Periodic Solutions of Autonomous Functional-Differential Equations with State Dependent Deviations

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Abstract. In this paper, we consider periodic solutions of the functional-differential equation $x'' + x(t - kx) = 0$. The structure of the set $\mathcal{A}_k$ ($k \in (0, \infty)$) of all its non-trivial periodic solutions x satisfying $x' < 1/k$ on $\mathbf{R}$ is described. It is proved that for each $k \in (0, \infty)$ and $T_* \in (2\pi, \infty)$, there exists $x \in \mathcal{A}_k$ having the period T_* and for each $k \in (0, \infty)$ and $a \in (0, 1/k)$, there exists a unique $x \in \mathcal{A}_k$ such that $x(0) = 0$ and $x'(0) = a$.

Key Words and Phrases. Second order functional-differential equation, State dependent deviation, Periodic solution.

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1. Introduction

In this paper, we discuss periodic solutions x of the functional-differential equation

$$(1.1) \quad x''(t) + x(t - kx(t)) = 0, \quad k \in (0, \infty)$$

for which

$$(1.2) \quad x'(t) < \frac{1}{k}, \quad t \in \mathbf{R}.$$

We say that a function $x \in C^2(\mathbf{R})$ is a *periodic solution of problem* (1.1), (1.2) if x satisfies (1.1) and (1.2) for $t \in \mathbf{R}$, $x \neq 0$ and x is a periodic function.

If x is a periodic solution of problem (1.1), (1.2) with a period P , then x changes its sign on any interval of the length P . (By the period P of a nonconstant periodic function x we mean the number $P = \inf\{a : a \in (0, \infty), x(t+a) = x(t) \text{ for } t \in \mathbf{R}\}$.) Otherwise $x \geq 0$ or $x \leq 0$ on $\mathbf{R}$ and then $x'' \leq 0$ or $x'' \geq 0$ on $\mathbf{R}$ which implies that x' is a monotone function and since x is P -periodic, $x' = 0$. Hence x is a constant function and from $x''(t) + x(t - kx(t)) = 0$ we see that $x = 0$, which is a contradiction.

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Notice that (1.1) is a functional-differential equation with a deviation depending on solutions. Since any periodic solution of (1.1) changes its sign, equation (1.1) does not belong to the class of functional-differential equations with state dependent delay. In the literature periodic solutions are considered for functional-differential equations with state dependent delays of the form $r(t, x)$ where r is nonnegative and bounded on a subset of $\mathbf{R}^2$. For the first-order scalar functional-differential equations see e.g. [1], [2], [4] and references therein. But in [5] and [6] periodic solutions are considered for iterative functional-differential equations, which belong to the class of functional-differential equations with a state dependent deviation. In [5] the existence and properties of the first-order iterative functional-differential equation $x' = \omega(t)(ax - bx(x))$, $a > b > 0$, are studied. Conditions for the existence of periodic solutions of the second-order functional-differential equation $x'' + g(x(x)) = p(t)$ are given in [6]. Remember also paper [3] where lower bounds for periods of periodic solutions to an autonomous iterative functional-differential equation

$$x'(t) = G(x(t), x^{[2]}(t), \dots, x^{[m]}(t))$$

are presented. Here $G: \mathbf{R}^m \rightarrow \mathbf{R}$ satisfies a Lipschitz condition and $x^{[i]}(t) = (x \circ \dots \circ x)(t)$ is the i th iteration of x .

For $k \in (0, \infty)$, denote by $\mathcal{A}_k$ the set of all periodic solutions x of problem (1.1), (1.2).

The aim of the paper is to consider properties and the structure of the set $\mathcal{A}_k$, $k \in (0, \infty)$.

Together with problem (1.1), (1.2), we consider the associated problem

$$(1.3) \quad y''(t) = y(t) - y(y(t)),$$

$$(1.4) \quad y'(t) > 0, \quad t \in \mathbf{R},$$

and denote by Ω the set of all solutions y on $\mathbf{R}$, $y(t) \not\equiv t$, to problem (1.3), (1.4) for which the function $x(t) = y(t) - t$ is periodic.

The discussion of periodic solutions to problem (1.1), (1.2), that is the discussion of the structure of the set $\mathcal{A}_k$, is based on the following result.

Lemma 1.1. *Let $k \in (0, \infty)$. Then $x \in \mathcal{A}_k$ if and only if the function*

$$(1.5) \quad y(t) = t - kx(t), \quad t \in \mathbf{R},$$

belongs to Ω .

Proof. Let $x \in \mathcal{A}_k$ and y be defined by (1.5). Then $y' > 0$ on $\mathbf{R}$, $t - y(t)$ is periodic, $t - y(t) \not\equiv 0$ and

$$y''(t) = -kx''(t) = kx(t - kx(t)) = kx(y(t)) = y(t) - y(y(t)), \quad t \in \mathbf{R},$$

which shows that y is a solution of (1.3). Hence $y \in \Omega$.

Let $y \in \Omega$ and set $x(t) = (t - y(t))/k$, $t \in \mathbf{R}$. Then x is periodic, $x' = (1 - y')/k < 1/k$ and

$$\begin{aligned} x''(t) &= -\frac{y''(t)}{k} = -\frac{1}{k}[y(t) - y(y(t))] \\ &= -\frac{y(t)}{k} + \left(\frac{y(t)}{k} - x(y(t))\right) = -x(t - kx(t)) \end{aligned}$$

for $t \in \mathbf{R}$. Therefore $x \in \mathcal{A}_k$. □

The rest of the paper is organized as follows. Section 2 is devoted to the study of properties of solutions of (1.3). In Section 3, we introduce a function $T(\alpha)$, ($\alpha \in (0, 1)$) by a (unique) solution y of (1.3) satisfying the initial conditions $y(0) = 0$, $y'(0) = \alpha$, and consider properties of the function $T(\alpha)$ and the set Ω . Main results of the paper are given in Section 4. It is described the structure of the set $\mathcal{A}_k$ and it is proved that for any $k > 0$ and each $T_* \in (2\pi, \infty)$, there exists a T_* -periodic solution of problem (1.1), (1.2). Also it is shown that for any $k > 0$ and $a \in (0, 1/k)$, there exists a unique periodic solution x of problem (1.1), (1.2) satisfying $x(0) = 0$, $x'(0) = a$.

2. Properties of solutions of $y''(t) = y(t) - y(y(t))$

The next lemma is obvious.

Lemma 2.1. *Let $y(t)$ be a solution of (1.3) on an interval J . Then*

$$y : J \rightarrow J.$$

Let $y(t)$ and $z(t)$ be solutions of (1.3) on intervals J and I , respectively. We say that $z(t)$ is a *continuation* of $y(t)$ if $J \subset I$, $J \neq I$, and $y(t) = z(t)$ for $t \in J$. If $t \geq s$ ($t \leq s$) for $t \in I \setminus J$ and any $s \in J$, then we say that $z(t)$ is a *right* (*left*) *continuation* of $y(t)$. If a solution $y(t)$ of (1.3) has no continuation, then $y(t)$ is said to be a *maximal solution* of (1.3).

It is easily seen that for each solution y of (1.3) there exists a maximal solution of (1.3) which is a continuation of y .

Lemma 2.2. *Let $y(t)$ be a maximal solution of (1.3) on an interval J . Then for each $c \in \mathbf{R}$ and $\mu \in \{-1, 1\}$, the function*

$$(2.1) \quad z_{\mu,c}(t) = c + \mu y(\mu(t - c)), \quad t \in I_{\mu,c} = \{t : \mu(t - c) \in J\},$$

is a maximal solution of (1.3) on the interval $I_{\mu,c}$.

Proof. Let $c \in \mathbf{R}$, $\mu \in \{-1, 1\}$ and $z_{\mu,c}$ be defined by (2.1). Then

$$\begin{aligned}
z''_{\mu,c}(t) &= \mu y''(\mu(t-c)) = \mu[y(\mu(t-c)) - y(y(\mu(t-c)))] \\
&= z_{\mu,c}(t) - c - \mu y(\mu(z_{\mu,c}(t) - c)) \\
&= z_{\mu,c}(t) - z_{\mu,c}(z_{\mu,c}(t))
\end{aligned}$$

for $t \in I_{\mu,c}$. Hence $z_{\mu,c}$ is a solution of (1.3) on $I_{\mu,c}$.

If $z_{\mu,c}$ is not a maximal solution of (1.3) for some $c \in \mathbf{R}$ and $\mu \in \{-1, 1\}$, then there is a continuation z of $z_{\mu,c}$ on an interval I , $I_{\mu,c} \subset I$, $I_{\mu,c} \neq I$. Therefore the function $u(t) = \mu[z(c + \mu t) - c]$, $t \in J_* = \{t : c + \mu t \in I\}$, is a continuation of y , which is impossible. $\square$

Lemma 2.3. *Let $y(t)$ be a maximal solution of (1.3) on an interval J . Then there exists $t_0 \in J$ such that*

$$(2.2) \quad y(t_0) = t_0.$$

Proof. First assume $y(t) < t$ for $t \in J$. Then Lemma 2.1 implies that $y(y(t)) < y(t)$ for $t \in J$ which gives $y'' > 0$ on J and y' is increasing. From $y(t) - y(y(t)) = y'(\xi)(t - y(t))$ for $t \in J$ with a $\xi \in (y(t), t)$, we see that $y'(\xi) > 0$. Since y' is increasing and $\xi < t$, we have $0 < y'(\xi) < y'(t)$. Hence $y' > 0$ on J . We now show that $J = \mathbf{R}$. Let

$$b = \sup\{t : t \in J\} < \infty.$$

If $b \in J$ then $y(b) < b$, and consider the initial value problem

$$(2.3) \quad \begin{cases} z'' = z - y(z), \\ z^{(j)}(b) = y^{(j)}(b), \quad j = 0, 1. \end{cases}$$

Since $y \in C^2(J)$, there exists a unique solution z of (2.3) on a neighbourhood $\mathcal{U}$ of $t = b$. Clearly, $z(t) = y(t)$ for $t \in J \cap \mathcal{U}$ and the function $w : J \cup \mathcal{U} \rightarrow \mathbf{R}$ defined by

$$w(t) = \begin{cases} y(t) & \text{for } t \in J \\ z(t) & \text{for } t \in \mathcal{U} \setminus J \end{cases}$$

is a solution of (1.3) on $J \cup \mathcal{U}$. Therefore w is a right continuation of y , which is impossible due to the fact that y is a maximal solution of (1.3). Let $b \notin J$. Since y and y' are increasing on J , there exist $\lim_{t \rightarrow b^-} y(t) = A$ and $\lim_{t \rightarrow b^-} y'(t) = B$. By Lemma 2.1, $A \leq b$. Since

$$\lim_{t \rightarrow b^-} y''(t) = A - \lim_{t \rightarrow b^-} y(y(t)) = A - \lim_{t \rightarrow A^-} y(t)$$

and

$$\lim_{t \rightarrow A^-} y(t) = \begin{cases} A & \text{if } A = b \\ y(A) & \text{if } A < b, \end{cases}$$

we see that $\lim_{t \rightarrow b^-} y''(t)$ is a finite number and $B \in (0, \infty)$. Set

$$u(t) = \begin{cases} y(t) & \text{for } t \in J \\ A & \text{for } t = b. \end{cases}$$

Then

$$\lim_{t \rightarrow b^-} (y(t) - y(y(t))) = \begin{cases} 0 & \text{if } A = b \\ A - y(A) & \text{if } A < b \end{cases}$$

and

$$u(b) - u(u(b)) = A - u(A) = \begin{cases} 0 & \text{if } A = b \\ A - y(A) & \text{if } A < b. \end{cases}$$

Letting $t \rightarrow b^-$ in the equality ($t_0 \in J$)

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \int_{t_0}^t (t - s)[y(s) - y(y(s))]ds, \quad t \in J,$$

we get

$$u(b) = y(t_0) + y'(t_0)(b - t_0) + \int_{t_0}^b (b - s)[u(s) - u(u(s))]ds.$$

Hence

$$u(t) = y(t_0) + y'(t_0)(t - t_0) + \int_{t_0}^t (t - s)[u(s) - u(u(s))]ds, \quad t \in J \cup \{b\},$$

and u is a solution of (1.3) on $J \cup \{b\}$. Therefore u is a right continuation of y , which is impossible. Consequently $b = \infty$. Similar reasoning as in the first part of the proof shows that $\inf\{t : t \in J\} = -\infty$. Hence $J = \mathbf{R}$. Therefore y' is increasing on $\mathbf{R}$. If $\lim_{t \rightarrow \infty} y'(t) > 1$, then there exist $t_1 > 0$ and $\gamma > 1$ such that $y'(t) \geq \gamma$ for $t \in [t_1, \infty)$. It follows that

$$y(t) = y(t_1) + \int_{t_1}^t y'(s)ds \geq y(t_1) + \gamma(t - t_1)$$

and

$$\frac{y(t)}{t} \geq \frac{y(t_1)}{t} + \gamma \left(1 - \frac{t_1}{t}\right)$$

for $t \in [t_1, \infty)$, which yields

$$(2.4) \quad \liminf_{t \rightarrow \infty} \frac{y(t)}{t} \geq \gamma.$$

On the other hand from our assumption $y(t) < t$ for $t \in \mathbf{R}$ we get $y(t)/t < 1$ for $t > 0$, contrary to (2.4). Hence $\lim_{t \rightarrow \infty} y'(t) \leq 1$ and then y' being increasing on $\mathbf{R}$ implies $y'(t) \leq d$ for $t \in (-\infty, -1]$ with a $d < 1$. Consequently,

$$y(-1) - y(t) = \int_t^{-1} y'(s) ds \leq -d(t+1)$$

and

$$\frac{y(t)}{t} \leq \frac{y(-1)}{t} + d \left(1 + \frac{1}{t}\right)$$

for $t \in (-\infty, -1]$. Hence

$$(2.5) \quad \limsup_{t \rightarrow -\infty} \frac{y(t)}{t} \leq d.$$

But $y(t) < t$ for $t \in \mathbf{R}$ gives $\liminf_{t \rightarrow -\infty} (y(t)/t) \geq 1$, contrary to (2.5). Summarizing, we have shown that y does not satisfy $y(t) < t$ for $t \in J$.

Assume now that $y(t) > t$ for $t \in J$. Set $v(t) = -y(-t)$ for $t \in I = \{t : -t \in J\}$. Then v is a maximal solution of (1.3) by Lemma 2.2 (see (2.1) with $c = 0$ and $\mu = -1$). Since $y(-t) > -t$ for $t \in I$, we have $v(t) < t$ for $t \in I$. By the first part of the proof, this situation for v is impossible, and so the inequality $y(t) > t$ for $t \in J$ is not true. We have proved that (2.2) holds with a $t_0 \in J$. $\square$

Lemma 2.4. *A function $y(t)$ is a solution of (1.3) on an interval I and $y(0) = 0$, $y'(0) = \alpha$ if and only if $y(t)$ is a solution of the equation*

$$y(t) = \alpha \sinh t - \int_0^t \sinh(t-s) y(y(s)) ds$$

in the set $C^0(I)$.

Proof. The validity of Lemma 2.4 follows immediately by a simple calculation. $\square$

Lemma 2.5. *Let $\alpha \in (0, 1)$. Then there exists a unique solution $y(t)$ of (1.3) on an interval $[-\varepsilon, \varepsilon]$, $\varepsilon > 0$, such that*

$$y(0) = 0, \quad y'(0) = \alpha.$$

Proof. Let ε be a positive number satisfying the inequalities

$$(2.6) \quad \alpha \cosh \varepsilon + \varepsilon^2 \cosh \varepsilon + \varepsilon \sinh(2\varepsilon) \leq 1,$$

$$(2.7) \quad \varepsilon \sinh \varepsilon < \frac{1}{2}$$

and set

$$\mathcal{B} = \{x : x \in C^0([-\varepsilon, \varepsilon]), x(0) = 0, |x(t_2) - x(t_1)| \leq |t_2 - t_1| \text{ for } t_1, t_2 \in [-\varepsilon, \varepsilon]\}.$$

Then $\mathcal{B}$ is a nonempty and closed subset of the Banach space $C^0([-\varepsilon, \varepsilon])$ endowed with the sup-norm $\|x\| = \max\{|x(t)| : t \in [-\varepsilon, \varepsilon]\}$.

For each $x \in \mathcal{B}$, we have

$$(2.8) \quad |x(t)| \leq |t|, \quad t \in [-\varepsilon, \varepsilon],$$

and therefore $x : [-\varepsilon, \varepsilon] \rightarrow [-\varepsilon, \varepsilon]$. Next, for $x, y \in \mathcal{B}$ and $t \in [-\varepsilon, \varepsilon]$,

$$(2.9) \quad \begin{aligned} |x(x(t)) - y(y(t))| &\leq |x(x(t)) - x(y(t))| + |x(y(t)) - y(y(t))| \\ &\leq |x(t) - y(t)| + \|x - y\| \\ &\leq 2\|x - y\|. \end{aligned}$$

Define the operator $\mathcal{F} : \mathcal{B} \rightarrow C^0([-\varepsilon, \varepsilon])$ by

$$(\mathcal{F}x)(t) = \alpha \sinh t - \int_0^t \sinh(t-s)x(x(s))ds.$$

By Lemma 2.4 and the Banach fixed point theorem, in order to prove our lemma it suffices to verify that

- (i) $\mathcal{F} : \mathcal{B} \rightarrow \mathcal{B}$ and
- (ii) $\mathcal{F}$ is contractive.

In order to prove (i), let $x \in \mathcal{B}$ and $t_1, t_2 \in [-\varepsilon, \varepsilon]$. Then

$$\begin{aligned} &|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| \\ &= \left| \alpha(\sinh t_2 - \sinh t_1) - \int_0^{t_1} [\sinh(t_2 - s) - \sinh(t_1 - s)]x(x(s))ds \right. \\ &\quad \left. - \int_{t_1}^{t_2} \sinh(t_2 - s)x(x(s))ds \right| \\ &\leq \alpha|t_2 - t_1| \cosh \xi + \left| (t_2 - t_1) \int_0^{t_1} \cosh \eta x(x(s))ds \right| \\ &\quad + \left| \int_{t_1}^{t_2} \sinh(t_2 - s)x(x(s))ds \right| \end{aligned}$$

where ξ lies between t_1 and t_2 and η lies between $t_2 - s$ and $t_1 - s$. Now, by (2.6) and (2.8),

$$|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| \leq [\alpha \cosh \varepsilon + \varepsilon^2 \cosh \varepsilon + \varepsilon \sinh(2\varepsilon)]|t_2 - t_1| \leq |t_2 - t_1|,$$

which shows that (i) is true.

Finally, for $x, y \in \mathcal{B}$, we have (see (2.9))

$$|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| = \left| \int_0^t \sinh(t-s)[x(x(s)) - y(y(s))]ds \right| \leq 2\varepsilon \sinh \varepsilon \|x - y\|$$

for $t \in [-\varepsilon, \varepsilon]$. Therefore

$$\|\mathcal{F}x - \mathcal{F}y\| \leq \beta \|x - y\|, \quad x, y \in \mathcal{B},$$

where $\beta = 2\varepsilon \sinh \varepsilon < 1$ by (2.7). Hence (ii) is satisfied. $\square$

Lemma 2.6. *Let $\alpha \in (0, 1)$. Then*

(a) *there exists a maximal solution of (1.3) satisfying the condition*

$$(2.10) \quad y(0) = 0, \quad y'(0) = \alpha,$$

(b) *there exists a unique $t_0 \in (0, \infty)$ such that any maximal solution $y(t)$ of the initial value problem (1.3), (2.10) satisfies $y(t_0) = t_0$, $y(t) < t$ for $t \in (0, t_0)$ and*

$$(2.11) \quad y(t) = -y(-t) \quad \text{for } t \in [-t_0, 0],$$

(c) *there exists a unique solution of the initial value problem (1.3), (2.10) on the interval $[-t_0, t_0]$.*

Proof. By Lemma 2.5, there exists a unique solution v of (1.3) on an interval $[-\varepsilon, \varepsilon]$, $\varepsilon > 0$, satisfying (2.10) (with $y = v$). Let y be a maximal solution of (1.3) on an interval J which is a continuation of v . Hence (a) is true. From (2.10) we deduce that $y(t) < t$ for $t \in (0, \varepsilon_0)$ with an $\varepsilon_0 \in (0, \varepsilon]$. Let $J_1 = \{t : t \in J, 0 < y(s) < s \text{ for } 0 < s \leq t\}$. Then $y(y(t)) < y(t)$ for $t \in J_1$, and therefore $y''(t) = y(t) - y(y(t)) > 0$ on this interval. Hence y' is increasing and $y' > \alpha$ on J_1 by (2.10). If $J_1 = (0, \infty)$, then $\lim_{t \rightarrow \infty} y'(t) \leq 1$ (see the proof of Lemma 2.3) and from

$$y'''(t) = y'(t)[1 - y'(y(t))] > 0, \quad t \in (0, \infty)$$

and $y''(0) = 0$ we see that y'' is increasing and $y'' > 0$ on $(0, \infty)$, which contradicts $\lim_{t \rightarrow \infty} y'(t) \leq 1$. Therefore

$$\sup\{t : t \in J_1\} = t_0 < \infty, \quad t_0 \geq \varepsilon_0.$$

We now show that $y(t_0) = t_0$. First we prove that $t_0 \in J$. If not, using the same procedure as in the proof of Lemma 2.3, there is a right continuation of y on $J \cup \{t_0\}$, contrary to y being a maximal solution of (1.3). Hence $t_0 \in J$. If $y(t_0) < t_0$, then using the same procedure as in the proof of Lemma 2.3, there exists $\tau > 0$ such that $J_1 \cup (t_0, t_0 + \tau] \subset J$ and $y(t) < t$ for $t \in J_1 \cup (t_0, t_0 + \tau]$, contrary to the definition of J_1 and t_0 . Hence $y(t_0) = t_0$. Let u be another

maximal solution of the initial value problem (1.3), (2.10) and let $u(t_1) = t_1$ and $u(t) < t$ for $t \in (0, t_1)$ with a $t_1 > 0$. Without loss of generality we can assume that $t_1 \leq t_0$. We now prove that $u(t) = y(t)$ for $t \in [0, t_1]$. Assume $y(t) \neq u(t)$ for $t \in [0, t_1]$. Lemma 2.5 shows that $y(t) = u(t)$ for $t \in [0, \varepsilon]$ with an $\varepsilon > 0$. Let

$$\chi = \max\{t : t \in [0, t_1], y(s) = u(s) \text{ for } s \in [0, t]\}.$$

Set

$$(2.12) \quad \varrho(t) = \max\{|y(s) - u(s)| : \chi \leq s \leq t\}, \quad t \in [\chi, t_1],$$

and $Q = \max\{|u'(t)| : 0 \leq t \leq t_1\}$. Then $\chi \geq \varepsilon$, $\varrho \in C^0([\chi, t_1])$ is nondecreasing, $\varrho(\chi) = 0$ and $\varrho > 0$ on $(\chi, t_1]$. Since $y(\chi) < \chi$ and y is increasing on $[0, t_0]$, there exists $\mu \in (0, t_1 - \chi]$ such that $y(t) \leq \chi$ for $t \in [0, \chi + \mu]$ and then

$$u(y(t)) - y(y(t)) = 0, \quad t \in [0, \chi + \mu].$$

Hence

$$(2.13) \quad \begin{aligned} |u(u(t)) - y(y(t))| &= |[u(u(t)) - u(y(t))] - [u(y(t)) - y(y(t))]| \\ &= |u(u(t)) - u(y(t))| = |u'(\xi)(u(t) - y(t))| \\ &\leq Q|u(t) - y(t)| \end{aligned}$$

for $t \in [0, \chi + \mu]$ where ξ lies between $u(t)$ and $y(t)$. Since (see Lemma 2.4)

$$|y(t) - u(t)| = \left| \int_0^t \sinh(t-s)[u(u(s)) - y(y(s))]ds \right|, \quad t \in [0, t_1],$$

it follows from the properties of y and u and (2.13) that

$$|y(t) - u(t)| \leq Q \int_{\chi}^t \sinh(t-s)|u(s) - y(s)|ds, \quad t \in [\chi, \chi + \mu].$$

Hence (see (2.12))

$$|y(s) - u(s)| \leq Q \sinh \mu \varrho(t)(t - \chi), \quad \chi \leq s \leq t \leq \chi + \mu$$

and then

$$\varrho(t) \leq Q \sinh \mu \varrho(t)(t - \chi), \quad t \in [\chi, \chi + \mu].$$

Consequently $1 \leq Q \sinh \mu(t - \chi)$ for $t \in (\chi, \chi + \mu]$ since $\varrho > 0$ on $(\chi, t_1]$, which is impossible. Summarizing, we have proved the existence of a unique $t_0 \in (0, \infty)$ such that any maximal solution $y(t)$ of the initial value problem (1.3), (2.10) satisfies $y(t_0) = t_0$, $y(t) < t$ for $t \in (0, t_0)$ and also that the above problem has a unique solution on $[0, t_0]$.

Let $y(t)$ be a maximal solution of (1.3) on an interval J satisfying (2.10). For (b), it remains to verify (2.11). Similar reasoning as in the first part of our proof shows that there is a $\hat{t} < 0$ such that $[\hat{t}, 0] \subset J$, $y(\hat{t}) = \hat{t}$ and $y(t) > t$ for $t \in (\hat{t}, 0)$. Set

$$w(t) = -y(-t), \quad t \in \hat{J} = \{t : -t \in J\}.$$

By Lemma 2.1 (with $c = 0$ and $\mu = -1$), w is a solution on $\hat{J} \supset [-t_0, 0]$. Since $w(0) = 0$ and $w'(0) = y'(0) = \alpha$, using Lemma 2.5, we have $w(t) = y(t)$ for $t \in [-\varepsilon, 0]$. If $\varepsilon \geq t_0$, (2.11) is true. Let $\varepsilon < t_0$. There is no loss of generality in assuming $\hat{t} \leq -t_0$. Suppose that $w(t) \not\equiv y(t)$ on $[-t_0, -\varepsilon]$ and let

$$d = \min\{t : w(s) = y(s) \text{ for } s \in [t, 0]\}.$$

Then $d \in (-t_0, -\varepsilon]$ and from the equalities

$$y(t) = y(d) + y'(d)(t - d) + \int_d^t (t - s)[y(s) - y(y(s))]ds,$$

$$w(t) = w(d) + w'(d)(t - d) + \int_d^t (t - s)[w(s) - w(w(s))]ds$$

for $t \in [-t_0, d]$ and $y(d) = w(d)$, $y'(d) = w'(d)$, we get

$$(2.14) \quad y(t) - w(t) = \int_d^t (t - s)[y(s) - w(s) - y(y(s)) + w(w(s))]ds,$$

$$t \in [-t_0, d].$$

Since $w(t) > t$ on a neighbourhood of $t = d$, there exists $\varepsilon_* > 0$ such that $w(t) \geq d$ for $t \in [d - \varepsilon_*, d] \subset [-t_0, d]$. Then $w(w(t)) = y(w(t))$ for $t \in [d - \varepsilon_*, d]$, and consequently

$$\begin{aligned} y(y(t)) - w(w(t)) &= [y(y(t)) - y(w(t))] + [y(w(t)) - w(w(t))] \\ &= y(y(t)) - y(w(t)) = y'(\xi)(y(t) - w(t)) \end{aligned}$$

for $t \in [d - \varepsilon_*, d]$ where ξ lies between $y(t)$ and $w(t)$. Hence

$$|y(y(t)) - w(w(t))| \leq S|y(t) - w(t)|, \quad t \in [d - \varepsilon_*, d]$$

where $S = \max\{|y'(t)| : t \in [-t_0, 0]\}$. Now (2.14) gives

$$(2.15) \quad |y(t) - w(t)| \leq \varepsilon_*(1 + S) \int_t^d |y(s) - w(s)|ds, \quad t \in [d - \varepsilon_*, d].$$

Let

$$\gamma(t) = \max\{|y(s) - w(s)| : s \in [t, d]\}, \quad t \in [d - \varepsilon_*, d].$$

Then $\gamma \in C^0([d - \varepsilon_*, d])$ is nonincreasing, $\gamma(d) = 0$ and $\gamma > 0$ on $[d - \varepsilon_*, d)$. From (2.15) we now deduce that

$$|y(s) - w(s)| \leq \varepsilon_*(1 + S)\gamma(t)(d - t), \quad d - \varepsilon_* \leq t \leq s \leq d.$$

Therefore

$$\gamma(t) \leq \varepsilon_*(1 + S)\gamma(t)(d - t), \quad t \in [d - \varepsilon_*, d]$$

and $1 \leq \varepsilon_*(1 + S)(d - t)$ for $t \in [d - \varepsilon_*, d)$, which is impossible. We have proved that $w(t) = y(t)$ for $t \in [-t_0, 0]$. Therefore (b) is true.

Since we have verified that the initial value problem (1.3), (2.10) has a unique solution on $[0, t_0]$, the validity of (c) follows now immediately from (b). $\square$

Lemma 2.7. *Let $y(t)$ be a maximal solution of (1.3) on an interval J , $y(0) = 0$ and $y'(0) = 1$. Then $J = \mathbf{R}$ and $y(t) = t$ for $t \in \mathbf{R}$.*

Proof. The function $u(t) = t$ for $t \in \mathbf{R}$ is a maximal solution of (1.3) on $\mathbf{R}$ satisfying $u(0) = 0$, $u'(0) = 1$. In order to prove our lemma we have to show that $y = u$. Assume on the contrary that $y \neq u$ and let $y(t) \not\equiv u(t)$ for $t \in [0, \infty) \cap J$ (the case when $y(t) \not\equiv u(t)$ for $t \in (-\infty, 0] \cap J$ is treated in a similar fashion). Then there is an $\varepsilon \geq 0$ such that $y(t) = u(t)$ for $t \in [0, \varepsilon]$ and the function $\delta : [\varepsilon, \infty) \cap J \rightarrow \mathbf{R}$ defined by

$$(2.16) \quad \delta(t) = \max\{|y(s) - s| : s \in [\varepsilon, t]\}$$

is continuous and nondecreasing on $[\varepsilon, \infty) \cap J$, $\delta(\varepsilon) = 0$ and $\delta(t) > 0$ for $t \in (\varepsilon, \infty) \cap J$. Let $\gamma = \sup\{t : t \in J\}$. Now from the equality

$$y(t) = t + \int_{\varepsilon}^t (t - s)[y(s) - y(y(s))]ds, \quad t \in [\varepsilon, \infty) \cap J$$

we deduce that

$$(2.17) \quad |y(t) - t| = \int_{\varepsilon}^t (t - s)|y'(s)| |y(s) - s|ds \leq (t - \varepsilon)V \int_{\varepsilon}^t |y(s) - s|ds$$

for $t \in [\varepsilon, \varepsilon + v]$ where $v = \min\{1, (\gamma - \varepsilon)/2\}$, η lies between s and $y(s)$ and

$$V = \max\left\{|y'(t)| : \varepsilon \leq t \leq \max\left\{\varepsilon + v, \max_{s \in [\varepsilon, \varepsilon + v]} y(s)\right\}\right\}.$$

From (2.16) and (2.17) it follows that $\delta(t) \leq (t - \varepsilon)^2 V \delta(t)$ for $t \in [\varepsilon, \varepsilon + v]$ and then $1 \leq (t - \varepsilon)^2 V$ for $t \in (\varepsilon, \varepsilon + v]$, which is impossible. Hence $J = \mathbf{R}$ and $y(t) = t$. $\square$

Lemma 2.8. *Let $y(t)$ be a maximal solution of (1.3) on an interval J and let*

$$(2.18) \quad y(0) = 0, \quad y'(0) > 1.$$

Then there exists $t_0 \in (0, \infty) \cap J$ such that $y(t_0) = t_0$, $y(t) > t$ for $t \in (0, t_0)$ and $y'(t_0) \in (0, 1)$.

Proof. (2.18) implies that $y(t) > t$ on a right neighbourhood of $t = 0$. Assume that $y(t) > t$ for $t \in J_1 = (0, \infty) \cap J$. If $\sup\{t : t \in J_1\} = a < \infty$ then, by Lemma 2.1, $t < y(t) \leq a$ for $t \in J_1$, therefore $\lim_{t \rightarrow a^-} y(t) = a$ and $y(t) > t$ for $t \in J_1$ implies $a \notin J_1$. Then we can verify (see the proof of Lemma 2.3) that the function $z : J \cup \{a\} \rightarrow \mathbf{R}$,

$$z(t) = \begin{cases} y(t) & \text{for } t \in J \\ a & \text{for } t = a \end{cases}$$

is a solution of (1.3) on $J \cup \{a\}$, and so z is a right continuation of y , contrary to y being a maximal solution of (1.3). Hence $J_1 = (0, \infty)$ and then $y(y(t)) - y(t) > 0$ for $t \in J_1$ implies $y'' < 0$ on J_1 . Therefore y' is decreasing on J_1 and $y(t) > t$ gives $y' > 1$ on J_1 . Then $y'''(t) = y'(t)(1 - y'(y(t))) < 0$ and $y'' \leq -K$ on an interval $[t_*, \infty)$, $t_* > 0$, with a positive constant K , contrary to $y' > 1$ on J_1 . It follows that $y(t_0) = t_0$ for a $t_0 > 0$ and $y(t) > t$ for $t \in (0, t_0)$. Now $y'' < 0$ on $(0, t_0)$ implies that y' is decreasing on $[0, t_0]$ and then $t_0 = y(t_0) = y(t_0) - y(0) = y'(\xi)t_0$ with a $\xi \in (0, t_0)$ gives $y'(\xi) = 1$. Hence $y'(t_0) < y'(\xi) = 1$ and $y'(t_0) \in (0, 1)$. $\square$

3. The structure of the set Ω and properties of the function $T(\alpha)$

We first show that for any $\alpha \in (0, 1)$ there exists a unique element y in the set Ω defined in Section 1 satisfying (2.10).

Lemma 3.1. *For each $\alpha \in (0, 1)$ there exists a unique $y \in \Omega$ satisfying (2.10). In addition, $y(-t) = -y(t)$ for $t \in \mathbf{R}$.*

Proof. Fix $\alpha \in (0, 1)$. By Lemma 2.6, there is a unique solution u of (1.3) on an interval $[-t_0, t_0]$, $t_0 > 0$, such that $u(0) = 0$, $u'(0) = \alpha$, $u(t_0) = t_0$, $u(t) < t$ for $t \in (0, t_0)$ and $u(t) = -u(-t)$ for $t \in [-t_0, 0]$. Then $u(-t_0) = -u(t_0) = -t_0$, $u'(-t_0) = u'(t_0)$ and $u''(-t_0) = -u''(t_0) = 0$. Also $u' > 0$ on $[-t_0, t_0]$ since u' is decreasing on $[-t_0, 0]$ and increasing on $[0, t_0]$. Let $w : [-t_0, t_0] \rightarrow \mathbf{R}$ be the function defined by

$$w(t) = t - u(t).$$

Then $w(-t_0) = w(t_0) = 0$, $w'(-t_0) = w'(t_0) = 1 - u'(t_0)$, $w''(-t_0) = w''(t_0) = 0$ and $w''(t) = -w(t - w(t))$ for $t \in [-t_0, t_0]$. Write w_* for the $2t_0$ -periodic continuation of w on $\mathbf{R}$ and set

$$y(t) = t - w_*(t), \quad t \in \mathbf{R}.$$

Then $y(-t) = -y(t)$ for $t \in \mathbf{R}$, $t - y(t)$ is a $2t_0$ -periodic function, $y' > 0$ on $\mathbf{R}$ since y' is $2t_0$ -periodic and $y'(t) = u'(t) > 0$ for $t \in [-t_0, t_0]$, and from $w_*''(t) = -w_*(t - w_*(t))$ for $t \in \mathbf{R}$, we have

$$y''(t) = -w_*''(t) = w_*(t - w_*(t)) = w_*(y(t)) = y(t) - y(y(t)), \quad t \in \mathbf{R}.$$

Hence y is the unique element of Ω satisfying (2.10). $\square$

For each $\alpha \in (0, 1)$, denote by y_α the unique element of Ω satisfying (2.10). The existence and uniqueness of y_α is guaranteed by Lemma 3.1. Next Lemma 2.6 shows that for each y_α with $\alpha \in (0, 1)$, there exists a unique $T(\alpha) > 0$ such that

$$(3.1) \quad y_\alpha(T(\alpha)) = T(\alpha) \quad \text{and} \quad y_\alpha(t) < t \quad \text{for } t \in (0, T(\alpha)).$$

Thus we obtain the function $T : (0, 1) \rightarrow (0, \infty)$. Using this function T , we can describe the period of $y_\alpha(t) - t$, namely the number $2T(\alpha)$ is its period.

We also note that $y(t) \equiv t$ is a unique solution of problem (1.3), (1.4) on $\mathbf{R}$ satisfying (2.10) with $\alpha = 1$ by Lemma 2.7, but $y \notin \Omega$ which follows from the definition of the set Ω .

In the next lemma, we describe all elements of the set Ω by y_α with $\alpha \in (0, 1)$.

Lemma 3.2. *Let $\Gamma = \{c + y_\alpha(t - c) : c \in \mathbf{R}, \alpha \in (0, 1)\}$. Then $\Omega = \Gamma$.*

Proof. We know that $y_\alpha \in \Omega$ for each $\alpha \in (0, 1)$. Next Lemma 2.2 implies that $c + y_\alpha(t - c)$ is a solution of (1.3) on $\mathbf{R}$ for each $c \in \mathbf{R}$. Since $(c + y_\alpha(t - c))' = y_\alpha'(t - c) > 0$ for $t \in \mathbf{R}$ and $y_\alpha(t - c) + c - t$ is a periodic function, $c + y_\alpha(t - c) \in \Omega$ for each $c \in \mathbf{R}$ and $\alpha \in (0, 1)$. Hence $\Gamma \subset \Omega$.

Let now $u \in \Omega$. Then $u(t_0) = t_0$ with a $t_0 \in \mathbf{R}$ by Lemma 2.3. Setting $u_0(t) = t_0 - u(t_0 - t)$ for $t \in \mathbf{R}$, u_0 is a solution of (1.3) on $\mathbf{R}$ (see Lemma 2.2), $u_0(0) = 0$, $u_0' > 0$ on $\mathbf{R}$ and $u_0(t) - t$ is periodic. If $u_0'(0) = \alpha \in (0, 1)$, then $u_0 = y_\alpha$ by Lemma 2.6, and since y_α is an odd function, we have

$$u(t) = t_0 - u_0(t_0 - t) = t_0 - y_\alpha(t_0 - t) = t_0 + y_\alpha(t - t_0), \quad t \in \mathbf{R}.$$

Hence $u \in \Gamma$. Let $u_0'(0) > 1$. Then Lemma 2.8 guarantees that $u_0(t_*) = t_*$ with a $t_* > 0$, $u_0(t) > t$ for $t \in (0, t_*)$ and $u_0'(t_*) = \beta \in (0, 1)$. Set

$$\hat{u}(t) = t_* - u_0(t_* - t), \quad t \in \mathbf{R}.$$

Then $\hat{u}$ is a solution of (1.3) on $\mathbf{R}$ by Lemma 2.2 (with $c = t_*$ and $\mu = -1$), $\hat{u}' > 0$ on $\mathbf{R}$, $\hat{u}(t) - t$ is periodic and $\hat{u}(0) = 0$, $\hat{u}'(0) = u_0'(t_*) = \beta$. Hence $\hat{u} = y_\beta$ and then

$$u(t) = t_0 - u_0(t_0 - t) = t_0 - t_* + y_\beta(t - t_0 + t_*), \quad t \in \mathbf{R}.$$

Therefore $u \in \Gamma$. We have verified that $\Omega \subset \Gamma$. Consequently, $\Omega = \Gamma$. $\square$

Lemma 3.3. *The period $2T(\alpha)$ of $y_\alpha(t)$, $\alpha \in (0, 1)$, satisfies the inequality*

$$(3.2) \quad \ln \frac{1}{\alpha^2} < 2T(\alpha) < \frac{2\pi}{\sqrt{\alpha}}.$$

Proof. Since $y_\alpha(0) = 0$, $y'_\alpha(0) = \alpha$ and y'_α is increasing on $[0, T(\alpha)]$, integrating the equality

$$2y'_\alpha(t)y''_\alpha(t) = 2y_\alpha(t)y'_\alpha(t) - 2y_\alpha(y_\alpha(t))y'_\alpha(t)$$

from 0 to $t \in (0, T(\alpha)]$, we get

$$y_\alpha^2(t) = \alpha^2 + y_\alpha^2(t) - 2 \int_0^t y_\alpha(y_\alpha(s))y'_\alpha(s)ds, \quad t \in [0, T(\alpha)].$$

Then

$$y_\alpha^2(t) < \alpha^2 + y_\alpha^2(t) < (\alpha + y_\alpha(t))^2, \quad t \in (0, T(\alpha)]$$

and from the inequality $y'_\alpha(t) < \alpha + y_\alpha(t)$ we deduce that

$$(3.3) \quad y_\alpha(t) < \alpha(e^t - 1), \quad t \in (0, T(\alpha)].$$

Consider the function $p(t) = \alpha(e^t - 1) - t$ on the interval $[0, \infty)$. Since $p(0) = 0$, $p'(0) = \alpha - 1 < 0$, $\lim_{t \rightarrow \infty} p(t) = \infty$ and $p'(\xi) = \alpha e^\xi - 1 = 0$ if and only if $\xi = \ln(1/\alpha)$, we see that there exists a unique $t_* > 0$ such that $p(t_*) = 0$ and $t_* > \xi = \ln(1/\alpha)$. Now (3.3) gives $y_\alpha(t) < p(t) + t \leq t_*$ for $t \in (0, t_*]$, and therefore $t_* < T(\alpha)$ which proves that $2T(\alpha) > 2t_* > \ln(1/\alpha^2)$.

It remains to prove $T(\alpha) < \pi/\sqrt{\alpha}$. Since y'_α is increasing on $[0, T(\alpha)]$ and $y'_\alpha(0) = \alpha$, we have $y'_\alpha > \alpha$ on $(0, T(\alpha)]$ and then

$$(3.4) \quad y''_\alpha(t) = y_\alpha(t) - y_\alpha(y_\alpha(t)) = y'_\alpha(\eta)(t - y_\alpha(t)) > \alpha(t - y_\alpha(t))$$

for $t \in (0, T(\alpha)]$ where $\eta \in (y_\alpha(t), t)$. Set

$$w(t) = t - y_\alpha(t), \quad t \in [0, T(\alpha)].$$

Then $w(0) = w(T(\alpha)) = 0$, $w'(0) = 1 - \alpha$, $w > 0$ on $(0, T(\alpha))$ and (see (3.4)) $w''(t) = -y''_\alpha(t) < -\alpha w(t)$ for $t \in (0, T(\alpha)]$. Hence $w''(t) = -\alpha w(t) + q(t)$ for $t \in [0, T(\alpha)]$ where $q \in C^0([0, T(\alpha)])$ is a negative function on $(0, T(\alpha))$, and so

$$(3.5) \quad w(t) = \frac{1 - \alpha}{\sqrt{\alpha}} \sin(\sqrt{\alpha}t) + \frac{1}{\sqrt{\alpha}} \int_0^t \sin(\sqrt{\alpha}(t - s))q(s)ds, \quad t \in [0, T(\alpha)].$$

Since

$$\int_0^{\pi/\sqrt{\alpha}} \sin\left(\sqrt{\alpha}\left(\frac{\pi}{\sqrt{\alpha}} - s\right)\right) q(s) ds < 0$$

and then (see (3.5))

$$w\left(\frac{\pi}{\sqrt{\alpha}}\right) = \frac{1}{\sqrt{\alpha}} \int_0^{\pi/\sqrt{\alpha}} \sin\left(\sqrt{\alpha}\left(\frac{\pi}{\sqrt{\alpha}} - s\right)\right) q(s) ds < 0,$$

we deduce from the inequality $w(t) \geq 0$ on $[0, T(\alpha)]$ that $T(\alpha) < \pi/\sqrt{\alpha}$, which completes the proof. $\square$

Lemma 3.4. *The function $T(\alpha)$ is continuous on $(0, 1)$ and*

$$(3.6) \quad \lim_{\alpha \rightarrow 0^+} T(\alpha) = \infty, \quad \lim_{\alpha \rightarrow 1^-} T(\alpha) \leq \pi.$$

Proof. The continuity of $T(\alpha)$ on $(0, 1)$ is proved by a way of contradiction. Assume that $T(\alpha)$ is discontinuous at a point $\alpha = \alpha_0$, $\alpha_0 \in (0, 1)$. Then there exist $\varepsilon_0 > 0$ and a sequence $\{\alpha_n\} \subset (\alpha_0/2, (1 + \alpha_0)/2)$, $\lim_{n \rightarrow \infty} \alpha_n = \alpha_0$ such that

$$(3.7) \quad |T(\alpha_n) - T(\alpha_0)| \geq \varepsilon_0 \quad \text{for } n \in \mathbf{N}.$$

The next part of the proof is divided into two cases.

Case (i). There is a subsequence $\{\alpha_{k_n}\}$ of $\{\alpha_n\}$ such that

$$(3.8) \quad T(\alpha_{k_n}) \geq T(\alpha_0) + \varepsilon_0 \quad \text{for } n \in \mathbf{N}.$$

From (3.2) and $\alpha_{k_n} > \alpha_0/2$ we obtain

$$T(\alpha_{k_n}) < \frac{\pi}{\sqrt{\alpha_{k_n}}} < \sqrt{\frac{2}{\alpha_0}} \pi, \quad n \in \mathbf{N},$$

and consequently $\{T(\alpha_{k_n})\}$ is bounded. Hence we can assume that $\{T(\alpha_{k_n})\}$ is convergent and let $\lim_{n \rightarrow \infty} T(\alpha_{k_n}) = \delta$. Then (see (3.8))

$$(3.9) \quad \delta \geq T(\alpha_0) + \varepsilon_0.$$

As

$$0 < y_{\alpha_{k_n}}(t) < t, \quad 0 < y_{\alpha_{k_n}}(y_{\alpha_{k_n}}(t)) < t, \quad 0 < y_{\alpha_{k_n}}''(t) = y_{\alpha_{k_n}}(t) - y_{\alpha_{k_n}}(y_{\alpha_{k_n}}(t)) < t$$

for $t \in (0, T(\alpha_0)]$ and $n \in \mathbf{N}$, $\{y_{\alpha_{k_n}}'(t)\}$ is equicontinuous on $[0, T(\alpha_0)]$ and since $y_{\alpha_{k_n}}'(0) = \alpha_{k_n}$, $\{y_{\alpha_{k_n}}\}$ is bounded in $C^1([0, T(\alpha_0)])$. Applying the Arzelà-Ascoli theorem we can assume without loss of generality that $\{y_{\alpha_{k_n}}\}$ is convergent in $C^1([0, T(\alpha_0)])$ and let $\lim_{n \rightarrow \infty} y_{\alpha_{k_n}} = \beta$. Then from the equalities

$$\begin{aligned} y_{\alpha_{k_n}}(y_{\alpha_{k_n}}(t)) - \beta(\beta(t)) &= [y_{\alpha_{k_n}}(y_{\alpha_{k_n}}(t)) - y_{\alpha_{k_n}}(\beta(t))] + [y_{\alpha_{k_n}}(\beta(t)) - \beta(\beta(t))] \\ &= y'_{\alpha_{k_n}}(\xi)(y_{\alpha_{k_n}}(t) - \beta(t)) + [y_{\alpha_{k_n}}(\beta(t)) - \beta(\beta(t))] \end{aligned}$$

where ξ lies between $y_{\alpha_{k_n}}(t)$ and $\beta(t)$, we deduce that $\lim_{n \rightarrow \infty} y_{\alpha_{k_n}}(y_{\alpha_{k_n}}(t)) = \beta(\beta(t))$ uniformly on $[0, T(\alpha_0)]$. Letting $n \rightarrow \infty$ in the equality (for $t \in [0, T(\alpha_0)]$ and $n \in N$)

$$(3.10) \quad y_{\alpha_{k_n}}(t) = \alpha_{k_n}t + \int_0^t (t-s)[y_{\alpha_{k_n}}(s) - y_{\alpha_{k_n}}(y_{\alpha_{k_n}}(s))]ds,$$

we have

$$\beta(t) = \alpha_0 t + \int_0^t (t-s)[\beta(s) - \beta(\beta(s))]ds, \quad t \in [0, T(\alpha_0)].$$

Hence $\beta(0) = 0$, $\beta'(0) = \alpha_0$ and $\beta''(t) = \beta(t) - \beta(\beta(t))$ for $t \in [0, T(\alpha_0)]$, and therefore $\beta(t) = y_{\alpha_0}(t)$ for $t \in [0, T(\alpha_0)]$ by Lemma 2.6. We have proved $\lim_{n \rightarrow \infty} y_{\alpha_{k_n}}^{(j)}(t) = y_{\alpha_0}^{(j)}(t)$ uniformly on $[0, T(\alpha_0)]$ for $j = 0, 1$. Then

$$(3.11) \quad \lim_{n \rightarrow \infty} y_{\alpha_{k_n}}(T(\alpha_0)) = y_{\alpha_0}(T(\alpha_0)) = T(\alpha_0).$$

From $T(\alpha_0) = y_{\alpha_0}(T(\alpha_0)) = y_{\alpha_0}(T(\alpha_0)) - y_{\alpha_0}(0) = y'_{\alpha_0}(\eta)T(\alpha_0)$ where $\eta \in (0, T(\alpha_0))$, it follows that $y'_{\alpha_0}(\eta) = 1$. Since $0 < y_{\alpha_0}(t) < t$ for $t \in (0, T(\alpha_0))$, we have $y''_{\alpha_0}(t) = y_{\alpha_0}(t) - y_{\alpha_0}(y_{\alpha_0}(t)) > 0$ on this interval. Hence y'_{α_0} is increasing on $(0, T(\alpha_0))$ and $y'_{\alpha_0}(\eta) = 1$ shows that $y'_{\alpha_0}(T(\alpha_0)) > 1$. Now the equality $\lim_{n \rightarrow \infty} y'_{\alpha_{k_n}}(T(\alpha_0)) = y'_{\alpha_0}(T(\alpha_0))$ implies

$$(3.12) \quad y'_{\alpha_{k_n}}(T(\alpha_0)) \geq \frac{1 + y'_{\alpha_0}(T(\alpha_0))}{2} (>1)$$

for $n \geq n_0$ with an $n_0 \in N$. We know that $y'_{\alpha_{k_n}}$ is increasing on $[0, T(\alpha_{k_n})]$, and therefore

$$y'_{\alpha_{k_n}}(t) \geq \frac{1 + y'_{\alpha_0}(T(\alpha_0))}{2}, \quad t \in [T(\alpha_0), T(\alpha_{k_n})], \quad n \geq n_0.$$

Then (for $n \geq n_0$)

$$\begin{aligned} (3.13) \quad T(\alpha_{k_n}) - y_{\alpha_{k_n}}(T(\alpha_0)) &= y_{\alpha_{k_n}}(T(\alpha_{k_n})) - y_{\alpha_{k_n}}(T(\alpha_0)) \\ &= y'_{\alpha_{k_n}}(\eta)(T(\alpha_{k_n}) - T(\alpha_0)) \\ &\geq \frac{1 + y'_{\alpha_0}(T(\alpha_0))}{2}(T(\alpha_{k_n}) - T(\alpha_0)) \end{aligned}$$

where $\eta \in (T(\alpha_0), T(\alpha_{k_n}))$. Letting $n \rightarrow \infty$ in (3.13) and using $\lim_{n \rightarrow \infty} T(\alpha_{k_n}) = \delta$, (3.9) and (3.11), we get

$$1 \geq \frac{1 + y'_{\alpha_0}(T(\alpha_0))}{2},$$

contrary to $y'_{\alpha_0}(T(\alpha_0)) > 1$ (see (3.12)).

Case (ii). There is a subsequence $\{\alpha_{l_n}\}$ of $\{\alpha_n\}$ such that

$$(3.14) \quad T(\alpha_{l_n}) \leq T(\alpha_0) - \varepsilon_0 \quad \text{for } n \in \mathbf{N}.$$

Then (for $n \in \mathbf{N}$)

$$\begin{aligned} 0 < y_{\alpha_{l_n}}(t) < 3T(\alpha_0), & \quad t \in [0, 2T(\alpha_0)], \\ 0 < y_{\alpha_{l_n}}(t) < 2T(\alpha_0), & \quad t \in [0, T(\alpha_0)] \end{aligned}$$

and

$$|y''_{\alpha_{l_n}}(t)| \leq T(\alpha_{l_n}) < T(\alpha_0), \quad t \in \mathbf{R}.$$

Since $y'_{\alpha_{l_n}}(0) = \alpha_{l_n}$, applying the Arzelà-Ascoli theorem, we can assume (taking a subsequence if necessary) that $\{y_{\alpha_{l_n}}\}$ is convergent in $C^1([0, 2T(\alpha_0)])$ and let $\lim_{n \rightarrow \infty} y_{\alpha_{l_n}} = \gamma$. From

$$y_{\alpha_{l_n}}(y_{\alpha_{l_n}}(t)) - \gamma(\gamma(t)) = y'_{\alpha_{l_n}}(\tau)(y_{\alpha_{l_n}}(t) - \gamma(t)) + y_{\alpha_{l_n}}(\gamma(t)) - \gamma(\gamma(t))$$

where τ lies between $y_{\alpha_{l_n}}(t)$ and $\gamma(t)$, it follows that $\lim_{n \rightarrow \infty} y_{\alpha_{l_n}}(y_{\alpha_{l_n}}(t)) = \gamma(\gamma(t))$ uniformly on $[0, T(\alpha_0)]$. Taking the limit in (3.10) (with $y_{\alpha_{l_n}}$ and α_{l_n} instead of $y_{\alpha_{k_n}}$ and α_{k_n} , respectively) as $n \rightarrow \infty$, we have

$$\gamma(t) = \alpha_0 t + \int_0^t (t-s)(\gamma(t) - \gamma(\gamma(s)))ds, \quad t \in [0, T(\alpha_0)].$$

Using the same procedure as in *Case (i)*, $\gamma(t) = y_{\alpha_0}(t)$ for $t \in [0, T(\alpha_0)]$, and consequently $\lim_{n \rightarrow \infty} y_{\alpha_{l_n}}^{(j)}(t) = y_{\alpha_0}^{(j)}(t)$ uniformly on $[0, T(\alpha_0)]$ for $j = 0, 1$. Now using the inequality (see (3.2))

$$T(\alpha_{l_n}) > \ln \frac{1}{\alpha_{l_n}} > \ln \frac{2}{1 + \alpha_0},$$

and $y_{\alpha_0}(t) < t$ on $(0, T(\alpha_0))$, we see that

$$(3.15) \quad y_{\alpha_0}(t) < t - v \quad \text{for } t \in \left[\ln \frac{2}{1 + \alpha_0}, T(\alpha_0) - \varepsilon_0 \right]$$

with a positive constant v . In addition, since $\lim_{n \rightarrow \infty} y_{\alpha_{l_n}}(t) = y_{\alpha_0}(t)$ uniformly on $[0, T(\alpha_0)]$, there exists $n_* \in \mathbf{N}$ such that

$$(3.16) \quad |y_{\alpha_{l_n}}(t) - y_{\alpha_0}(t)| < \frac{v}{2} \quad \text{for } t \in \left[\ln \frac{2}{1 + \alpha_0}, T(\alpha_0) - \varepsilon_0 \right], n \geq n_*.$$

Then (see (3.15) and (3.16))

$$\begin{aligned} \frac{\nu}{2} &> y_{\alpha_{l_n}}(T(\alpha_{l_n})) - y_{\alpha_0}(T(\alpha_{l_n})) = T(\alpha_{l_n}) - y_{\alpha_0}(T(\alpha_{l_n})) \\ &> T(\alpha_{l_n}) - T(\alpha_{l_n}) + \nu = \nu \end{aligned}$$

which is impossible.

We have proved that $T(\alpha)$ is continuous on $(0, 1)$. From the inequality (3.2) we deduce the validity of (3.6). $\square$

4. Periodic solutions of problem (1.1), (1.2)

In Section 1, the set of all periodic solutions of problem (1.1), (1.2) is denoted by $\mathcal{A}_k$. The following theorem describes the structure of the set $\mathcal{A}_k$ by means of solutions y_α to problem (1.3), (1.4) with $\alpha \in (0, 1)$. Notice that the functions y_α are introduced in Section 3.

Theorem 4.1. *For each $k \in (0, \infty)$,*

$$\mathcal{A}_k = \left\{ \frac{1}{k}(t - c - y_\alpha(t - c)) : c \in \mathbf{R}, \alpha \in (0, 1) \right\}.$$

Proof. Fix $k \in (0, \infty)$. By Lemma 3.2, $\Omega = \{c + y_\alpha(t - c) : c \in \mathbf{R}, \alpha \in (0, 1)\}$ is the set of all solutions $y(t)$ of problem (1.3), (1.4) on $\mathbf{R}$ for which $x(t) = t - y(t)$ is periodic. The assertion of our theorem now follows from Lemma 1.1 which guarantees that $x \in \mathcal{A}_k$ if and only if $x(t) = (t - y(t))/k$ for $t \in \mathbf{R}$ where $y \in \Omega$. $\square$

Corollary 4.2. *For each $k \in (0, \infty)$ and each $T_* \in (2\pi, \infty)$, there exists a T_* -periodic solution of problem (1.1), (1.2).*

Proof. Fix $k \in (0, \infty)$ and $T_* \in (2\pi, \infty)$. As the function $T(\alpha)$ defined in (3.1) is continuous by Lemma 3.4 and satisfies (3.6), there exists $\alpha_* \in (0, 1)$ such that $T(\alpha_*) = T_*/2$. Then T_* is the period of the function $t - y_{\alpha_*}(t)$ and since $(t - y_{\alpha_*}(t))/k \in \mathcal{A}_k$ by Theorem 4.1, T_* is the period of the periodic solution $(t - y_{\alpha_*}(t))/k$ to problem (1.1), (1.2). $\square$

Corollary 4.3. *For each $k \in (0, \infty)$ and each $a \in (0, 1/k)$, there exists a unique periodic solution $x(t)$ of problem (1.1), (1.2) such that $x(0) = 0$, $x'(0) = a$.*

Proof. Fix $k \in (0, \infty)$ and $a \in (0, 1/k)$. Set $\alpha_* = 1 - ak \in (0, 1)$ and $x(t) = (t - y_{\alpha_*}(t))/k$ for $t \in \mathbf{R}$. Then $x \in \mathcal{A}_k$ by Theorem 4.1 and $x(0) = 0$, $x'(0) = (1 - y'_{\alpha_*}(0))/k = (1 - \alpha_*)/k = a$. Hence x is a periodic solution of problem (1.1), (1.2) and $x(0) = 0$, $x'(0) = a$. In order to prove the uniqueness of x with the above properties assume that u is another solution, $u \neq x$. Then, by Theorem 4.1, there exist $c_0 \in \mathbf{R}$ and $\alpha_0 \in (0, 1)$ such that

$$u(t) = \frac{1}{k}(t - c_0 - y_{\alpha_0}(t - c_0)), \quad t \in \mathbf{R}.$$

Now $u(0) = 0$, $u'(0) = a$ imply $y_{\alpha_0}(-c_0) = -c_0$ and $y'_{\alpha_0}(-c_0) = 1 - ak = \alpha_*$. From these equalities we deduce that $-c_0 = 2lT(\alpha_0)$ where l is an integer and $\alpha_0 = \alpha_*$. Since $2T(\alpha_*)$ is the period of the function $t - y_{\alpha_*}(t)$, we have

$$u(t) = \frac{1}{k}[t + 2lT(\alpha_*) - y_{\alpha_*}(t + 2lT(\alpha_*))] = \frac{1}{k}(t - y_{\alpha_*}(t)), \quad t \in \mathbf{R}.$$

Therefore $u = x$, which is a contradiction. $\square$

Remark 4.4. We can also discuss a more general state-dependent functional-differential equation

$$(4.1) \quad y''(t) + a^2 y\left(t - \frac{k}{a} y(t)\right) = 0$$

where $a, k \in (0, \infty)$. It is easy to check that x is a solution of (1.1) on $\mathbf{R}$ if and only if $y(t) = x(at)$, $t \in \mathbf{R}$, is a solution of (4.1) on $\mathbf{R}$. Hence our results for periodic solutions of (1.1) can be immediately transformed for periodic solutions of (4.1).

References

- [1] Domoshnitsky, A. and Drakhlin, M., Periodic solutions of differential equations with delay depending on solution, *Nonlin. Anal.*, **30** (1997), 2665–2672.
- [2] Domoshnitsky, A., Drakhlin, M. and Litsyn, E., On equations with delay depending on solution, *Nonlin. Anal.*, **49** (2002), 689–701.
- [3] Medveď, M., On minimal periods of functional-differential equations and difference inclusions, *Ann. Polon. Math.*, **54** (1991), 263–270.
- [4] Magal, P. and Arino, O., Existence of periodic solutions for a state dependent delay differential equation, *J. Differ. Equations*, **165** (2000), 61–95.
- [5] Xiang, X. and Ge, W., On the periodic solutions to the differential-iterative equation $x'(t) = \omega(t)(ax(t) - bx(x(t)))$ ($a > b > 0$), *J. Syst. Sci. Math. Sci.*, **19** (1999), 457–464. (Chinese)
- [6] Yang, W. and Ge, W., Periodic solutions for the differential-iterative equation $x'' + g(x(x)) = p(t)$. *J. Beijing Inst. Technol., Chin. Ed.*, **22** (2002), 537–539. (Chinese)

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