Coupled Painlevé V Systems in Dimension 4

By

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Abstract. We find and study coupled Painlevé V systems in dimension 4, which are different from the systems of type $A_5^{(1)}$. We also give the augmented phase spaces for these systems.

Key Words and Phrases. Painlevé equations, Accessible singularity, Hamiltonian system, $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ -flop.

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0. Introduction

The purpose of this paper is to characterize higher order Painlevé equations from the viewpoint of algebraic vector fields with favorable properties in regard to accessible singularities (cf. Definition 1.1) and local index (cf. Definition 1.2) [8]. As two examples of higher order Painlevé equations with favorable properties, we studied the systems of type $A_4^{(1)}$ and $A_5^{(1)}$ [3], which can be considered as a generalization of Painlevé equations P_{IV} and P_V to fourth-order. Let us summarize important properties of these two systems as follows:

Notation.

- $H \in \mathcal{C}(t)[x, y, z, w]$,
- $\deg(H)$: degree with respect to x, y, z, w ,
- $\mathcal{O}_{\mathbf{P}^4}(-\log \mathcal{H})$: subsheaf of $\mathcal{O}_{\mathbf{P}^4}$ whose local section v satisfies $v(f) \in (f)$ for any local equation f of the boundary divisor $\mathcal{H}$ of $\mathbf{P}^4$,
- $H_V(X, Y, T)$ (resp. $H_{IV}(X, Y, T)$): the Hamiltonian of P_V (resp. P_{IV}),
- $\dim. \text{ of sol.}$: dimension of the parameter space of meromorphic solutions which pass through an accessible singular point.

symmetry	$W(A_5^{(1)})$	$W(A_4^{(1)})$
Hamiltonian	$H_V(x, y, t)$ $+ H_V(z, w, t)$ $- 2yzw + \frac{2xyzw}{t}$	$H_{IV}(x, y, t)$ $+ H_{IV}(z, w, t) - 2yzw$
form of equations	coupled Painlevé V	coupled Painlevé IV

degree of Hamiltonian H	4	3
$v \in H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(-\log \mathcal{H})(n\mathcal{H}))$	$n = 2$	$n = 1$
number of parameters	5	4

type of accessible singularity	point	point	point
type of local index	$(+1, +3, +1, +1)$	$(+1, +3, -1, +1)$	$(+1, +3, -3, +5)$
dim. of sol.	dim. 3	dim. 2	dim. 2

Unlike the second-order case, where only accessible singularities of point type occur, in the fourth-order case there exist the following possibilities for the type of accessible singularity: (i) point, (ii) curve, (iii) surface.

These properties suggest the possibility that there exist higher order versions of P_V (resp. P_{IV}) as well, and furthermore, suggest a procedure for searching for such higher order versions with different types of accessible singularities from those of the systems of type $A_4^{(1)}$ and $A_5^{(1)}$. In this vein, the goal of this work is to find a fourth-order version of the Painlevé V (resp. Painlevé IV) system other than the systems of type $A_4^{(1)}$ and $A_5^{(1)}$. Here, we consider the following problem.

Problem. Can we classify the coupled Painlevé V (resp. Painlevé IV) systems in dimension 4 that are Hamiltonian systems with Hamiltonian $H \in \mathcal{C}(t)[x, y, z, w]$ of $\deg(H) = 4$ (resp. $\deg(H) = 3$), and, moreover, have given some accessible singularities?

To answer this, in [8], we constructed a 3-parameter family of fourth-order algebraic ordinary differential equations of Painlevé IV type. In the present paper, we construct a 4-parameter family of fourth-order algebraic ordinary differential equations which can be considered as coupled Painlevé V systems in dimension 4, which are given as follows:

$$(1) \quad \begin{cases} \frac{dx}{dt} = \frac{-2x^2y - 2xzw + x^2 + xz}{t} - 2xy + \left(1 + \frac{\alpha_1 + \alpha_3}{t}\right)x - \alpha_2 \\ \frac{dy}{dt} = \frac{2xy^2 + 2yzw - 2xy - yz - zw}{t} + y^2 - \left(1 + \frac{\alpha_1 + \alpha_3}{t}\right)y + \frac{\alpha_1}{t} \\ \frac{dz}{dt} = \frac{-2z^2w - 2xyz + z^2 + xz}{t} - 2zw + \left(1 + \frac{\alpha_1 + \alpha_3}{t}\right)z - \alpha_4 \\ \frac{dw}{dt} = \frac{2zw^2 + 2xyw - 2zw - xy - xw}{t} + w^2 - \left(1 + \frac{\alpha_1 + \alpha_3}{t}\right)w + \frac{\alpha_1}{t}. \end{cases}$$

Here x, y, z and w denote unknown complex variables and $\alpha_1, \alpha_2, \alpha_3$ and α_4 are complex parameters.

Our differential system is equivalent to a Hamiltonian system. The Hamiltonian is given as follows:

$$H = H_V(x, y, t; \alpha_1, \alpha_2, \alpha_3) + H_V(z, w, t; \alpha_2, \alpha_3, \alpha_4) - \frac{2xyzw}{t} + \frac{xzw + xyz}{t}.$$

Here the symbol $H_V(q, p, t; \gamma_1, \gamma_2, \gamma_3)$ denotes the Hamiltonian of the second-order Painlevé V system, which is given as follows:

$$H_V(q, p, t; \gamma_1, \gamma_2, \gamma_3) = \frac{q^2p - q^2p^2 - \gamma_1q + (\gamma_1 + \gamma_3)qp}{t} + qp - qp^2 - \gamma_2p.$$

From the viewpoint of symmetry, it is interesting to point out that our differential system is invariant under the transformations s_1, s_2, s_3 and π in the following table.

	α_1	α_2	α_3	α_4	x	y	z	w
s_1	$\alpha_1 + \alpha_2$	$-\alpha_2$	$\alpha_2 + \alpha_3$	α_4	x	$y + \frac{\alpha_2}{x}$	z	w
s_2	$\alpha_1 + \alpha_4$	α_2	$\alpha_3 + \alpha_4$	$-\alpha_4$	x	y	z	$w + \frac{\alpha_4}{z}$
s_3	$1 - \alpha_2 - \alpha_3 - \alpha_4$	α_2	$1 - \alpha_1 - \alpha_2 - \alpha_4$	α_4	x	$y + \frac{\alpha_5}{x+z+t}$	z	$w + \frac{\alpha_5}{x+z+t}$
π	α_1	α_4	α_3	α_2	z	w	x	y

Here, $\alpha_5 = 1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$. The above list should be read as $s_1(\alpha_1) = \alpha_1 + \alpha_2$, $s_1(\alpha_2) = -\alpha_2$, $s_1(\alpha_3) = \alpha_2 + \alpha_3$, $s_1(\alpha_4) = \alpha_4$, $s_1(x) = x$, $s_1(y) = y + \frac{\alpha_2}{x}$, $s_1(z) = z$, $s_1(w) = w$.

In 1979, K. Okamoto constructed the spaces of initial conditions of Painlevé equations, which can be considered as the parameter spaces of all meromorphic solutions (including holomorphic solutions). We call these spaces *augmented* phase spaces, in accordance with [9]. For these differential equations in (1), we can construct the following phase spaces.

Theorem 0.1. *The augmented phase space $\mathcal{X}$ over $B = \mathbf{C}^* = \mathbf{C} - \{0\}$ for the vector field v in (1) is obtained by gluing sixteen copies of $\mathbf{C}^4 \times \mathbf{C}^*$:*

$$U_0 \times \mathbf{C}^* = \mathbf{C}^4 \times \mathbf{C}^* \ni (x, y, z, w, t),$$

$$U_j \times \mathbf{C}^* = \mathbf{C}^4 \times \mathbf{C}^* \ni (x_j, y_j, z_j, w_j, t) \quad (j = 1, 2, \dots, 15),$$

via the following symplectic transformations

- 1) $x_1 = \frac{1}{x}, y_1 = -x(xy + zw - \alpha_1), z_1 = \frac{z}{x}, w_1 = xw,$
- 2) $x_2 = \frac{x}{z}, y_2 = yz, z_2 = \frac{1}{z}, w_2 = -z(zw + xy - \alpha_1),$
- 3) $x_3 = -y(xy + \alpha_2), y_3 = \frac{1}{y}, z_3 = z, w_3 = w,$
- 4) $x_4 = x, y_4 = y, z_4 = -w(zw + \alpha_4), w_4 = \frac{1}{w},$
- 5) $x_5 = -yz(xy + \alpha_2), y_5 = \frac{1}{yz}, z_5 = \frac{1}{z}, w_5 = -z(xy + zw - \alpha_1),$
- 6) $x_6 = \frac{1}{x}, y_6 = -x(zw + xy - \alpha_1), z_6 = -xw(zw + \alpha_4), w_6 = \frac{1}{xw},$
- 7) $x_7 = -y(xy + \alpha_2), y_7 = \frac{1}{y}, z_7 = -w(zw + \alpha_4), w_7 = \frac{1}{w},$
- 8) $x_8 = \frac{1}{x}, y_8 = -x(-x + xy - z + zw - \alpha_3), z_8 = \frac{z}{x}, w_8 = (w - 1)x,$
- 9) $x_9 = \frac{x}{z}, y_9 = (y - 1)z, z_9 = \frac{1}{z}, w_9 = -z(-z + zw - x + xy - \alpha_3),$
- 10) $x_{10} = -y(1 + ty + xy + yz - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4), y_{10} = \frac{1}{y}, z_{10} = - (w - y)(zw - yz + \alpha_4), w_{10} = \frac{1}{w - y},$
- 11) $x_{11} = -(y - w)(xy - xw + \alpha_2), y_{11} = \frac{1}{y - w}, z_{11} = -w(1 + tw + xw + zw - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4), w_{11} = \frac{1}{w},$
- 12) $x_{12} = -y(1 + ty + xy + yz - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4), y_{12} = \frac{1}{y}, z_{12} = z, w_{12} = w - y,$
- 13) $x_{13} = x, y_{13} = y - w, z_{13} = -w(1 + tw + xw + zw - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4), w_{13} = \frac{1}{w},$
- 14) $x_{14} = -(-1 + y)z(-x + xy + \alpha_2), y_{14} = \frac{1}{z(y - 1)}, z_{14} = \frac{1}{z}, w_{14} = -z(-x + xy - z + zw - \alpha_3),$
- 15) $x_{15} = \frac{1}{x}, y_{15} = -x(-z + zw - x + xy - \alpha_3), z_{15} = -(-1 + w)x(-z + zw + \alpha_4), w_{15} = \frac{1}{x(w - 1)}.$

Theorem 0.2. *On the affine open set $(x_i, y_i, z_i, w_i, t) \in U_i \times B$ in Theorem 0.1, each Hamiltonian H_i on $U_i \times B$ is expressed as a polynomial in x_i, y_i, z_i, w_i and a rational function in t and satisfies the following conditions:*

$$dx \wedge dy + dz \wedge dw - dH \wedge dt = dx_i \wedge dy_i + dz_i \wedge dw_i - dH_i \wedge dt.$$

This paper is organized as follows. In Section 1, we review the notion of accessible singularity and local index. We will explain our approach to obtain the system (1) in Section 2. We will prove Proposition 2.1 in Section 3. In Section 4, we will prove Theorem 0.1 by explicit blowing-up and blowing-down. In the final section, we will prove Theorem 0.2.

1. Accessible singularity and local index

1.1. Accessible singularity and local index

Let us review the notion of accessible singularity in accordance with [5]. Let B be a connected open domain in $\mathcal{C}$ and $\pi: \mathcal{Z} \rightarrow B$ a smooth proper

holomorphic map. We assume that $\mathcal{K} \subset \mathcal{Z}$ is a normal crossing divisor which is flat over B . Let us consider a rational vector field $\tilde{v}$ on $\mathcal{Z}$ satisfying the condition

$$\tilde{v} \in H^0(\mathcal{Z}, \Theta_{\mathcal{Z}}(-\log \mathcal{K})(\mathcal{K})).$$

Fixing $t_0 \in B$ and $P \in \mathcal{Z}_{t_0}$, we can take a local coordinate system $(x_1, x_2, \dots, x_n)$ of $\mathcal{Z}_{t_0}$ centered at P such that $\mathcal{K}_{smooth}$ can be defined by the local equation $x_1 = 0$. Since $\tilde{v} \in H^0(\mathcal{Z}, \Theta_{\mathcal{Z}}(-\log \mathcal{K})(\mathcal{K}))$, we can write down the vector field $\tilde{v}$ near $P = (0, 0, \dots, 0, t_0)$ as follows:

$$\tilde{v} = \frac{\partial}{\partial t} + a_1 \frac{\partial}{\partial x_1} + \frac{a_2}{x_1} \frac{\partial}{\partial x_2} + \dots + \frac{a_n}{x_1} \frac{\partial}{\partial x_n}$$

This vector field defines the following system of differential equations

$$(2) \quad \begin{cases} \frac{dx_1}{dt} = a_1(x_1, x_2, \dots, x_n, t) \\ \frac{dx_2}{dt} = \frac{a_2(x_1, x_2, \dots, x_n, t)}{x_1} \\ \cdot \\ \cdot \\ \frac{dx_n}{dt} = \frac{a_n(x_1, x_2, \dots, x_n, t)}{x_1} \end{cases}$$

Here $a_i(x_1, x_2, \dots, x_n, t)$ ($i = 1, 2, \dots, n$) are holomorphic functions near $P = (0, 0, \dots, 0, t_0)$.

Definition 1.1. Under the above notation, assume that the rational vector field $\tilde{v}$ on $\mathcal{Z}$ satisfies the condition

$$\tilde{v} \in H^0(\mathcal{Z}, \Theta_{\mathcal{Z}}(-\log \mathcal{K})(\mathcal{K})).$$

We say that $\tilde{v}$ has an accessible singularity at $P = (0, 0, \dots, 0, t_0)$ if

$$x_1 = 0 \quad \text{and} \quad a_i(0, 0, \dots, 0, t_0) = 0 \quad \text{for every } i, 2 \leq i \leq n.$$

In particular, on the point type of accessible singularity, we call it an accessible singular point.

Remark 1.1. The above definition 1.1 is a generalization of the definition in [1] applied to the system (2). With either definition, if $P \in \mathcal{K}_{smooth}$ is not an accessible singular point, all solutions of the ordinary differential equation passing through P are vertical solutions, that is, the solutions are contained in

the fiber $\mathcal{L}_{t_0}$ over $t = t_0$. With accessible singular points as defined in [1], a point in the intersection between the closure of a solution and the boundary divisor $\mathcal{K}$ is called an accessible singular point. With accessible singular points as defined in definition 1.1, if $P \in \mathcal{K}_{smooth}$ is an accessible singular point, there may be a solution of (2) which passes through P and goes into the interior $\mathcal{L} - \mathcal{K}$ of $\mathcal{L}$. Thus, an accessible singular point P in definition 1.1 has two cases: there may or may not exist a solution passing through P .

Let us recall the notion of local index. When we construct the phase spaces of the higher order Painlevé equations, an object that we call the local index is the key for determining when we need to make a blowing-up of an accessible singularity or a blowing-down to a minimal phase space. In the case of equations of higher order with favorable properties, for example the systems of type $A_4^{(1)}$, the local index at the accessible singular point corresponds to the set of orders that appears in the free parameters of formal solutions passing through that point [9].

Definition 1.2. Let v be an algebraic vector field which is given by (2) and (X, Y, Z, W) be a boundary coordinate system in a neighborhood of an accessible singular point $P = (0, 0, 0, 0, t)$. Assume that the system is written as

$$\begin{cases} \frac{dX}{dt} = a + f_1(X, Y, Z, W, t) \\ \frac{dY}{dt} = \frac{bY + f_2(X, Y, Z, W, t)}{X} \\ \frac{dZ}{dt} = \frac{cZ + f_3(X, Y, Z, W, t)}{X} \\ \frac{dW}{dt} = \frac{dW + f_4(X, Y, Z, W, t)}{X} \end{cases}$$

near the accessible singular point P , where a, b, c and d are nonzero constants. We say that the vector field v has local index (a, b, c, d) at P if $f_1(X, Y, Z, W, t)$ is a polynomial which vanishes at $P = (0, 0, 0, 0, t)$ and $f_i(X, Y, Z, W, t)$ ($i = 2, 3, 4$) are polynomials of order 2 in X, Y, Z, W . Here $f_i \in \mathbb{C}(t)[X, Y, Z, W]$ ($i = 1, 2, 3, 4$).

Remark 1.2. We are interested in the case with local index $(1, b/a, c/a, d/a) \in \mathbb{Z}^4$. If each component of $(1, b/a, c/a, d/a)$ has the same sign, we may resolve the accessible singularity by blowing-up finitely many times. But when different signs appear, we may need to both blow up and blow down.

1.2. Examples

In order to clarify the meaning of local index, let us consider the following examples.

Example 1.1. Let v be an algebraic vector field on $\mathbf{C}^4 \times (\mathbf{C} - \{0\})$ which is given by

$$\frac{dx}{dt} = -\frac{2x^2y}{t} - 2xy, \quad \frac{dy}{dt} = \frac{2xy^2}{t} + y^2, \quad \frac{dz}{dt} = 0, \quad \frac{dw}{dt} = 0.$$

In the boundary coordinate system $(X_3, Y_3, Z_3, W_3) = (x, 1/y, z, w) \in V_3 \subset \mathcal{S}$, this system is written as follows:

$$\frac{dX_3}{dt} = -\frac{2X_3}{Y_3} - \frac{2X_3^2}{tY_3}, \quad \frac{dY_3}{dt} = -1 - \frac{2X_3}{t}, \quad \frac{dZ_3}{dt} = 0, \quad \frac{dW_3}{dt} = 0.$$

We see that $S = \{(X_3, Y_3, Z_3, W_3) \mid X_3 = Y_3 = 0\} \cong \mathbf{C}^2$ is the accessible singular locus. In particular, $P = \{(X_3, Y_3, Z_3, W_3) = (0, 0, 0, 0)\}$ is an accessible singular point with the local index $(-2, -1, 0, 0)$. Then we see that the solutions which pass through the point P are given by

$$X_3(t) = a_2 t^2, \quad Y_3(t) = -t - a_2 t^2, \quad Z_3(t) = 0, \quad W_3(t) = 0,$$

where a_2 is a free parameter. In this case, the local index at the accessible singular point P corresponds to the set of orders that appears in the free parameter of formal solutions passing through P .

Next, let us consider a resolution of accessible singular point P . It is easy to see that by the following 2 steps, we can resolve the accessible singular point P .

Step 1: We blow up along the surface $\{(X_3, Y_3, Z_3, W_3) \mid X_3 = Y_3 = 0\}$

$$X'_3 = \frac{X_3}{Y_3}, \quad Y'_3 = Y_3, \quad Z'_3 = Z_3, \quad W'_3 = W_3.$$

Step 2: We blow up along the surface $\{(X'_3, Y'_3, Z'_3, W'_3) \mid X'_3 = Y'_3 = 0\}$

$$X''_3 = \frac{X'_3}{Y'_3}, \quad Y''_3 = Y'_3, \quad Z''_3 = Z'_3, \quad W''_3 = W'_3.$$

Moreover, the transition functions between (x, y, z, w) and $(X'''_3, Y'''_3, Z'''_3, W'''_3) := (-X''_3, Y''_3, Z''_3, W''_3)$ are symplectic, that is,

$$dx \wedge dy + dz \wedge dw = dX'''_3 \wedge dY'''_3 + dZ'''_3 \wedge dW'''_3.$$

	Type of local index	Canonical divisor
Step 1	$(-2, -1, 0, 0)$	$K_{\mathcal{S}} = -2\mathcal{H}_1 - 2\mathcal{H}_2 - 3\mathcal{H}_3$
Step 2	$(-1, -1, 0, 0)$	$K_{\mathcal{S}_1} = -2\mathcal{H}_1 - E_1 - 2\mathcal{H}_2 - 3\mathcal{H}_3$

Table 1

Here E_1 is the exceptional divisor which is obtained by Step 1.

Example 1.2. Let v be an algebraic vector field on $C^4 \times (C - \{0\})$ which is given by

$$\frac{dx}{dt} = -\frac{2x^2y}{t} + x^2, \quad \frac{dy}{dt} = \frac{2xy^2}{t} - 2xy - zw, \quad \frac{dz}{dt} = xz, \quad \frac{dw}{dt} = -xw.$$

In the boundary coordinate system $(X_1, Y_1, Z_1, W_1) = (1/x, y, z/x, w) \in V_1 \subset \mathcal{S}$, this system is written as follows:

$$\begin{aligned} \frac{dX_1}{dt} &= -1 + \frac{2Y_1}{t}, & \frac{dY_1}{dt} &= -\frac{2Y_1}{X_1} - \frac{-2Y_1^2 + tZ_1W_1}{tX_1}, \\ \frac{dZ_1}{dt} &= \frac{2Y_1Z_1}{tX_1}, & \frac{dW_1}{dt} &= -\frac{W_1}{X_1}. \end{aligned}$$

We see that $C = \{(X_1, Y_1, Z_1, W_1) \mid X_1 = Y_1 = W_1 = 0\}$ is the accessible singular locus. In particular, $P_1 = \{(X_1, Y_1, Z_1, W_1) = (0, 0, 0, 0)\}$ is an accessible singular point with the local index $(-1, -2, 0, -1)$. Then we see that the solutions which pass through the point P_1 are given by

$$X_1(t) = -t + b_2t^2, \quad Y_1(t) = b_2t^2, \quad Z_1(t) = 0, \quad W_1(t) = \frac{d_1t}{1 - b_2t},$$

where b_2 and d_1 are free parameters. In this case, the local index at the accessible singular point P_1 corresponds to the set of orders that appears in the free parameters of formal solutions passing through the point P_1 .

Next, let us consider a resolution of accessible singular locus C . It is easy to see that by the following 2 steps, we can resolve the accessible singular locus C .

Step 1: We blow up along the curve $\{(X_1, Y_1, Z_1, W_1) \mid X_1 = Y_1 = W_1 = 0\}$

$$X'_1 = X_1, \quad Y'_1 = \frac{Y_1}{X_1}, \quad Z'_1 = Z_1, \quad W'_1 = \frac{W_1}{X_1}.$$

Step 2: We blow up along the surface $\{(X'_1, Y'_1, Z'_1, W'_1) \mid X'_1 = Y'_1 + Z'_1W'_1 = 0\}$

$$X''_1 = X'_1, \quad Y''_1 = \frac{Y'_1 + Z'_1W'_1}{X'_1}, \quad Z''_1 = Z'_1, \quad W''_1 = W'_1.$$

Moreover, the transition functions between (x, y, z, w) and $(X_1''', Y_1''', Z_1''', W_1''')$ $:= (X_1'', -Y_1'', Z_1'', W_1'')$ are symplectic, that is,

$$dx \wedge dy + dz \wedge dw = dX_1''' \wedge dY_1''' + dZ_1''' \wedge dW_1'''.$$

	Type of local index	Canonical divisor
Step 1	$(-1, -2, 0, -1)$	$K_{\mathcal{S}} = -2\mathcal{H}_1 - 2\mathcal{H}_2 - 3\mathcal{H}_3$
Step 2	$(-1, -1, 0, 0)$	$K_{\mathcal{S}_1} = -2\mathcal{H}_1 - 2\mathcal{H}_2 - 3\mathcal{H}_3 - E_1$

Table 2

Here E_1 is the exceptional divisor which is obtained by Step 1.

2. Construction of coupled Painlevé V systems (1)

Let us consider the coupled Painlevé V systems with the Hamiltonian H of degree 4 involving unknown parameters, which is given as follows:

$$(3) \quad H = H_V(x, y, t; \alpha_1, \alpha_2, \alpha_3) + H_V(z, w, t; \beta_1, \beta_2, \beta_3) + K.$$

Here $K \in \mathcal{C}(t)[x, y, z, w]$, and $H_V(x, y, t; \alpha_1, \alpha_2, \alpha_3) + H_V(z, w, t; \beta_1, \beta_2, \beta_3)$ and K have no common monomials.

If we take a relative compactification $\mathbf{P}^4 \times B$ of $\mathbf{C}^4 \times B$, the extended vector field $\tilde{v}$ with the Hamiltonian (3) satisfies the following conditions:

$$\begin{aligned} \tilde{v} &\notin H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(-\log Y)(2Y)), \\ \tilde{v} &\in H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(-\log Y)(3Y)). \end{aligned}$$

Here Y is the boundary divisor in $\mathbf{P}^4$ and $\mathcal{O}_{\mathbf{P}^4}(-\log Y)$ is the subsheaf of $\mathcal{O}_{\mathbf{P}^4}$ whose local section v satisfies $v(f) \in (f)$ for any local equation f of Y . In this case, the analysis of accessible singularities on Y becomes much more difficult. In order to obtain a favorable phase space, one may need to blow down the proper transform of Y after some birational transformations. In order to avoid this difficulty, we will take the relative compactification $\mathcal{S} \times B$ of $\mathbf{C}^4 \times B$, which is obtained by gluing twelve copies of $\mathbf{C}^4 \times B$ as follows.

In Figure 1, we have

$$\begin{aligned} V_0 \times B &= \mathbf{C}^4 \times \mathbf{C} \ni (x, y, z, w, t), \\ V_j \times B &= \mathbf{C}^4 \times \mathbf{C} \ni (X_j, Y_j, Z_j, W_j, t) \quad (j = 1, 2, \dots, 11), \end{aligned}$$

Fig. 1. Relative compactification $\mathcal{S} \times B$ of $\mathbf{C}^4 \times B$

via the following rational transformations

- 1) $X_1 = \frac{1}{x}$, $Y_1 = y$, $Z_1 = \frac{z}{x}$, $W_1 = w$, 2) $X_2 = \frac{x}{z}$, $Y_2 = y$, $Z_2 = \frac{1}{z}$, $W_2 = w$,
- 3) $X_3 = x$, $Y_3 = \frac{1}{y}$, $Z_3 = z$, $W_3 = w$, 4) $X_4 = x$, $Y_4 = y$, $Z_4 = z$, $W_4 = \frac{1}{w}$,
- 5) $X_5 = \frac{x}{z}$, $Y_5 = \frac{1}{y}$, $Z_5 = \frac{1}{z}$, $W_5 = w$, 6) $X_6 = \frac{1}{x}$, $Y_6 = y$, $Z_6 = \frac{z}{x}$, $W_6 = \frac{1}{w}$,
- 7) $X_7 = x$, $Y_7 = \frac{1}{y}$, $Z_7 = z$, $W_7 = \frac{1}{w}$, 8) $X_8 = \frac{x}{z}$, $Y_8 = y$, $Z_8 = \frac{1}{z}$, $W_8 = \frac{1}{w}$,
- 9) $X_9 = \frac{x}{z}$, $Y_9 = \frac{1}{y}$, $Z_9 = \frac{1}{z}$, $W_9 = \frac{1}{w}$, 10) $X_{10} = \frac{1}{x}$, $Y_{10} = \frac{1}{y}$, $Z_{10} = \frac{z}{x}$, $W_{10} = \frac{1}{w}$,
- 11) $X_{11} = \frac{1}{x}$, $Y_{11} = \frac{1}{y}$, $Z_{11} = \frac{z}{x}$, $W_{11} = w$.

Lemma 2.1. *By the following 4 steps, we can obtain the rational variety $\mathcal{S}$ (See Figure 2).*

Step 1: We blow up along the curves $C_1 \cong \mathbf{P}^1$ and $C_2 \cong \mathbf{P}^1$.

Step 2: We blow down the 3-fold $V_1 \cong \mathbf{F}_1 \times \mathbf{P}^1$.

Step 3: We blow up along the surfaces $F_1 \cong \mathbf{P}^2$ and $F_2 \cong \mathbf{P}^2$.

Step 4: We blow down the 3-fold $V_2 \cong \mathbf{P}^2 \times \mathbf{P}^1$.

Fig. 2. Birational transformation from Step 1 to Step 4

Here F_1 denotes a rational ruled surface $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1))$ which is isomorphic to $\mathbf{P}^2$ with one point blown up.

It is easy to see that the canonical divisor $K_{\mathcal{S}}$ of the variety $\mathcal{S}$ is given as follows:

$$K_{\mathcal{S}} = -2\mathcal{H}_1 - 2\mathcal{H}_2 - 3\mathcal{H}_3.$$

Here

$$\begin{aligned} \{Y_3 = 0\} &\subset \mathcal{H}_1 \cong \mathbf{P}^2 \times \mathbf{P}^1, & \{W_4 = 0\} &\subset \mathcal{H}_2 \cong \mathbf{P}^2 \times \mathbf{P}^1, \\ \{X_1 = 0\} \cup \{Z_2 = 0\} &\subset \mathcal{H}_3 \cong \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1. \end{aligned}$$

Let us extend the regular vector field v on $\mathbf{C}^4 \times B$ to a rational vector field $\tilde{v}$ on $\mathcal{S} \times B$. Then $\tilde{v}$ has the pole along the boundary divisor $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$. Moreover, $\tilde{v}$ has accessible singularities along subvarieties in the boundary divisor $\mathcal{H}$. In order to obtain our favorable vector fields, we consider the vector fields with the following conditions (A1) and (A2).

Conditions:

- (A1) $\tilde{v} \in H^0(\mathcal{S}, \Theta_{\mathcal{S}}(-\log \mathcal{H})(\mathcal{H}))$.
- (A2) The vector field $\tilde{v}$ has 3 disjoint accessible singular loci on $\mathcal{H}$ as follows:

$$\begin{cases} L_1 = C_1 \cup C_2 \\ \quad = \{(X_1, Y_1, Z_1, W_1) \mid X_1 = Y_1 = W_1 = 0\} \\ \quad \quad \cup \{(X_2, Y_2, Z_2, W_2) \mid Y_2 = Z_2 = W_2 = 0\}, \\ S_3 = \{(X_3, Y_3, Z_3, W_3) \mid X_3 = Y_3 = 0\}, \\ S_4 = \{(X_4, Y_4, Z_4, W_4) \mid Z_4 = W_4 = 0\}. \end{cases}$$

Here, $L_1(\cong \mathbf{P}^1) \subset V_1 \cup V_2$, $S_3(\cong \mathbf{C}^2) \subset V_3$, and $S_4(\cong \mathbf{C}^2) \subset V_4$.

Remark 2.1 (cf. [5]). Let us see the condition (A1) in the local coordinate system on V_1 . By using the local coordinate system (X_1, Y_1, Z_1, W_1) of V_1 , it is easy to see that the condition (A1) is equivalent to the following condition:

$$\begin{aligned} \tilde{v}|_{V_1} &= \frac{\partial}{\partial t} + F_1(X_1, Y_1, Z_1, W_1, t) \frac{\partial}{\partial X_1} + \frac{F_2(X_1, Y_1, Z_1, W_1, t)}{X_1} \frac{\partial}{\partial Y_1} \\ &\quad + \frac{F_3(X_1, Y_1, Z_1, W_1, t)}{X_1} \frac{\partial}{\partial Z_1} + \frac{F_4(X_1, Y_1, Z_1, W_1, t)}{X_1} \frac{\partial}{\partial W_1}, \end{aligned}$$

where $F_i(X_1, Y_1, Z_1, W_1, t) \in \mathbf{C}(t)[X_1, Y_1, Z_1, W_1]$ ($i = 1, 2, 3, 4$).

Proposition 2.1. *Let us consider the Hamiltonian vector field with the Hamiltonian of degree 4 as in (3). Assume that the conditions (A1) and (A2) hold. Then the vector field v on $\mathbf{C}^4 \times B$ can be written as follows:*

Fig. 3. Condition (A2) on the boundary divisor $\mathcal{H}$

$$(4) \quad \begin{cases} \frac{dx}{dt} = \frac{-2x^2y + b(t)xzw + x^2 + xz}{t} - 2xy + \left(1 + \frac{\alpha_1 + \alpha_3}{t}\right)x - \alpha_2 \\ \frac{dy}{dt} = \frac{2xy^2 - b(t)yzw - 2xy - yz - zw}{t} + y^2 - \left(1 + \frac{\alpha_1 + \alpha_3}{t}\right)y + \frac{\alpha_1}{t} \\ \frac{dz}{dt} = \frac{-2z^2w + b(t)xyz + z^2 + xz}{t} - 2zw + \left(1 + \frac{\beta_1 + \beta_3}{t}\right)z - \beta_2 \\ \frac{dw}{dt} = \frac{2zw^2 - b(t)xyz - 2zw - xy - xw}{t} + w^2 - \left(1 + \frac{\beta_1 + \beta_3}{t}\right)w + \frac{\beta_1}{t}. \end{cases}$$

Here x , y , z and w denote unknown complex variables and $b(t)$ is a rational function of t on B . Moreover, setting $\alpha_1 = \beta_1$, we can resolve the accessible singular loci given in (A2) by means of successive blowing-up.

We remark that besides 3 assigned accessible singular loci, there may be other accessible singularities of equations in (4). It is still an open question whether all of the accessible singularities of (4) can be resolved. By doing Painlevé test, we can obtain $b(t) = \frac{-2}{t}$, $\alpha_3 = \beta_3$. By setting $\beta_2 := \alpha_2$, we can obtain the system (1).

3. Proof of Proposition 2.1

Lemma 3.1. *Let us take an arbitrary polynomial K in $\mathbf{C}(t)[x, y, z, w]$. If its degree is 4 with respect to x, y, z, w , then K is explicitly written as follows:*

$$\begin{aligned} K = & a_1x^4 + a_2y^4 + a_3z^4 + a_4w^4 + a_5xyzw + x^3(a_6y + a_7z + a_8w) \\ & + y^3(a_9x + a_{10}z + a_{11}w) + z^3(a_{12}x + a_{13}y + a_{14}w) + w^3(a_{15}x + a_{16}y + a_{17}z) \end{aligned}$$

$$\begin{aligned}
& + x^2(a_{18}y^2 + a_{19}z^2 + a_{20}w^2) + y^2(a_{21}z^2 + a_{22}w^2) + a_{23}z^2w^2 \\
& + x^2\{y(a_{24}z + a_{25}w) + a_{26}zw\} + y^2\{x(a_{27}z + a_{28}w) + a_{29}zw\} \\
& + z^2\{x(a_{30}y + a_{31}w) + a_{32}yw\} + w^2\{x(a_{33}y + a_{34}z) + a_{35}yz\} + b_1xyz \\
& + b_2xyw + b_3xzw + b_4yzw + b_5x^3 + b_6y^3 + b_7z^3 + b_8w^3 \\
& + x^2(b_9y + b_{10}z + b_{11}w) + y^2(b_{12}x + b_{13}z + b_{14}w) + z^2(b_{15}x + b_{16}y + b_{17}w) \\
& + w^2(b_{18}x + b_{19}y + b_{20}z) + c_1x^2 + c_2y^2 + c_3z^2 + c_4w^2 + c_5xy + c_6xz + c_7xw \\
& + c_8yz + c_9yw + c_{10}zw + c_{11}x + c_{12}y + c_{13}z + c_{14}w + c_{15}.
\end{aligned}$$

We will see that K has 70 unknown parameters. Since $H_V(x, y, t; \alpha_1, \alpha_2, \alpha_3) + H_V(z, w, t; \beta_1, \beta_2, \beta_3)$ and K have no common monomials, we obtain the following conditions:

$$a_{18} = a_{23} = b_9 = b_{12} = b_{17} = b_{20} = c_5 = c_{10} = c_{11} = c_{12} = c_{13} = c_{14} = 0.$$

Now we are ready to prove Proposition 2.1. At first, we prepare the following notation.

Notation.

$$P_i = \{(X_i, Y_i, Z_i, W_i) \mid X_i = Y_i = Z_i = W_i = 0\} \quad (i = 1, 2, 3, 4).$$

3.1. Vanishing conditions on the coefficients in K

Lemma 3.2. *If the rational vector field $\tilde{v}$ is written as in condition (A1) on each $V_i \times B$ ($i = 1, 2, 3, 4$), the coefficients in K satisfy the following vanishing conditions:*

On $V_i \times B$	Vanishing conditions on the coefficients in K
$V_1 \times B$	$a_1 = a_3 = a_6 = a_7 = a_8 = a_{12} = a_{13} = a_{14} = a_{19} = a_{24} = a_{26} = a_{30} = a_{31} = b_5 = b_7 = b_{10} = b_{15} = 0$
$V_2 \times B$	$a_1 = a_3 = a_6 = a_7 = a_8 = a_{12} = a_{13} = a_{14} = a_{19} = a_{24} = a_{26} = a_{30} = a_{31} = b_5 = b_7 = b_{10} = b_{15} = 0$
$V_3 \times B$	$a_2 = a_9 = a_{10} = a_{11} = a_{21} = a_{22} = a_{27} = a_{28} = a_{29} = b_6 = b_{13} = b_{14} = 0$
$V_4 \times B$	$a_4 = a_{15} = a_{16} = a_{17} = a_{20} = a_{22} = a_{33} = a_{34} = a_{35} = b_8 = b_{18} = b_{19} = 0$

Proof. For the open set $V_1 \times B \subset \mathcal{S} \times B$, we use the coordinate system $(X_1, Y_1, Z_1, W_1) = (1/x, y, z/x, w)$ centered at P_1 . By this coordinate system, we can write v in a neighborhood of P_1 as follows:

$$(5) \quad \begin{cases} \frac{dX_1}{dt} = -\frac{a_6 + a_{24}Z_1 + a_{30}Z_1^2 + a_{13}Z_1^3}{X_1} + f_1(X_1, Y_1, Z_1, W_1, t), \\ \frac{dY_1}{dt} = -\frac{4a_1 + 3a_7Z_1 + 2a_{19}Z_1^2 + a_{12}Z_1^3}{X_1^3} \\ \quad - \frac{3b_5 + 3a_8W_1 + 3a_6Y_1 + 2b_{10}Z_1 + 2a_{26}Z_1W_1 + 2a_{24}Y_1Z_1 + b_{15}Z_1^2 + a_{31}Z_1^2W_1 + a_{30}Y_1Z_1^2}{X_1^2} \\ \quad + \frac{f_5(X_1, Y_1, Z_1, W_1, t)}{X_1} + f_2(X_1, Y_1, Z_1, W_1, t), \\ \frac{dZ_1}{dt} = \frac{a_8 + a_{26}Z_1 - a_6Z_1 - a_{24}Z_1^2 + a_{31}Z_1^2 - a_{13}Z_1^4 + a_{14}Z_1^3 - a_{30}Z_1^3}{X_1^2} \\ \quad + \frac{f_6(X_1, Y_1, Z_1, W_1, t)}{X_1} + f_3(X_1, Y_1, Z_1, W_1, t), \\ \frac{dW_1}{dt} = -\frac{a_7 + 2a_{19}Z_1 + 3a_{12}Z_1^2 - 4a_3Z_1^3}{X_1^3} \\ \quad - \frac{b_{10} + a_{26}W_1 + a_{24}Y_1 + 2b_{15}Z_1 + 2a_{31}Z_1W_1 + 2a_{30}Y_1Z_1 + 3b_7Z_1^2 + 3a_{14}Z_1^2W_1 - 3a_{13}Y_1Z_1^2}{X_1^2} \\ \quad + \frac{f_7(X_1, Y_1, Z_1, W_1, t)}{X_1} + f_4(X_1, Y_1, Z_1, W_1, t), \end{cases}$$

where $f_i(X_1, Y_1, Z_1, W_1, t) \in \mathcal{C}(t)[X_1, Y_1, Z_1, W_1]$ ($i = 1, 2, 3, 4, 5, 6, 7$). Now it is easy to see that the system (5) is written in the following form on $V_1 \times B$:

$$\begin{aligned} \tilde{v}|_{V_1} = & \frac{\partial}{\partial t} + F_1(X_1, Y_1, Z_1, W_1, t) \frac{\partial}{\partial X_1} + \frac{F_2(X_1, Y_1, Z_1, W_1, t)}{X_1} \frac{\partial}{\partial Y_1} \\ & + \frac{F_3(X_1, Y_1, Z_1, W_1, t)}{X_1} \frac{\partial}{\partial Z_1} + \frac{F_4(X_1, Y_1, Z_1, W_1, t)}{X_1} \frac{\partial}{\partial W_1}, \end{aligned}$$

where $F_i(X_1, Y_1, Z_1, W_1, t) \in \mathcal{C}(t)[X_1, Y_1, Z_1, W_1]$ ($i = 1, 2, 3, 4$) if and only if

$$\begin{aligned} a_1 &= a_3 = a_6 = a_7 = a_8 = a_{12} = a_{13} = a_{14} = a_{19} = a_{24} = a_{26} = a_{30} = a_{31} \\ &= b_5 = b_7 = b_{10} = b_{15} = 0. \end{aligned}$$

For other cases, the proof is similar. $\square$

3.2. Accessible singularities and conditions on the coefficients in K

Lemma 3.3. *If L_1 , S_3 and S_4 are accessible singular loci of $\tilde{v}$, the coefficients in K satisfy the following conditions:*

On $V_i \times B$	Conditions on the coefficients in K
$V_1 \times B$	$b_1 = b_3 = \frac{1}{t}, b_{11} = b_{16} = c_1 = c_3 = c_6 = 0$
$V_2 \times B$	$b_1 = b_3 = \frac{1}{t}, b_{11} = b_{16} = c_1 = c_3 = c_6 = 0$
$V_3 \times B$	$a_{32} = b_4 = b_{16} = c_2 = c_8 = c_9 = 0$
$V_4 \times B$	$a_{25} = b_2 = b_{11} = c_4 = c_7 = c_9 = 0$

Proof. For the open set $V_1 \times B$, we use the coordinate system $(X_1, Y_1, Z_1, W_1) = (1/x, y, z/x, w)$ centered at P_1 . By this coordinate system, we can write v in a neighborhood of P_1 as follows:

$$(6) \quad \begin{cases} \frac{dX_1}{dt} = f_1(X_1, Y_1, Z_1, W_1, t), \\ \frac{dY_1}{dt} = \frac{-2c_1 - 2b_{11}W_1 - 2a_{25}Y_1W_1 - c_6Z_1 - b_3Z_1W_1 - b_1Y_1Z_1 - a_5Y_1Z_1W_1}{X_1} \\ \quad + \frac{-2Y_1 + 2Y_1^2}{tX_1} + f_2(X_1, Y_1, Z_1, W_1, t), \\ \frac{dZ_1}{dt} = \frac{b_{11} + a_{25}Y_1 + b_3Z_1 - a_{25}Z_1W_1 + a_5Y_1Z_1 - b_1Z_1^2 - a_5Z_1^2W_1 + a_{32}Y_1Z_1^2 - b_{16}Z_1^3 - a_{32}Z_1^3W_1}{X_1} \\ \quad + \frac{-Z_1 + 2Y_1Z_1 + Z_1^2 - 2Z_1^2W_1}{tX_1} + f_3(X_1, Y_1, Z_1, W_1, t), \\ \frac{dW_1}{dt} = \frac{-c_6 - b_3W_1 - b_1Y_1 - a_5Y_1W_1 - 2c_3Z_1 - 2b_{16}Y_1Z_1 - 2a_{32}Y_1Z_1W_1}{X_1} \\ \quad + \frac{-2Z_1W_1 + 2Z_1W_1^2}{tX_1} + f_4(X_1, Y_1, Z_1, W_1, t), \end{cases}$$

where $f_i(X_1, Y_1, Z_1, W_1, t) \in \mathbf{C}(t)[X_1, Y_1, Z_1, W_1]$ ($i = 1, 2, 3, 4$). If L_1 is an accessible singular locus of the system (6), it is easy to see that

$$b_1 = b_3 = \frac{1}{t}, \quad b_{11} = b_{16} = c_1 = c_3 = c_6 = 0.$$

For other cases, the proof is similar. □

3.3. Resolution of accessible singular loci L_1, S_3 and S_4

3.3.1. Resolution of the accessible singular locus C_1

By the following 2 steps, we can resolve the accessible singular locus C_1 . At first, we take the coordinate system (X_1, Y_1, Z_1, W_1) centered at P_1 .

Step 1: We blow up along the curve $C_1 = \{(X_1, Y_1, Z_1, W_1) \mid X_1 = Y_1 = W_1 = 0\}$

$$X'_1 = X_1, \quad Y'_1 = \frac{Y_1}{X_1}, \quad Z'_1 = Z_1, \quad W'_1 = \frac{W_1}{X_1}.$$

Step 2: We blow up along the surface $S_1 = \{(X'_1, Y'_1, Z'_1, W'_1) \mid X'_1 = Y'_1 + Z'_1 W'_1 - \alpha_1 = 0\}$

$$X''_1 = X'_1, \quad Y''_1 = \frac{Y'_1 + Z'_1 W'_1 - \alpha_1}{X'_1}, \quad Z''_1 = Z'_1, \quad W''_1 = W'_1.$$

3.3.2. Resolution of the accessible singular locus C_2

By the following 2 steps, we can resolve the accessible singular locus C_2 . At first, we take the coordinate system (X_2, Y_2, Z_2, W_2) centered at P_2 .

Step 1: We blow up along the curve $C_2 = \{(X_2, Y_2, Z_2, W_2) \mid Y_2 = Z_2 = W_2 = 0\}$

$$X'_2 = X_2, \quad Y'_2 = \frac{Y_2}{Z_2}, \quad Z'_2 = Z_2, \quad W'_2 = \frac{W_2}{Z_2}.$$

Step 2: We blow up along the surface $S_1 = \{(X'_2, Y'_2, Z'_2, W'_2) \mid W'_2 + X'_2 Y'_2 - \alpha_1 = Z'_2 = 0\}$

$$X''_2 = X'_2, \quad Y''_2 = Y'_2, \quad Z''_2 = Z'_2, \quad W''_2 = \frac{W'_2 + X'_2 Y'_2 - \alpha_1}{Z'_2}.$$

Here, we set $\alpha_1 = \beta_1$ in order to take the common center of blowing-up.

3.3.3. Resolution of the accessible singular locus S_3

By the following 2 steps, we can resolve the accessible singular locus S_3 . At first, we take the coordinate system (X_3, Y_3, Z_3, W_3) centered at P_3 .

Step 1: We blow up along the surface $S_3 = \{(X_3, Y_3, Z_3, W_3) \mid X_3 = Y_3 = 0\}$

$$X'_3 = \frac{X_3}{Y_3}, \quad Y'_3 = Y_3, \quad Z'_3 = Z_3, \quad W'_3 = W_3.$$

Step 2: We blow up along the surface $S'_3 = \{(X'_3, Y'_3, Z'_3, W'_3) \mid X'_3 + \alpha_2 = Y'_3 = 0\}$

$$X''_3 = \frac{X'_3 + \alpha_2}{Y'_3}, \quad Y''_3 = Y'_3, \quad Z''_3 = Z'_3, \quad W''_3 = W'_3.$$

3.3.4. Resolution of the accessible singular locus S_4

By the following 2 steps, we can resolve the accessible singular locus S_4 . At first, we take the coordinate system (X_4, Y_4, Z_4, W_4) centered at P_4 .

Step 1: We blow up along the surface $S_4 = \{(X_4, Y_4, Z_4, W_4) \mid Z_4 = W_4 = 0\}$

$$X'_4 = X_4, \quad Y'_4 = Y_4, \quad Z'_4 = \frac{Z_4}{W_4}, \quad W'_4 = W_4.$$

Step 2: We blow up along the surface $S'_4 = \{(X'_4, Y'_4, Z'_4, W'_4) \mid Z'_4 + \beta_2 = W'_4 = 0\}$

$$X''_4 = X'_4, \quad Y''_4 = Y'_4, \quad Z''_4 = \frac{Z'_4 + \beta_2}{W'_4}, \quad W''_4 = W'_4.$$

Remark 3.1. $\tilde{v}$ has more accessible singularities on $\mathcal{H} \times B$ than just the three accessible singular loci in condition (A2).

4. Resolution of the accessible singularities

To prove Theorem 0.1, the latter part of this paper is devoted to giving an explicit resolution of accessible singularities of the systems (1) and to construct a family of phase spaces for the systems. For second-order Painlevé equations, we can obtain the entire space of initial conditions by adding subvarieties of codimension 1 (equivalently, of dimension 1) to the space of initial conditions of holomorphic solutions [2]. But in the case of fourth-order differential equations, we need to add codimension 2 subvarieties to the space in addition to codimension 1 subvarieties. In order to resolve singularities, we need to both blow up and blow down. Moreover, to obtain a smooth variety by blowing-down, we need to resolve for a pair of singularities.

4.1. Accessible singularities of the systems (1)

Let P be an accessible singular point in the boundary divisor $\mathcal{H}$. Rewriting the systems in the local coordinate system, the right hand side of each differential equation has poles along $\mathcal{H}$. If we resolve the accessible singular point P and the right hand side of each differential equation becomes holomorphic in new coordinate system, then we can use Cauchy's existence and uniqueness theorem of solutions. In order to consider a family of phase spaces for the system (1), let us take the compactification $\mathcal{S} \times B$ of $\mathbf{C}^4 \times B$. Fixing parameter α_i and setting $B = \text{Spec } \mathbf{C}[t, t^{-1}] \cong \mathbf{C}^* = \mathbf{C} - \{0\}$, consider the product $\mathcal{S} \times B$ and extend the regular vector field on $\mathbf{C}^4 \times B$ to a rational vector

field $\tilde{v}$ on $\mathcal{S} \times B$. The following Lemma 4.1 shows that this rational vector field $\tilde{v}$ has thirteen accessible singular loci on the boundary divisor $\mathcal{H} \times \{t\} \subset \mathcal{S} \times \{t\}$ for each $t \in B$.

Fig. 4. Accessible singularities of the system (1)

Lemma 4.1. *The rational vector field $\tilde{v}$ has the following accessible singular loci:*

$$\left\{ \begin{array}{l} C_1 = \{(X_1, Y_1, Z_1, W_1) \mid X_1 = Y_1 = W_1 = 0\}, \\ C_2 = \{(X_2, Y_2, Z_2, W_2) \mid Y_2 = Z_2 = W_2 = 0\}, \\ S_3 = \{(X_3, Y_3, Z_3, W_3) \mid X_3 = Y_3 = 0\}, \\ S_4 = \{(X_4, Y_4, Z_4, W_4) \mid Z_4 = W_4 = 0\}, \\ P_5 = \{(X_5, Y_5, Z_5, W_5) \mid X_5 = Y_5 = Z_5 = W_5 = 0\}, \\ P_6 = \{(X_6, Y_6, Z_6, W_6) \mid X_6 = Y_6 = Z_6 = W_6 = 0\}, \\ P_7 = \{(X_7, Y_7, Z_7, W_7) \mid X_7 = Y_7 = Z_7 = W_7 = 0\}, \\ C_8 = \{(X_1, Y_1, Z_1, W_1) \mid X_1 = 0, Y_1 = W_1 = 1\}, \\ C_9 = \{(X_2, Y_2, Z_2, W_2) \mid Z_2 = 0, Y_2 = W_2 = 1\}, \\ P_{10} = \{(X_3, Y_3, Z_3, W_3) \mid Y_3 = Z_3 = 0, X_3 = -t, W_3 = 1/2\}, \\ P_{11} = \{(X_4, Y_4, Z_4, W_4) \mid X_4 = W_4 = 0, Z_4 = -t, Y_4 = 1/2\}, \\ P_{12} = \{(X_7, Y_7, Z_7, W_7) \mid Y_7 = Z_7 = W_7 = 0, X_7 = -t\}, \\ P_{13} = \{(X_7, Y_7, Z_7, W_7) \mid X_7 = Y_7 = W_7 = 0, Z_7 = -t\}, \\ P_{14} = \{(X_5, Y_5, Z_5, W_5) \mid X_5 = Y_5 = Z_5 = 0, W_5 = 1\}, \\ P_{15} = \{(X_6, Y_6, Z_6, W_6) \mid X_6 = Z_6 = W_6 = 0, Y_6 = 1\}, \end{array} \right.$$

where $C_1 \cup C_2 = \mathbf{P}^1$ and $C_8 \cup C_9 = \mathbf{P}^1$ and $S_3 \cong \mathbf{C}^2$ and $S_4 \cong \mathbf{C}^2$ (See Figure 4).

This lemma can be proven by direct calculation. □

Remark 4.1. By the symmetry

$$\pi : (x, y, z, w; \alpha_1, \alpha_2, \alpha_3, \alpha_4) \rightarrow (z, w, x, y; \alpha_1, \alpha_4, \alpha_3, \alpha_2),$$

it is easy to see that

$$\begin{aligned} \pi(C_1) &= C_2, & \pi(S_3) &= S_4, & \pi(P_5) &= P_6, & \pi(C_8) &= C_9, \\ \pi(P_{10}) &= P_{11}, & \pi(P_{12}) &= P_{13}, & \pi(P_{14}) &= P_{15}. \end{aligned}$$

4.2. Summary of resolution process

Let us start by summarizing the steps which are needed to resolve the accessible singularities of $\tilde{v}$:

- (1) Step 1: We blow up along the curves $L_1 = C_1 \cup C_2$ and $L_2 = C_8 \cup C_9$.
- (2) Step 2: We blow up along the surfaces F_1, F_2, F_3 and F_4 .
- (3) Step 3: We blow down the 3-folds V_1, V_2, V_3 and V_4 .
- (4) Step 4: We blow up along the surfaces F_5 and F_6 .
- (5) Step 5: We blow up along the surfaces F_7 and F_8 .
- (6) Step 6: We blow up along the surfaces F_9 and F_{10} .
- (7) Step 7: We blow up along the surfaces F_{11} and F_{12} .
- (8) Step 8: We blow up along the surfaces F_{13} and F_{14} .
- (9) Step 9: We blow up along the surfaces F_{15} and F_{16} .
- (10) Step 10: We blow up along the surfaces F_{17} and F_{18} .

Now we are ready to prove Theorem 0.1.

4.3. Resolution of the accessible singular point P_{12}

In this subsection, we will give an explicit resolution process for the accessible singular point P_{12} by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_{12} .

Step 0: We take the coordinate system centered at P_{12}

$$x_{12}^{(1)} = X_7 + t, \quad y_{12}^{(1)} = Y_7, \quad z_{12}^{(1)} = Z_7, \quad w_{12}^{(1)} = W_7.$$

Here we make a change of variables

$$x_{12}^{(2)} = x_{12}^{(1)}, \quad y_{12}^{(2)} = y_{12}^{(1)}, \quad z_{12}^{(2)} = z_{12}^{(1)}, \quad w_{12}^{(2)} = \frac{1}{w_{12}^{(1)}} - \frac{1}{y_{12}^{(1)}}.$$

For the process from Step 1 to Step 3, the accessible singular point P_{12} is not yet transformed.

Step 4: We blow up along the surface F_5

$$x_{12}^{(3)} = \frac{x_{12}^{(2)} + z_{12}^{(2)}}{y_{12}^{(2)}}, \quad y_{12}^{(3)} = y_{12}^{(2)}, \quad z_{12}^{(3)} = z_{12}^{(2)}, \quad w_{12}^{(3)} = w_{12}^{(2)}.$$

Step 5: We blow up along the surface F_7

$$x_{12}^{(4)} = \frac{x_{12}^{(3)} + 1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4}{y_{12}^{(3)}}, \quad y_{12}^{(4)} = y_{12}^{(3)},$$

$$z_{12}^{(4)} = z_{12}^{(3)}, \quad w_{12}^{(4)} = w_{12}^{(3)}.$$

4.4. Resolution of the accessible singular point P_{10}

In this subsection, we will give an explicit resolution process for the accessible singular point P_{10} by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_{10} .

Step 0: We take the coordinate system centered at P_{10}

$$x_{10}^{(1)} = X_3 + t, \quad y_{10}^{(1)} = Y_3, \quad z_{10}^{(1)} = Z_3, \quad w_{10}^{(1)} = W_3 - \frac{1}{2}.$$

Here we make a change of variable

$$x_{10}^{(2)} = x_{10}^{(1)}, \quad y_{10}^{(2)} = y_{10}^{(1)}, \quad z_{10}^{(2)} = z_{10}^{(1)}, \quad w_{10}^{(2)} = \frac{1}{w_{10}^{(1)} + \frac{1}{2} - \frac{1}{y_{10}^{(1)}}}.$$

For the process from Step 1 to Step 3, the accessible singular point P_{10} is not yet transformed.

Step 4: We blow up along the surface F_5

$$x_{10}^{(3)} = \frac{x_{10}^{(2)} + z_{10}^{(2)}}{y_{10}^{(2)}}, \quad y_{10}^{(3)} = y_{10}^{(2)}, \quad z_{10}^{(3)} = z_{10}^{(2)}, \quad w_{10}^{(3)} = w_{10}^{(2)}.$$

Step 5: We blow up along the surface F_7

$$x_{10}^{(4)} = \frac{x_{10}^{(3)} + 1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4}{y_{10}^{(3)}}, \quad y_{10}^{(4)} = y_{10}^{(3)},$$

$$z_{10}^{(4)} = z_{10}^{(3)}, \quad w_{10}^{(4)} = w_{10}^{(3)}.$$

For the process from Step 6 to Step 8, the accessible singular locus $\{(x_{10}^{(4)}, y_{10}^{(4)}, z_{10}^{(4)}, w_{10}^{(4)}) \mid z_{10}^{(4)} = w_{10}^{(4)} = 0\}$ is not yet transformed.

Step 9: We blow up along the surface F_{15}

$$x_{10}^{(5)} = x_{10}^{(4)}, \quad y_{10}^{(5)} = y_{10}^{(4)}, \quad z_{10}^{(5)} = \frac{z_{10}^{(4)}}{w_{10}^{(4)}}, \quad w_{10}^{(5)} = w_{10}^{(4)}.$$

Step 10: We blow up along the surface F_{17}

$$x_{10}^{(6)} = x_{10}^{(5)}, \quad y_{10}^{(6)} = y_{10}^{(5)}, \quad z_{10}^{(6)} = \frac{z_{10}^{(5)} + \alpha_4}{w_{10}^{(5)}}, \quad w_{10}^{(6)} = w_{10}^{(5)}.$$

4.5. Resolution of the accessible singular locus C_8

In this subsection, we will give an explicit resolution process for the accessible singular locus C_8 by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular locus C_8 .

Step 0: We take the coordinate system centered at $\{(X_1, Y_1, Z_1, W_1) \mid X_1 = Z_1 = 0, Y_1 = W_1 = 1\}$

$$x_8^{(1)} = X_1, \quad y_8^{(1)} = Y_1 - 1, \quad z_8^{(1)} = Z_1, \quad w_8^{(1)} = W_1 - 1.$$

Step 1: We blow up along the curve $L_2 \cong \mathbf{P}^1$

$$x_8^{(2)} = x_8^{(1)}, \quad y_8^{(2)} = \frac{y_8^{(1)}}{x_8^{(1)}}, \quad z_8^{(2)} = z_8^{(1)}, \quad w_8^{(2)} = \frac{w_8^{(1)}}{x_8^{(1)}}.$$

For the process from Step 2 to Step 7, the accessible singular locus $\{(x_8^{(2)}, y_8^{(2)}, z_8^{(2)}, w_8^{(2)}) \mid y_8^{(2)} + z_8^{(2)}w_8^{(2)} - \alpha_3 = x_8^{(2)} = 0\}$ is not yet transformed.

Step 8: We blow up along the surface F_{14}

$$x_8^{(3)} = x_8^{(2)}, \quad y_8^{(3)} = \frac{y_8^{(2)} + z_8^{(2)}w_8^{(2)} - \alpha_3}{x_8^{(2)}}, \quad z_8^{(3)} = z_8^{(2)}, \quad w_8^{(3)} = w_8^{(2)}.$$

4.6. Resolution of the accessible singular point P_5

In this subsection, we will give an explicit resolution process for the accessible singular point P_5 by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_5 . For the process Step 1, the accessible singular point P_5 is not yet transformed.

Step 2: We blow up along the surface $F_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$

$$x_5^{(1)} = X_5, \quad y_5^{(1)} = Y_5, \quad z_5^{(1)} = Z_5, \quad w_5^{(1)} = \frac{W_5}{Z_5}.$$

Step 3: We blow down the 3-fold $V_1 \cong \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$

$$x_5^{(2)} = x_5^{(1)}, \quad y_5^{(2)} = y_5^{(1)}z_5^{(1)}, \quad z_5^{(2)} = z_5^{(1)}, \quad w_5^{(2)} = w_5^{(1)}.$$

The resolution process from Step 2 to Step 3 is well-known as $\mathbf{P}^1 \times \mathbf{P}^1$ -flop. The surface F_1 is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. By Step 2, each point on F_1 is transformed to $\mathbf{P}^1$. The 3-fold V_1 is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. In order to resolve the accessible singular point P_5 and obtain a holomorphic coordinate system, we need to blow down the 3-fold V_1 along another $\mathbf{P}^1$ -fiber. (See Figure 5)

Fig. 5. Resolution process from Step 2 to Step 3

For the process from Step 4 to Step 5, the accessible singular point P_5 is not yet transformed.

Step 6: We blow up along the surface F_9

$$x_5^{(3)} = \frac{x_5^{(2)}}{y_5^{(2)}}, \quad y_5^{(3)} = y_5^{(2)}, \quad z_5^{(3)} = z_5^{(2)}, \quad w_5^{(3)} = w_5^{(2)}.$$

Step 7: We blow up along the surface F_{11}

$$x_5^{(4)} = \frac{x_5^{(3)} + \alpha_2}{y_5^{(3)}}, \quad y_5^{(4)} = y_5^{(3)}, \quad z_5^{(4)} = z_5^{(3)}, \quad w_5^{(4)} = w_5^{(3)}.$$

Step 8: We blow up along the surface F_{13}

$$x_5^{(5)} = x_5^{(4)}, \quad y_5^{(5)} = y_5^{(4)}, \quad z_5^{(5)} = z_5^{(4)}, \quad w_5^{(5)} = \frac{w_5^{(4)} + x_5^{(4)}y_5^{(4)} - \alpha_1 - \alpha_2}{z_5^{(4)}}.$$

4.7. Resolution of the accessible singular point P_{14}

In this subsection, we will give an explicit resolution process for the accessible singular point P_{14} by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_{14} .

Step 0: We take the coordinate system centered at P_{14}

$$x_{14}^{(1)} = X_5, \quad y_{14}^{(1)} = Y_5, \quad z_{14}^{(1)} = Z_5, \quad w_{14}^{(1)} = W_5 - 1.$$

For the process Step 1, the accessible singular point P_{14} is not yet transformed.

Step 2: We blow up along the surface F_3

$$x_{14}^{(2)} = x_{14}^{(1)}, \quad y_{14}^{(2)} = y_{14}^{(1)}, \quad z_{14}^{(2)} = z_{14}^{(1)}, \quad w_{14}^{(2)} = \frac{w_{14}^{(1)}}{z_{14}^{(1)}}.$$

Step 3: We blow down the 3-fold $V_3 \cong \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$

$$x_{14}^{(3)} = x_{14}^{(2)}, \quad y_{14}^{(3)} = \frac{y_{14}^{(2)} z_{14}^{(2)}}{1 - y_{14}^{(2)}}, \quad z_{14}^{(3)} = z_{14}^{(2)}, \quad w_{14}^{(3)} = w_{14}^{(2)}.$$

For the process from Step 4 to Step 5, the accessible singular point P_{14} is not yet transformed.

Step 6: We blow up along the surface F_9

$$x_{14}^{(4)} = \frac{x_{14}^{(3)}}{y_{14}^{(3)}}, \quad y_{14}^{(4)} = y_{14}^{(3)}, \quad z_{14}^{(4)} = z_{14}^{(3)}, \quad w_{14}^{(4)} = w_{14}^{(3)}.$$

Step 7: We blow up along the surface F_{11}

$$x_{14}^{(5)} = \frac{x_{14}^{(4)} + \alpha_2}{y_{14}^{(4)}}, \quad y_{14}^{(5)} = y_{14}^{(4)}, \quad z_{14}^{(5)} = z_{14}^{(4)}, \quad w_{14}^{(5)} = w_{14}^{(4)}.$$

Step 8: We blow up along the surface F_{14}

$$x_{14}^{(6)} = x_{14}^{(5)}, \quad y_{14}^{(6)} = y_{14}^{(5)}, \quad z_{14}^{(6)} = z_{14}^{(5)},$$

$$w_{14}^{(6)} = \frac{w_{14}^{(5)} + x_{14}^{(5)} y_{14}^{(5)} - \alpha_2 - \alpha_3}{z_{14}^{(5)}}.$$

4.8. Resolution of the accessible singular point P_7

In this subsection, we will give an explicit resolution process for the accessible singular point P_7 by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_7 . For the process from Step 1 to Step 5, the accessible singular point P_7 is not yet transformed.

Step 6: We blow up along the surfaces F_9 and F_{10}

$$x_7^{(1)} = \frac{X_7}{Y_7}, \quad y_7^{(1)} = Y_7, \quad z_7^{(1)} = \frac{Z_7}{W_7}, \quad w_7^{(1)} = W_7.$$

Step 7: We blow up along the surfaces F_{11} and F_{12}

$$x_7^{(2)} = \frac{x_7^{(1)} + \alpha_2}{y_7^{(1)}}, \quad y_7^{(2)} = y_7^{(1)}, \quad z_7^{(2)} = \frac{z_7^{(1)} + \alpha_4}{w_7^{(1)}}, \quad w_7^{(2)} = w_7^{(1)}.$$

5. Hamiltonian system

At first, we recall the definition of a symplectic transformation and its properties (cf. [7]). Let

$$\begin{aligned} \varphi : x &= x(X, Y, Z, W, t), & y &= y(X, Y, Z, W, t), \\ z &= z(X, Y, Z, W, t), & w &= w(X, Y, Z, W, t) \\ t &= t \end{aligned}$$

be a biholomorphic mapping from a domain D in $\mathbf{C}^5 \ni (X, Y, Z, W, t)$ into $\mathbf{C}^5 \ni (x, y, z, w, t)$. We say that the mapping is symplectic if

$$dx \wedge dy + dz \wedge dw = dX \wedge dY + dZ \wedge dW,$$

where t is considered as a constant or a parameter, namely, if, for $t = t_0$, $\varphi_{t_0} = \varphi|_{t=t_0}$ is a symplectic mapping from the t_0 -section D_{t_0} of D to $\varphi(D_{t_0})$. Suppose that the mapping is symplectic. Then any Hamiltonian system $dx/dt = \partial H/\partial y$, $dy/dt = -\partial H/\partial x$, $dz/dt = \partial H/\partial w$, $dw/dt = -\partial H/\partial z$ is transformed to $dX/dt = \partial K/\partial Y$, $dY/dt = -\partial K/\partial X$, $dZ/dt = \partial K/\partial W$, $dW/dt = -\partial K/\partial Z$ where

$$dx \wedge dy + dz \wedge dw - dH \wedge dt = dX \wedge dY + dZ \wedge dW - dK \wedge dt.$$

Here t is considered as a variable. By the equation, the function K is determined from H uniquely modulo functions of t , namely, modulo functions independent of X , Y , Z and W .

The system (1) is written as a Hamiltonian system:

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z}.$$

Here the Hamiltonian is given as follows:

$$H = \frac{x^2y - x^2y^2 + xzw + xyz - 2xyzw + z^2w - z^2w^2 - \alpha_1x + \alpha_1xy - \alpha_1z + \alpha_1zw}{t} \\ + \frac{\alpha_3xy + \alpha_3zw}{t} + xy - xy^2 + zw - zw^2 - \alpha_2y - \alpha_4w.$$

We list below the Hamiltonian for each coordinate system (x_i, y_i, z_i, w_i) ($i = 1, 3, 5, 7, 8, 10, 12, 14$).

The Hamiltonian H_1 in U_1 . We obtain

$$H_1 = -\frac{1}{t}(y_1 + tx_1y_1 + x_1^2y_1^2 + tx_1^3y_1^2 + tx_1z_1w_1^2 + y_1z_1 + 2tx_1^2y_1z_1w_1 + tx_1z_1^2w_1^2 \\ - t\alpha_1 - \alpha_1x_1y_1 - 2\alpha_1tx_1^2y_1 - 2\alpha_1tx_1z_1w_1 + t\alpha_1^2x_1 - t\alpha_2x_1^2y_1 - t\alpha_2x_1z_1w_1 \\ + t\alpha_1\alpha_2x_1 + \alpha_3x_1y_1 - \alpha_1\alpha_3 + t\alpha_4x_1w_1),$$

where

$$x_1 = \frac{1}{x}, \quad y_1 = -x(xy + zw - \alpha_1), \quad z_1 = \frac{z}{x}, \quad w_1 = xw.$$

The Hamiltonian H_1 and coordinate system (x_1, y_1, z_1, w_1) above satisfy the following condition:

$$dx_1 \wedge dy_1 + dz_1 \wedge dw_1 - dH_1 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

The Hamiltonian H_3 in U_3 . We obtain

$$H_3 = \frac{1}{t}(tx_3 - tx_3y_3 - x_3^2y_3^2 + x_3^2y_3^3 + tz_3w_3 - tz_3w_3^2 - x_3y_3z_3 + 2x_3y_3z_3w_3 \\ - x_3y_3^2z_3w_3 + z_3^2w_3 - z_3^2w_3^2 - \alpha_1x_3y_3 + \alpha_1x_3y_3^2 - \alpha_1z_3 + \alpha_1z_3w_3 - t\alpha_2 \\ - 2\alpha_2x_3y_3 + 2\alpha_2x_3y_3^2 - \alpha_2z_3 + 2\alpha_2z_3w_3 - \alpha_2y_3z_3w_3 - \alpha_1\alpha_2 + \alpha_1\alpha_2y_3 \\ - \alpha_2^2 + \alpha_2^2y_3 - \alpha_3x_3y_3 + \alpha_3z_3w_3 - \alpha_2\alpha_3 - t\alpha_4w_3),$$

where

$$x_3 = -y(xy + \alpha_2), \quad y_3 = \frac{1}{y}, \quad z_3 = z, \quad w_3 = w.$$

The Hamiltonian H_3 and coordinate system (x_3, y_3, z_3, w_3) above satisfy the following condition:

$$dx_3 \wedge dy_3 + dz_3 \wedge dw_3 - dH_3 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

The Hamiltonian H_5 in U_5 . We obtain

$$\begin{aligned}
H_5 = \frac{-1}{t} & (-w_5 - x_5 y_5^2 w_5 - t z_5 w_5 + t x_5 z_5 + t x_5^2 y_5^2 z_5 + z_5^2 w_5^2 - 2 t x_5 y_5 z_5^2 w_5 \\
& + t z_5^3 w_5^2 - t \alpha_1 + \alpha_1 z_5 w_5 - 2 t \alpha_1 x_5 y_5 z_5 + 2 t \alpha_1 z_5^2 w_5 + t \alpha_1^2 z_5 + \alpha_2 y_5 w_5 \\
& - 2 t \alpha_2 x_5 y_5 z_5 + 2 t \alpha_2 z_5^2 w_5 + 2 t \alpha_1 \alpha_2 z_5 + t \alpha_2^2 z_5 - \alpha_3 z_5 w_5 - \alpha_1 \alpha_3 \\
& - t \alpha_4 x_5 y_5 z_5 + t \alpha_4 z_5^2 w_5 + t \alpha_1 \alpha_4 z_5 + t \alpha_2 \alpha_4 z_5),
\end{aligned}$$

where

$$x_5 = -yz(xy + \alpha_2), \quad y_5 = \frac{1}{yz}, \quad z_5 = \frac{1}{z}, \quad w_5 = -z(xy + zw - \alpha_1).$$

The Hamiltonian H_5 and coordinate system (x_5, y_5, z_5, w_5) above satisfy the following condition:

$$dx_5 \wedge dy_5 + dz_5 \wedge dw_5 - dH_5 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

The Hamiltonian H_7 in U_7 . We obtain

$$\begin{aligned}
H_7 = \frac{1}{t} & (tx_7 - tx_7 y_7 - x_7^2 y_7^2 + x_7^2 y_7^3 + tz_7 - tz_7 w_7 - 2x_7 y_7 z_7 w_7 + x_7 y_7 z_7 w_7^2 \\
& + x_7 y_7^2 z_7 w_7 - z_7^2 w_7^2 + z_7^2 w_7^3 - \alpha_1 x_7 y_7 + \alpha_1 x_7 y_7^2 - \alpha_1 z_7 w_7 + \alpha_1 z_7 w_7^2 \\
& - t \alpha_2 - 2 \alpha_2 x_7 y_7 + 2 \alpha_2 x_7 y_7^2 - 2 \alpha_2 z_7 w_7 + \alpha_2 z_7 w_7^2 + \alpha_2 y_7 z_7 w_7 - \alpha_1 \alpha_2 \\
& + \alpha_1 \alpha_2 y_7 - \alpha_2^2 + \alpha_2^2 y_7 - \alpha_3 x_7 y_7 - \alpha_3 z_7 w_7 - \alpha_2 \alpha_3 - t \alpha_4 - 2 \alpha_4 x_7 y_7 \\
& + \alpha_4 x_7 y_7 w_7 + \alpha_4 x_7 y_7^2 - 2 \alpha_4 z_7 w_7 + 2 \alpha_4 z_7 w_7^2 - \alpha_1 \alpha_4 + \alpha_1 \alpha_4 w_7 \\
& - 2 \alpha_2 \alpha_4 + \alpha_2 \alpha_4 w_7 + \alpha_2 \alpha_4 y_7 - \alpha_3 \alpha_4 - \alpha_4^2 + \alpha_4^2 w_7),
\end{aligned}$$

where

$$x_7 = -y(xy + \alpha_2), \quad y_7 = \frac{1}{y}, \quad z_7 = -w(zw + \alpha_4), \quad w_7 = \frac{1}{w}.$$

The Hamiltonian H_7 and coordinate system (x_7, y_7, z_7, w_7) above satisfy the following condition:

$$dx_7 \wedge dy_7 + dz_7 \wedge dw_7 - dH_7 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

The Hamiltonian H_8 in U_8 . We obtain

$$\begin{aligned}
H_8 = -\frac{1}{t} & (y_8 + tx_8 y_8 + x_8^2 y_8^2 + tx_8^3 y_8^2 + tx_8 z_8 w_8^2 + y_8 z_8 - 2tx_8^2 y_8 z_8 w_8 \\
& + tx_8 z_8^2 w_8^2 - \alpha_1 x_8 y_8 + t \alpha_2 + t \alpha_2 x_8^2 y_8 - t \alpha_2 x_8 z_8 w_8 + t \alpha_3 + \alpha_3 x_8 y_8 \\
& + 2t \alpha_3 x_8^2 y_8 - 2t \alpha_3 x_8 z_8 w_8 - \alpha_1 \alpha_3 + t \alpha_2 \alpha_3 x_8 + t \alpha_3^2 x_8 + t \alpha_4 + t \alpha_4 x_8 w_8),
\end{aligned}$$

where

$$x_8 = \frac{1}{x}, \quad y_8 = -x(-x + xy - z + zw - \alpha_3), \quad z_8 = \frac{z}{x}, \quad w_8 = (w - 1)x.$$

The Hamiltonian H_8 and coordinate system (x_8, y_8, z_8, w_8) above satisfy the following condition:

$$dx_8 \wedge dy_8 + dz_8 \wedge dw_8 - dH_8 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

The Hamiltonian H_{10} in U_{10} . We obtain

$$\begin{aligned} H_{10} = & \frac{1}{t}(-1 + t - tx_{10} + y_{10} - 2x_{10}y_{10} + tx_{10}y_{10} + 2x_{10}y_{10}^2 - x_{10}^2y_{10}^2 \\ & + x_{10}^2y_{10}^3 + tz_{10} - 2z_{10}w_{10} + y_{10}z_{10}w_{10} - 2x_{10}y_{10}z_{10}w_{10} \\ & + x_{10}y_{10}^2z_{10}w_{10} - z_{10}^2w_{10}^2 + \alpha_1 - \alpha_1y_{10} + \alpha_1x_{10}y_{10} - \alpha_1x_{10}y_{10}^2 \\ & + \alpha_1z_{10}w_{10} - \alpha_1y_{10}z_{10}w_{10} + 2\alpha_2 - t\alpha_2 - 2\alpha_2y_{10} + 2\alpha_2x_{10}y_{10} \\ & - 2\alpha_2x_{10}y_{10}^2 + 2\alpha_2z_{10}w_{10} - \alpha_2y_{10}z_{10}w_{10} - \alpha_1\alpha_2 + \alpha_1\alpha_2y_{10} - \alpha_2^2 \\ & + \alpha_2^2y_{10} + \alpha_3 - \alpha_3t - 2\alpha_3y_{10} + \alpha_3x_{10}y_{10} - 2\alpha_3x_{10}y_{10}^2 + \alpha_3z_{10}w_{10} \\ & - \alpha_3y_{10}z_{10}w_{10} + \alpha_1\alpha_3y_{10} - \alpha_2\alpha_3 + 2\alpha_2\alpha_3y_{10} + \alpha_3^2y_{10} - t\alpha_4 \\ & - \alpha_4y_{10} - \alpha_4x_{10}y_{10}^2 - \alpha_4y_{10}z_{10}w_{10} + \alpha_2\alpha_4y_{10} + \alpha_3\alpha_4y_{10}), \end{aligned}$$

where

$$\begin{aligned} x_{10} = & -y(1 + ty + xy + yz - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4), \quad y_{10} = \frac{1}{y}, \\ z_{10} = & -(w - y)(z(w - y) + \alpha_4), \quad w_{10} = \frac{1}{w - y}. \end{aligned}$$

The Hamiltonian H_{10} and coordinate system $(x_{10}, y_{10}, z_{10}, w_{10})$ above satisfy the following condition:

$$dx_{10} \wedge dy_{10} + dz_{10} \wedge dw_{10} - dH_{10} \wedge dt = dx \wedge dy + dz \wedge dw - d(H + y) \wedge dt.$$

The Hamiltonian H_{12} in U_{12} . We obtain

$$\begin{aligned} H_{12} = & \frac{1}{t}(-1 + t - tx_{12} + y_{12} - 2x_{12}y_{12} + tx_{12}y_{12} + 2x_{12}y_{12}^2 - x_{12}^2y_{12}^2 + x_{12}^2y_{12}^3 \\ & + \alpha_1 - \alpha_1y_{12} + \alpha_1x_{12}y_{12} - \alpha_1x_{12}y_{12}^2 + 2\alpha_2 - t\alpha_2 - 2\alpha_2y_{12} + 2\alpha_2x_{12}y_{12} \\ & - 2\alpha_2x_{12}y_{12}^2 - \alpha_1\alpha_2 + \alpha_1\alpha_2y_{12} - \alpha_2^2 + \alpha_2^2y_{12} + \alpha_3 - t\alpha_3 - 2\alpha_3y_{12} \end{aligned}$$

$$\begin{aligned}
& + \alpha_3 x_{12} y_{12} - 2\alpha_3 x_{12} y_{12}^2 + \alpha_1 \alpha_3 y_{12} - \alpha_2 \alpha_3 + 2\alpha_2 \alpha_3 y_{12} + \alpha_3^2 y_{12} + 2\alpha_4 \\
& - t\alpha_4 - 2\alpha_4 y_{12} + 2\alpha_4 x_{12} y_{12} - 2\alpha_4 x_{12} y_{12}^2 - \alpha_1 \alpha_4 + \alpha_1 \alpha_4 y_{12} - 2\alpha_2 \alpha_4 \\
& + 2\alpha_2 \alpha_4 y_{12} - \alpha_3 \alpha_4 + 2\alpha_3 \alpha_4 y_{12} - \alpha_4^2 + \alpha_4^2 y_{12} + 2z_{12} w_{12} - y_{12} z_{12} w_{12} \\
& + 2x_{12} y_{12} z_{12} w_{12} - x_{12} y_{12}^2 z_{12} w_{12} - \alpha_1 z_{12} w_{12} + \alpha_1 y_{12} z_{12} w_{12} - 2\alpha_2 z_{12} w_{12} \\
& + \alpha_2 y_{12} z_{12} w_{12} - \alpha_3 z_{12} w_{12} + \alpha_3 y_{12} z_{12} w_{12} - t\alpha_4 w_{12} - 2\alpha_4 z_{12} w_{12} \\
& + \alpha_4 y_{12} z_{12} w_{12} - tz_{12} w_{12}^2 - z_{12}^2 w_{12}^2),
\end{aligned}$$

where

$$x_{12} = -y(1 + ty + xy + yz - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4), \quad y_{12} = \frac{1}{y},$$

$$z_{12} = z, \quad w_{12} = w - y.$$

The Hamiltonian H_{12} and coordinate system $(x_{12}, y_{12}, z_{12}, w_{12})$ above satisfy the following condition:

$$dx_{12} \wedge dy_{12} + dz_{12} \wedge dw_{12} - dH_{12} \wedge dt = dx \wedge dy + dz \wedge dw - d(H + y) \wedge dt.$$

The Hamiltonian H_{14} in U_{14} . We obtain

$$\begin{aligned}
H_{14} = & -\frac{1}{t}(w_{14} + x_{14}y_{14}^2w_{14} + tz_{14}w_{14} + tx_{14}z_{14} + tx_{14}^2y_{14}^2z_{14} + z_{14}^2w_{14}^2 \\
& - 2tx_{14}y_{14}z_{14}^2w_{14} + tz_{14}^3w_{14}^2 - \alpha_1 z_{14}w_{14} + t\alpha_2 - \alpha_2 y_{14}w_{14} \\
& - 2t\alpha_2 x_{14}y_{14}z_{14} + 2t\alpha_2 z_{14}^2w_{14} + t\alpha_2^2 z_{14} + t\alpha_3 + \alpha_3 z_{14}w_{14} \\
& - 2t\alpha_3 x_{14}y_{14}z_{14} + 2t\alpha_3 z_{14}^2w_{14} - \alpha_1 \alpha_3 + 2t\alpha_2 \alpha_3 z_{14} + t\alpha_3^2 z_{14} \\
& + t\alpha_4 - t\alpha_4 x_{14}y_{14}z_{14} + t\alpha_4 z_{14}^2w_{14} + t\alpha_2 \alpha_4 z_{14} + t\alpha_3 \alpha_4 z_{14}),
\end{aligned}$$

where

$$x_{14} = -(-1 + y)z(-x + xy + \alpha_2), \quad y_{14} = \frac{1}{z(y - 1)},$$

$$z_{14} = \frac{1}{z}, \quad w_{14} = -z(-x + xy - z + zw - \alpha_3).$$

The Hamiltonian H_{14} and coordinate system $(x_{14}, y_{14}, z_{14}, w_{14})$ above satisfy the following condition:

$$dx_{14} \wedge dy_{14} + dz_{14} \wedge dw_{14} - dH_{14} \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

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