in \mathbb{N}$ we put

$$\begin{aligned}
 f_N(x, \xi; a) &= \sum_{|\alpha| < N} e^{ia\omega(\xi)} (\partial^\alpha e^{-ia\omega(\xi)}) p_{(\alpha)}(x, \xi) / \alpha!, \\
 g_N(x, \xi; a) &= \sum_{|\alpha| < (1-1/\kappa)N} [\partial_\eta^\alpha p_{(\alpha)}(x-aW(\xi, \eta), \xi)]_{\eta=0} / \alpha!.
 \end{aligned}$$

Then we have

$$f_N(x, \xi; a) \equiv g_N(x, \xi; a) \pmod{S_{1,0}^{m-(1-1/\kappa)N}}.$$

Note that

$$g_N(x, \xi; a) \equiv h_N(x, \xi; a) \pmod{S_{1,0}^{m-(1-1/\kappa)N}},$$

where

$$h_N(x, \xi; a) = \sum_{|\alpha| + (1-1/\kappa)|\beta| < (1-1/\kappa)N} [\partial_\eta^\alpha (-iaW(\xi, \eta))^\beta]_{\eta=0} p_{(\alpha+\beta)}(x, \xi) / (\alpha! \beta!).$$

Since $f_N(x, \xi; a)$ and $h_N(x, \xi; a)$ are polynomials of a of degree $(N-1)$, the coefficients of a^j must be coincident modulo $S_{1,0}^{m-(1-1/\kappa)N}$ ($0 \leq j \leq N-1$). This implies that

$$f_N(x, \xi; a) \equiv h_N(x, \xi; a) \pmod{S_{1,0}^{m-(1-1/\kappa)N}}$$

for any $a \in \mathbb{C}$, which proves the proposition. $\square$

$\mathcal{D}^{\{\kappa\}'}$ is a (FS) space (see, e.g., [12]). In particular, it is a Fréchet space. So we can apply the same argument as in [3] and obtain the following

Proposition A.2. *Let $\kappa > 1$, $x^0 \in \mathbf{R}^n$ and $p(x, \xi) \in S_{\{\kappa\}}^+$. Assume that for any $f \in \mathcal{D}^{\{\kappa\}'}$ there are an open neighborhood U of x^0 and $u \in \mathcal{X} \equiv \lim_{\leftarrow j \rightarrow \infty} L_{\kappa, -1/j}^2 (\subset \mathcal{F}_{\{\kappa\}})$ satisfying $p(x, D)u = f$ in U . Then $p(x, D)$ is locally solvable at x^0 in $\mathcal{D}^{\{\kappa\}'}$. In particular, $p(x, D)$ is locally solvable at x^0 in $\mathcal{D}^{\{\kappa\}'}$ if $p(x, D)$ is a properly supported pseudodifferential operator and locally solvable at x^0 in $\mathcal{D}^{\{\kappa\}'}$ in a germ sense.*

Proof. Put $V_k = \{x \in \mathbf{R}^n; |x - x^0| < 1/k\}$ and

$$E_k = \{(u, f) \in \mathcal{X} \times \mathcal{D}^{\{\kappa\}'}; p(x, D)u = f \text{ in } V_k\}$$

for $k \in \mathbf{N}$. Then E_k is a closed subspace of $\mathcal{X} \times \mathcal{D}^{\{\kappa\}'}$, which is a Fréchet space. Let $\pi_k : E_k \ni (u, f) \mapsto f \in \mathcal{D}^{\{\kappa\}'}$. Then, by assumption we have $\bigcup_{k=1}^{\infty} \pi_k(E_k) = \mathcal{D}^{\{\kappa\}'}$. Baire's theorem implies that there is $k_0 \in \mathbf{N}$ such that $\pi_{k_0}(E_{k_0})$ is of second category and the closure $\overline{\pi_{k_0}(E_{k_0})}$ of $\pi_{k_0}(E_{k_0})$ coincides with $\mathcal{D}^{\{\kappa\}'}$. So it follows from Banach's homomorphism theorem that $\pi_{k_0}(E_{k_0}) = \mathcal{D}^{\{\kappa\}'}$ (see, e.g., [15]). Therefore, for any $f \in \mathcal{D}^{\{\kappa\}'}$ there is $u \in \mathcal{X}$ such that $p(x, D)u = f$ in V_{k_0} , which proves the first part of the proposition. If $p(x, D)$ is properly supported, U is an open neighborhood of x^0 and $u \in \mathcal{F}_{\{\kappa\}}$, then there is $\tilde{u} \in \mathcal{E}^{\{\kappa\}'} (\subset \mathcal{X})$ such that $p(x, D)u = p(x, D)\tilde{u}$ in U . So the second part of the proposition is obvious. $\square$

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