On Global Attractors for a Nonlinear Parabolic Equation
of \(m\)-Laplacian Type in \(R^N\)

By
Mitsuhiro Nakao and Caisheng Chen
(Kyushu University, Japan and Hohai University, P.R. China)

Abstract. We prove the existence, some absorbing properties and some regularities of
\(L^2(R^N), L^p(R^N)\), \(2 \leq p < \infty\), global attractor for the \(m\)-Laplacian type quasilinear
parabolic equation in \(R^N\) with a perturbation \(g(x, u) + f(x)\).

Key Words and Phrases. Global attractor, Nonlinear parabolic equation, \(m\)-Laplacian.

2000 Mathematics Subject Classification Numbers. 35B35, 35B40, 35B41, 35K55.

1. Introduction

In this paper we consider the existence of some global attractors for nonlinear parabolic equations of the \(m\)-Laplacian type in \(R^N\):

\[
\begin{align*}
\frac{du}{dt} - \text{div}(\sigma(|Du|^2)Du) + \lambda u + g(x, u) &= f(x), \quad t > 0, x \in R^N, \\
u(x, 0) &= u_0(x), \quad x \in R^N, 
\end{align*}
\]

where \(\lambda > 0\) and \(\sigma(v^2)\) is a function like \(\sigma(v^2) = |v|^m, m \geq 0\).

For the case \(m = 0\) the existence of \(L^2\) global attractor for the problem is proved by Wang in [18] under appropriate assumptions on \(g\) and \(f\). Recently Khanmamedov [8] has discussed the existence of \((L^2, L^{p^*})\) global attractor in a weak sense of the problem \((1.1)–(1.2)\) with \(\lambda u\) replaced by \(\lambda |u|^m u\) where \(p^*\) is a certain special exponent. In these papers it is assumed that \(g_u(x, u) \geq -k(x)\) for an appropriate function \(k(x)\), which is essentially used to assure the uniqueness of solutions.

The object of this paper is to prove the existence of \((L^2, L^p)\) global attractor for any \(p \geq 2\), in particular, \(L^2\) global attractor in the usual sense of the problem \((1.1)–(1.2)\). We derive \(L^\infty\) estimate of solutions by Moser’s technique as in [1, 10, 14, 17] and due to this we need not make the assumption like \(g_u(x, u) \geq -k(x)\) to show the uniqueness.

The existence and some properties of global attractors of the nonlinear parabolic equations of \(m\)-Laplacian type in bounded domains have been studied by several authors (see Cholewa and Dlotko [5], Takeuchi and Yokota [15], Matsuura and Otani [9] and Nakao and Aris [11] etc., in this case the term \(\lambda u\) is
not necessary). But, there seem to be little results for the problem in $\mathbb{R}^N$. One of the difficulties comes from the lack of Poincare’s inequality like $k u k^2 < C k V u k_{m+2}$ and another is the lack of compactness of the embedding, say, $W^{1,m+2}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$. The first one is easily removed by adding the potential term $\lambda u$, $\lambda > 0$. To overcome the other one we need to prove the uniform smallness of the norm of $k u(t)k_{L(p,B(R)^c)}$ for large $t$ and large $R$, where $B(R)^c$ is the complement of $B(R) \equiv \{x \in \mathbb{R}^N | |x| < R\}$. In [8, 18] this is proved for $p = 2$ by a rather complicate argument using abstract semi-group theory. Here, we prove the fact for $p \geq 2$ by a direct cut-off technique.

We derive various estimates of solutions. Our estimates show that the global attractor $\mathcal{A}$ is a bounded set in $L^\infty(\mathbb{R}^N) \cap W^{1,m+2}(\mathbb{R}^N)$. Further, under an additional regularity assumption on $g$ and $f$ we derive an estimate of $\mathcal{A}$ in $W^{1,\infty}(\mathbb{R}^N)$. For the proof we use again Moser’s technique (cf. [1, 11, 12, 13]). Indeed similar estimates are derived in [11] for the problem in bounded domains and here we modify the argument there.

Global attractor is a basic concept to analyse the asymptotic behaviour of solutions of nonlinear evolution equations (cf. [3, 5, 6, 16]) and the parabolic equations of $m$-Laplacian type is one of the most typical nonlinear evolution equations, and so our argument and result here seem to be useful to consider other related problems.

2. Statement of the results

We use only standard function spaces and omit the definitions. We use $\| \cdot \|_p$ to denote $L^p(\mathbb{R}^N)$ norm.

We make the following assumptions:

Hyp.A. $\sigma(\cdot)$ is a function continuous on $[0, \infty)$, differentiable on $(0, \infty)$ and satisfying the conditions

$$k_0 |v|^m \leq \sigma(v^2) \leq k_1 |v|^m$$

and

$$0 \leq \sigma'(v^2)v^2 \leq k_1 |v|^m$$

for some $m \geq 0$ and $k_0, k_1 > 0$.

Hyp.B. $g(x,u)$ is measurable in $x \in \mathbb{R}^N$ for each $u \in \mathbb{R}$ and Lipschitz continuous in $u$ for a.e. $x \in \mathbb{R}^N$, satisfying the conditions:

$$0 \leq \int_0^u g(x,\eta) d\eta + L(x) |u| \leq k_2 (g(x,u)u + L(x)|u|),$$

with some $k_2 > 0$ and $L(\cdot) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,
(2.4) \( (g(x,u) - g(x,v))(u - v) \geq -k_3(1 + |u|^r + |v|^r)|u - v|^2 \)

with some \( k_3 > 0 \) and \( r \geq 0 \), and for any \( K > 0 \) there exists a function \( a_K(\cdot) \in L^2(\mathbb{R}^N) \), such that

\[
|g(x,u)| \leq a_K(x) \quad \text{if } |u| \leq K.
\]

For typical examples we can take \( g(x,u) = a(x)|u|^{\alpha}u - b(x)|u|^{\beta}u \) with \( a(x) \geq b(x) \geq 0 \), \( b \in L^2(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N) \), \( \alpha > \beta \geq 0 \) and \( g(x,u) = a(x)(u + \sin(|u|^{\beta})u) \) with \( a(\cdot) \geq 0 \), \( a \in L^2(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N) \), \( \beta \geq 0 \). For these examples (2.4) is valid with \( r = 0 \) and \( r = \beta \), respectively.

**Hyp.C.** \( f \) belongs to \( L^2(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N) \).

We set

\[
M = \|f\|_\infty + \|f\|_2 \quad \text{and} \quad L = \|L(\cdot)\|_\infty + \|L(\cdot)\|_2.
\]

We begin with existence and uniqueness theorem of global solutions for the initial data \( u_0 \in L^2(\mathbb{R}^N) \) which includes various information to construct global attractors. Our existence theorem seems to be new and meaningful aparting from global attractors. We seek for the solutions in the class

\[
X = C([0, \infty); L^2(\mathbb{R}^N)) \cap L^{\infty}_{\text{loc}}((0, \infty); W^{1,m+2}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N))
\]

\[
\cap W^{1,2}_{\text{loc}}((0, \infty); L^2(\mathbb{R}^N)).
\]

In the sequel we often omit \( \mathbb{R}^N \) in the notations of function spaces. We make the following definition of solutions.

**Definition 2.1.** A function \( u \in X \) is called a solution of (1.1)–(1.2) if

\[
\int_0^\infty \int_{\mathbb{R}^N} (u_t(x,t)\phi(x,t) + \sigma(|\nabla u|^2)\nabla u \cdot \nabla \phi(x,t) + (\lambda u + g(x,u) - f(x)) \times \phi(x,t)) \, dx \, dt = 0
\]

for any \( \phi \in X \) with \( \text{supp} \phi(x,t) \subset (0, \infty) \), and \( u(0) = u_0 \).

**Theorem 2.1.** Assume Hyp.A, Hyp.B with \( 0 \leq r < (4 + mN + 2m)/N \) and Hyp.C. Then, for \( u_0 \in L^2 \), the problem admits a unique solution \( u(\cdot) \in X \) which satisfies the following estimates:

\[
\|u(t)\|_2 \leq \|u_0\|_2 e^{-2t} + C \cdot (M + L),
\]

\[
\|u(t)\|_p \leq C_p \cdot (\|u_0\|_2 + M + L)^{1 - \frac{mN(p-2)}{p(2m+4+mN)}} e^{-N(p-2)/p(2m+4+mN)}
\]

\[
+ C_p \cdot (\|u_0\|_2 + M + L)^{1 - \frac{N(m+1)(p-2)}{p(2m+4+mN)+N(p-2)}}
\]

\[
\times (M + L)^{N(p-2)/p(2m+4+mN)+N(p-2)},
\]

\[
(2.5)
\]

\[
(2.6)
\]
\[
\|u(t)\|_\infty \leq \begin{cases} C(\|u_0\|_2, M + L)t^{-\mu}, & 0 < t \leq 1, \\ C(\|u_0\|_2, M + L)e^{-\lambda t} + C(M + L) < \infty, & 1 \leq t < \infty \end{cases}
\]

with \( \mu = N/(2(m + 2) + m) \),

\[
\|\nabla u(t)\|_{m+2} \leq C \cdot (\|u_0\|_2 + M + L)^{2/(m+2)} t^{-1/(m+2)} + C \cdot (M + L)^{2/(m+2)},
\]

\( 0 < t < \infty \),

and

\[
\int_t^\infty \|u_t(s)\|^2 ds \leq C \cdot (\|u_0\|_2 + M + L)^2 t^{-1} + C \cdot (M + L)^2
\]

for any \( 0 < t < \infty \), where \( C(a,b) \) denotes constants depending on \( a, b \) and \( C, C_p \) are constants independent of \( u_0, M \) and \( L \). (\( C_p \) depends on \( p \).)

Further, if we assume \( u_0 \in L^q \cap L^2 \), \( q \geq 2 \), the above solution belongs to \( C([0, \infty); L^q) \cap X \). The solution depends continuously on \( u_0 \) in \( L^2 \) or \( L^2 \cap L^q \) norm.

**Remark 2.1.** For the problem in a bounded domain with the Dirichlet boundary condition we have the estimate (cf. [11])

\[
\|\nabla u(t)\|_{m+2} \leq Cr^{-1/m} + C(M + L)^{1/(m+1)}, \quad 0 < t < \infty,
\]

which is independent of \( u_0 \) and a little different from (2.8).

We denote the solution \( u(t) \) in Theorem 2.1 by \( S(t)u_0 \). \( S(t) \) is a continuous semi-group on \( L^2 \) or \( L^2 \cap L^q \).

**Definition 2.2.** Let \( S(t) \) be a continuous semi-group in a Banach space \( X \). A subset \( \mathcal{A} \) in \( X \) is called the global attractor of \( S(t) \) if and only if

1. \( \mathcal{A} \) attracts \( S(t)B \) as \( t \to \infty \) for every bounded set \( B \) in \( X \),
2. \( \mathcal{A} \) is compact in \( X \)

and

3. \( \mathcal{A} \) is invariant, i.e., \( S(t)\mathcal{A} = \mathcal{A} \) for all \( t \geq 0 \).

More generally,

**Definition 2.3** (cf. [3]). Let \( S(t) \) be a continuous semi-group on a Banach space \( X \) and a semi-group on a Banach space \( Y \). A subset \( \mathcal{A} \) in \( X \cap Y \) is called \((X,Y)\) global attractor of \( S(t) \) if and only if

1. \( \mathcal{A} \) attracts \( S(t)B \) in \( Y \) norm for every bounded set \( B \) in \( X \),
2. \( \mathcal{A} \) is a compact set in \( Y \) and a bounded set in \( X \)

and

3. \( \mathcal{A} \) is invariant, i.e., \( S(t)\mathcal{A} = \mathcal{A} \) for any \( t \geq 0 \).
Note that \( \mathcal{A} \) is a \((X, X)\) global attractor if and only if \( \mathcal{A} \) is a global attractor in \( X \).

For a Banach space \( X \) we denote by \( B_X(R) \) the ball in \( X \) centered at 0 with radius \( R \). The existence of \((L^2, L^p)\) global attractor follows from Theorem 2.1 combined with a tail estimate on \( \|u(t)\|_{L^p(B(R)^c)} \).

**Theorem 2.2.** The problem \((1.1)–(1.2)\) has \((L^2, L^p)\) global attractor \( \mathcal{A} \) for any \( p \geq 2 \). \( \mathcal{A} \) is independent of \( p \) and satisfies the following properties:

1. \( \mathcal{A} \subset B_{L^2}(R) \cap B_{L^p}(R) \)
   with \( R = C \cdot (M + L) \).

2. \( \mathcal{A} \subset W^{1, m+2}(R^N) \)

and it holds that
\[
\|\nabla u\|_{m+2} \leq C \cdot (M + L)^{2/(m+2)} \quad \text{for } u \in \mathcal{A}.
\]

Further, according to the estimates in Theorem 2.1, \( \mathcal{A} \) or more precisely the absorbing sets \( B_{L^2}(R) \), \( B_{L^p}(R) \) have the following absorbing properties:

1. \( \text{dist}_{L^2}(S(t)B_0, B_{L^2}(R)) + \text{dist}_{L^p}(S(t)B_0, B_{L^p}(R)) \leq C(B_0) e^{-\lambda t} \quad \text{for } t \geq 1 \)
   with \( R = C \cdot (M + L) \) for any bounded set \( B_0 \) in \( L^2 \), and

2. \( \text{dist}_{W^{1, m+2}}(S(t)B_0, B_{W^{1, m+2}}(R)) \leq C(B_0) t^{-1/(m+2)} \quad \text{for } t \geq 1 \)
   with \( R = C(M + L)^{2/(m+2)} \) for any bounded set \( B_0 \) in \( L^2 \), where \( W^{1, m+2} \) is the completion of \( C_0^\infty(R^N) \) with respect to the norm \( \|\nabla \|_{m+2} \).

**Remark 2.2.** We say a set \( \mathcal{B} \) in \( L^p \) is an absorbing set of \( S(t) \) if and only if for any \( \varepsilon > 0 \) and for any bounded set \( B_0 \) in \( L^2 \) (or \( L^2 \cap L^q \)), there exists \( T = T(\varepsilon, B_0) > 0 \) such that
\[
S(t)B_0 \subset U_\varepsilon(\mathcal{B}) \quad \text{for } t > T
\]
where \( U_\varepsilon = \{ v \in L^p \mid \text{dist}_{L^p}(v, \mathcal{B}) < \varepsilon \} \). This definition may be a little different from a standard one, but, this seems to be more convenient for our use. (If \( \mathcal{B} \) is an absorbing set in the above sense \( U_\varepsilon(\mathcal{B}) \) with any \( \varepsilon > 0 \) is an absorbing set in a standard sense.)

**Corollary 2.1.** \( S(t) \) is a continuous semi-group in \( L^2 \cap L^q \) for any \( q \geq 2 \) and there exists a global attractor \( \mathcal{A} \) in \( L^2 \cap L^q \) which satisfies all of the properties stated in the previous Theorem.
If we assume (2.4) with \( r = 0 \), both of uniqueness and continuity of solutions in \( L^2 \) easily follow without \( L^\infty \) estimate. We also note that from the proof, \( M + L \) appearing in the estimates (2.5), (2.8) and (2.9) is replaced by \( M_2 + L_2 \), where we set \( M_p = \| f \|_2 + \| f \|_p \) and \( L_p = \| L(\cdot) \|_2 + \| L(\cdot) \|_p \). Thus we obtain the following Corollary.

**Corollary 2.2.** Assume that Hyp.\( A \) and Hyp.\( B \) are satisfied with \( r = 0 \), and \( L(\cdot) \) and \( f \) belong to \( L^2 \cap L^{m+p} \) with some \( 2 \leq p < \infty \). (If \( m = 0 \) we assume further \( f \in L^{p+\delta} \) with some \( \delta > 0 \).) Then, for \( u_0 \in L^2 \) the problem (1.1)–(1.2) admits a unique solution \( u(\cdot) \) in the class

\[
X_2 \equiv C([0, \infty); L^2(R^N)) \cap L^\infty((0, \infty); L^{p+m}(R^N) \cap W^{1,m+2}(R^N))
\]

and the estimates (2.5), (2.8) and (2.9) hold with \( M + L \) replaced by \( M_2 + L_2 \) and (2.6) holds with \( M + L \) replaced by \( M_{p+m} + L_{p+m} \) \( (M_{p+\delta} + L_{p+\delta} \text{ if } m = 0) \). Further we have

\[
(2.7)' \quad \| u(t) \|_p \leq \begin{cases} C(p, \| u_0 \|_2, M_2 + L_2)t^{-\mu_p}, & 0 < t \leq 1, \\ C(p, \| u_0 \|_2, M_2 + L_2)e^{-\lambda t} + C(M_p + L_p) < \infty, & 1 \leq t < \infty \end{cases}
\]

with \( \mu_p = N(p - 2)/p(2m + 4 + mN) \). Consequently, the problem has \( (L^2, L^p) \) global attractor \( A \) and we have the estimates:

1. \[ \text{dist}_{L^q}(S(t)B_0, B_{L^q}(R)) \leq C(B_0)e^{-\lambda t} \quad \text{for } t \geq 1 \]
2. \[ \text{dist}_{W^{1,m+2}}(S(t)B_0, B_{W^{1,m+2}}(R)) \leq C(B_0)t^{-1/(m+2)} \quad \text{for } t \geq 1 \]

with \( R = C(M_2 + L_2)^{2/(m+2)} \).

A typical example which Corollary 2.2 is applied to is \( g(x, u) = a(x)(|u|^\alpha u - |u|^\beta u), \alpha > \beta > 0 \), with \( a(\cdot) \in L^2 \cap L^\infty \). A related problem will be discussed in [4].

**Theorem 2.3.** In addition to Hyp.\( A \), Hyp.\( B \) and Hyp.\( C \) we assume

\[
f \in W^{1,\infty}(R^N) \cap W^{1,2}(R^N), \quad g \in C^{0,1}(R^N \times R)
\]

and

\[ |g_x(x, u)| + |g_u(x, u)| \leq k_4(1 + |u|^2) \]

with some \( k_4 > 0 \) and \( \alpha \geq 0 \). We set
\[ \mathcal{M} = \|\nabla f\|_2 + \|\nabla f\|_\infty. \]

Then the solution in Theorem 2.1 further satisfies the estimate:

\[ \|\nabla u(t)\|_\infty \leq \begin{cases} C(\|u_0\|_2 + M + L, \mathcal{M}, t_0^{-\mu_0})(t - t_0)^{-\mu} & \text{if} \ t_0 < t \leq t_0 + 1, \\ C(\|u_0\|_2 + M + L, \mathcal{M}) < \infty & \text{if} \ t \geq 1. \end{cases} \]

for any \( 0 < t_0 < 1 \) where \( \mu = N/(mN + 2m + 4) \) and \( \mu_0 = \max\{2\mu, m/(m + 2)\} \).

Consequently, the global attractor \( \mathcal{A} \) is also a bounded set in \( W^{1, \infty} \).

For the proof of Theorems we use elementary lemmas on differential inequalities.

**Lemma 2.1.** Let \( y(t) \) be a nonnegative continuous function on \([0, \infty)\), satisfying (in the distribution sense)

\[ y'(t) + Ay^{1+\theta}(t) \leq B, \quad t > 0, \]

for some \( A > 0, B \geq 0, \theta > 0 \). Then

\[ y(t) \leq (y(0))^{-\theta} + A\theta t^{-1/\theta} + (A^{-1}B)^{1/(1+\theta)} \]

and

\[ y(t) \leq \max\{y(0), (A^{-1}B)^{1/(1+\theta)}\}. \]

**Lemma 2.2.** Let \( y(t) \) be a nonnegative continuous function on \([0, T]\), satisfying

\[ y'(t) + At^{\theta - 1}y^{1+\theta}(t) \leq Bt^{-k}y(t) + Ct^{-\delta} \]

with \( A, \theta > 0, \lambda t \geq 1, B, C \geq 0, k \leq 1 \) and \( 0 \leq \delta < 1 \). Then we have

\[ y(t) \leq A^{-1/\theta}(2\lambda + 2BT^{1-k})^{1/\theta}t^{-\lambda} + 2C(\lambda + BT^{1-k})^{-1}t^{1-\delta}, \quad 0 < t \leq T. \]

For the proof of Lemma 2.2, see Ohara [14].

The following variant of Gagliardo-Nirenberg inequality is also very useful to prove smoothing effect and also derive estimate of solutions by use of Moser’s technique.

**Lemma 2.3** (Gagliardo-Nirenberg). Let \( \beta \geq 0, N \geq p \geq 1, \beta + 1 \leq q \) and \( 1 \leq r \leq q \leq (\beta + 1)Np/(N - p) \) \((1 \leq r \leq q \leq \infty \) if \( N \leq p \)). Then for \( u \) such that \( |u|^\beta u \in W^{1,p}(\mathbb{R}^N) \) and \( u \in L^r(\mathbb{R}^N) \), we have

\[ \|u\|_q \leq C^{1/(\beta+1)}\|u\|^{1-\theta}_r\|\nabla (|u|^\beta u)\|^{\theta/(\beta+1)}_p \]

with \( \theta = (\beta + 1)(r^{-1} - q^{-1})/(N^{-1} - p^{-1} + (\beta + 1)r^{-1}) \), where \( C \) is a constant independent of \( q, r, \beta \) and \( \theta \) if \( N \neq p \), and a constant depending on \( q/(\beta + 1) \) if \( N = p \). (We except for the case \( N = p \) and \( q = \infty \).)
3. Proof of Theorem 2.1

In this section we derive a priori estimates of the assumed solutions \( u(t) \) in \( X \), and we give a proof of Theorem 2.1.

The solutions are in fact given as limits of smooth solutions of appropriate approximate equations and we may assume for our estimations that the solutions under consideration are sufficiently smooth. We shall begin with the estimates of \( k u(t) k^2 \) and \( k u(t) k^p \), \( p > 2 \).

**Proposition 3.1.** Let \( u(t) \) be a solution of the problem (1.1)–(1.2). Then we have the estimates (2.5) and (2.6) for \( k u(t) k^2 \) and \( k u(t) k^p \), respectively.

**Proof.** We multiply the equation (1.1) by \( |u|^{p-2} u \), and integrate by parts to get

\[
\frac{1}{p} \frac{d}{dt} k u(t) k_p^p + k_0(p - 1) \| \nabla (|u|^{(p-2)/(m+2)} u) \|^m_{m+2} + \lambda k u(t) k_p^p \leq \int (|f| + |L|) |u|^{p-1} dx,
\]

where we have used the assumption on \( g \) and \( \sigma \). From this,

\[
\frac{d}{dt} k u(t) k_p^p + \lambda k u(t) k_2^2 \leq M + L
\]

which implies, in particular,

\[
\frac{d}{dt} k u(t) k_2^2 + \lambda k u(t) k_2^2 \leq M + L.
\]

Hence (cf. Lemma 2.1) we obtain (2.5). Further we see from Lemma 2.3

\[
\| u(t) \|_p \leq C k u(t) k_2^{1-\theta} \| \nabla (|u|^{(p-2)/(m+2)} u) \|^m_{m+2}
\]

with \( \theta = N(p - 2)(p + m)/p(2m + 4 + N(p + m - 2)) \). It follows from (2.5), (3.1) and (3.3) that

\[
\frac{d}{dt} k u(t) k_p^p + C \left( \frac{m + 2}{p + m} \right)^{m+1} \left( k 0 k_2^2 + M + L \right)^{- (p+m)(1-\theta)/\theta} k u(t) k_{p+m-p\theta}^{\theta+1} \leq M + L.
\]

Applying Lemma 2.1 to (3.4) we can derive (2.6).

**Proposition 3.2.** For a solution in \( X \) we have the estimate in (2.7) for \( k u(t) k_\infty \).
We take $p_1 = 2$ and $p_n = (m + 2)p_{n-1} - m$, $n = 2, 3, \ldots$. Then, by Gagliardo-Nirenberg inequality,

\begin{equation}
\|u\|_{p_n} \leq C^{(m+2)/(p_n+m)} \|u\|_{p_{n-1}}^{1-\theta_n} \|\nabla(u^{(p_n-2)/(m+2)}u)\|_{m+2}^{(m+2)\theta_n/(p_n+m)}
\end{equation}

with

$$\theta_n = (1 - p_{n-1}p_n^{-1})/(m + 2 + N(m + 1)).$$

Therefore we see from (3.1) that

\begin{equation}
\frac{d}{dt}\|u(t)\|_{p_n} + C^{-(m+2)/(p_n+m)} p_n^{-m-1} \|u\|_{p_{n-1}}^{m-\beta_n} \|u\|_{p_n}^{1+\beta_n} \leq C(M + L)
\end{equation}

where we set

$$\beta_n = (p_n + m)\theta_n^{-1} - p_n.$$

Note that

$$\lim_{n \to \infty} \beta_n/p_n = (m + 2)/N(m + 1).$$

We shall show, by induction,

\begin{equation}
\|u(t)\|_{p_n} \leq \eta_n t^{-\lambda_n}, \quad 0 < t \leq 1,
\end{equation}

where $\lambda_1 = 0$, $\eta_1 = \|u_0\|_2 + C(M + L)$. Replacing the right-hand side of (3.6) by $p_n\|u(t)\|_{p_n} + C(L + M)$ and applying Lemma 2.2 with $B = p_n$, $k = 0$, $\delta = 0$ we get

$$\|u(t)\|_{p_n} \leq (2C^{(m+2)/(\beta_n+p_n+m)} p_n^{(m+1)/\beta_n} \eta_n^{1-m/\beta_n}(p_n + (1 + \lambda_{n-1}(\beta_n - m)/\beta_n)^{1/\beta_n})^{-\lambda_n}$$

$$+ C p_n^{-1}(M + L) t^{-\lambda_n} \leq \eta_n t^{-\lambda_n}, \quad 0 < t \leq 1,$$

where we set

$$\lambda_n = ((\beta_n - m)\lambda_{n-1} + 1)/\beta_n = \frac{2N}{2(m+2)+mN} \left(1 - \frac{1}{p_n}\right)$$

and

\begin{equation}
\eta_n = \eta_n^{1-m/\beta_n}(Cp_n)^{C/p_n} + C p_n^{-1}(M + L).
\end{equation}

We can show that $\{\eta_n\}$ is bounded by a constant $C(\|u_0\|_2, M + L)$ independent of $n$ and also $\lim_{n \to \infty} \lambda_n = N/(mN + 2(m + 2)) = \mu$. Thus we conclude the estimate (2.7) for $0 < t \leq 1$.

To show the estimate for $\|u(t)\|_{\infty}$, $t \geq 1$, we return to the inequality (3.1) to get
\[
\frac{d}{dt} \|u(t)\|_p + \dot{\lambda} \|u(t)\|_p \leq M + L
\]

and hence
\[
\|u(t)\|_p \leq \|u(1)\|_p e^{-\lambda t} + C(M + L), \quad t \geq 1.
\]

Taking the limit as \(p \to \infty\) and using the estimate \(\|u(1)\|_\infty\), we have the latter estimate in (2.7).

We proceed to the estimation of \(\|Vu(t)\|_{m+2}\).

**Proposition 3.3.** Let \(u(t)\) be a solution in \(X\). Then we have the estimates (2.8) and (2.9) for \(\|Vu(t)\|_{m+2}\) and \(\|u(t)\|_2\), respectively.

**Proof.** Multiplying (1.1) by \(u_t\), we see
\[
\frac{1}{2} \frac{d}{dt} \Gamma(t) + \|u_t(t)\|_2^2 = 0
\]

where we set
\[
\Gamma(t) = \int_{\mathbb{R}^N} \left( \int_0^{[Vu(t)]^2} \sigma(\eta) d\eta + \dot{\lambda} |u(t)|^2 + 2 \int_0^u g(x, \eta) d\eta - 2fu \right) dx.
\]

We set
\[
\tilde{\Gamma}(t) = \Gamma(t) + C(M + L)^2
\]

for some \(C > 0\). Then, by the assumptions on \(\sigma\) and \(g(x, u)\) we easily see that
\[
\tilde{\Gamma}(t) \geq C \cdot (\|Vu\|_{m+2} + \|u\|_2^2)
\]

with some \(C > 0\). Next, multiplying (1.1) by \(u\) we have (see (2.3))
\[
C\tilde{\Gamma}(t) \leq \|Vu\|_{m+2} + \int_{\Omega} (g(x, u)u + L|u|) dx + \|u(t)\|_2^2 + C(M + L)^2
\]

\[
\leq C\|u(t)\|_2 \|u(t)\|_2 + C(M + L)^2.
\]

Since \((d\tilde{\Gamma}/dt)(t) = (d\Gamma/dt)(t)\) the inequalities (3.9) and (3.10) together with the estimate (2.1) yield
\[
\tilde{\Gamma}(t) \leq C \cdot (\|u_0\|_2 + M + L) \sqrt{-\frac{d}{dt} \tilde{\Gamma}(t) + C(M + L)^2}
\]

and hence,
\[
\frac{d}{dt} \tilde{\Gamma}(t) + C \cdot (\|u_0\|_2 + M + L)^{-2} \tilde{\Gamma}(t)^2 \leq C \cdot (\|u_0\|_2 + M + L)^{-2} (M + L)^4
\]
which implies
\[ \tilde{\Phi}(t) \leq C(\|u_0\|_2 + M + L)^2 t^{-1} + C(M + L)^2. \]
Thus we conclude (2.8). Finally we return to (3.9) to get
\[ \int_t^\infty \|u_t(s)\|_2^2 ds \leq C(\|u_0\|_2 + M + L)^2 t^{-1} + C(M + L)^2 \]
for any \( 0 < t < \infty \), which shows (2.9).

Completion of the proof of Theorem 2.1.

We are now in a position to complete the proof of Theorem 2.1. Let \( \varepsilon > 0 \) and consider the approximate problem
\begin{align*}
(3.12) \quad & u_t - \text{div}(\sigma_\varepsilon(|Vu|^2)Vu) + \lambda u + g_\varepsilon(x, u) = f_\varepsilon(x) \quad t > 0, \ x \in \mathbb{R}^N, \\
(3.13) \quad & u(x, 0) = u_{0, \delta}(x), \quad t > 0,
\end{align*}
where \( \sigma_\varepsilon \) is a smooth functions on \( [0, \infty) \) such that
\[ (1 + k\varepsilon)\sigma(v^2) + k\varepsilon \geq \sigma_\varepsilon(v^2) \geq (1 - \varepsilon)\sigma(v^2) + \varepsilon, \]
and \( u_{0, \delta} \in C_0^\infty(\mathbb{R}^N) \) is a sequence such that
\[ u_{0, \delta} \to u_0 \quad \text{in} \ L^2 \quad \text{as} \ \delta \to 0. \]
g_\varepsilon(u) and \( f_\varepsilon \) are smooth functions tending to \( g \) and \( f \), respectively, in convenient ways (see [11]).

The problem admits a unique smooth solution \( u_{\varepsilon, \delta}(t) \). (Note that this solution is given by a limit of approximate solutions \( u_{\varepsilon, \delta, R} \) of the problem in a bounded domain \( B(R) \) in \( \mathbb{R}^N \), where we take \( R \) such that \( \text{supp} \ u_{0, \delta} \subset B(R) \) and we impose the homogeneous Dirichlet boundary condition on \( \partial B(R) \).) It is easy to see that all of the estimates established for assumed solution \( u(t) \) are valid for \( u_{\varepsilon, \delta}(t) \) and the constants appearing there are essentially independent of \( \delta \) and \( \varepsilon \). First, by taking the limit as \( \varepsilon \to 0 \) we obtain the solution \( u_\delta(\cdot) \in X \). Next, by taking the limit as \( \delta \to 0 \) we obtain the desired solution \( u(\cdot) \in X \) (see [11]). We note that \( L^2 \) continuity of \( u(t) \) follows by the uniform convergence of \( u_{\delta}(t) \) on each interval \( [0, T] \), \( T > 0 \), as follows. First we easily see
\[ \int_0^T \|u_{\delta, t}(t)\|_2^2 dt \leq C(M + L, \|u_{0, \delta}\|_2, T, \|Vu_{0, \delta}\|_{m+2}) < \infty, \quad T > 0, \]
and hence
\[ u_\delta(\cdot) \in C([0, \infty); L^2). \]
Thus \( u_\delta(t) \) is a solution in \( X \) of the problem (1.1)–(1.2) with \( u_0 \) replaced by \( u_{0, \delta} \). Uniqueness of \( u_\delta(t) \) follows by the inequality
Indeed, for possible two solutions fact (2.7) is sufficient) follows from essentially the same argument as in (3.16). We can show by almost the same argument that \( u(t) \) converges to \( u(t) \) in \( L^2 \) uniformly on each compact interval \([0, T]\) as \( \delta \to 0 \) and hence, \( u \in C\left([0, \infty); L^2\right) \), i.e., \( u \in X \). Needless to say, \( u(0) = u_0 \).

The uniqueness of the solution in \( X \) satisfying the estimates (2.5)–(2.9) (in fact (2.7) is sufficient) follows from essentially the same argument as in (3.16). Indeed, for possible two solutions \( u(t), v(t) \in X \) with \( u(0) = v(0) = u_0 \) we have, for any \( \epsilon > 0 \),

\[
\|u(t) - v(t)\|_2 \leq \|u(\epsilon) - v(\epsilon)\|_2 + C \int_\epsilon^t (1 + s^{-\mu}) \|u(s) - v(s)\|_2^2 ds
\]

and hence,\n
\[
\|u(t) - v(t)\|_2 \leq C(T)\|u(\epsilon) - v(\epsilon)\|_2^2, \quad 0 < t \leq T,
\]

for any \( T > 0 \). Thus we conclude that \( u_0(t) \) converges to \( u(t) \) in \( L^2 \) uniformly on each compact interval \([0, T]\) as \( \delta \to 0 \) and hence, \( u \in C\left([0, \infty); L^2\right) \), i.e., \( u \in X \). Needless to say, \( u(0) = u_0 \).
4. Proof of Theorem 2.2 and Corollaries 2.1, 2.2

Let $B_0$ be a bounded set in $L^2$. We already know that $S(t)$ is a continuous semi-group in $L^2$ and $\bigcup_{t \geq t_0} S(t)B_0$ is a bounded set in $L^2 \cap L^\infty$ for arbitrary $t_0 > 0$. As a consequence we know that $S(t)$ is continuous semi-group on any $L^p$, $2 \leq p < \infty$ for $t > 0$. Then, by a standard argument, to show the existence of $(L^2, L^p)$, $2 \leq p < \infty$, global attractor it suffices to prove the following claims:

1. There exists a bounded set $B \subset L^p$ such that for any bounded set $B_0 \subset L^2$,

$$\text{dist}_{L^p}(S(t)B_0, B) \to 0 \quad \text{as} \quad t \to \infty.$$ 

2. For any bounded set $B_0$ in $L^2$, $S(t)B_0$ is asymptotically compact in $L^p$, that is, for any sequences $\{u_{0,n}\} \subset B_0$ and a real sequence $t_n$ with $t_n \to \infty$ as $n \to \infty$, there exists a subsequence $\{n'\}$ such that $\{S(t_{n'})u_{0,n'}\}$ is convergent in $L^p$.

Indeed we can show that if the above (1) and (2) are proved, the global attractor $\mathcal{A}$ is given by $\omega\text{-lim}_{t \to \infty} B = \bigcap_{\tau > 0} \bigcup_{t > \tau} S(t)B^{L^p}$. By Theorem 2.1 we see that the set

$$B = \{u \in L^p \mid \|u\|_p \leq C(M + L)\}$$

with some $C > 0$ is an absorbing set in the sense of (1). (It is easy to show that there exists a minimum constant $C > 0$ such that the set $B$ in (4.1) is an absorbing set.)

To show the asymptotic compactness we prepare:

**Proposition 4.1.** Let $f(\cdot), L(\cdot) \in L^2(R^N) \cap L^{p+m}(R^N)$. Then, for any $\varepsilon > 0$ there exist $T = T(\varepsilon, \|u_0\|, f, L) > 0$ and $R = R(\varepsilon, \|u_0\|, f, L) > 0$ such that

$$\|u(t)\|_{L^p(B(R)^c)} \leq \varepsilon, \quad t > T.$$ 

**Proof.** When $\sigma(v^2) = 1$ and $p = 2$ this fact is proved in [18] by use of an abstract theory of semi-group generated by evolution equation. Here we give a direct proof for a general case $m \geq 0$ and $p \geq 2$, though we utilize a smoothing effect in $L^{p+m}$. Take a function $k_0(s)$ as

$$k_0(s) = \begin{cases} 
0 & \text{if } 0 \leq s \leq 1 \\
 s - 1 & \text{if } 1 \leq s \leq 2 \\
1 & \text{if } s \geq 2
\end{cases}$$
and set \( k_\delta(s) = (\rho_\delta * k_0)(s) \) for \( 0 < \delta < 1 \), where \( \rho_\delta(s) \) is a standard mollifier on \( R \) with \( \text{supp } \rho_\delta(\cdot) \subseteq [-\delta, \delta] \). It is easy to see that

\[
k_\delta(\cdot) \in C^\infty(R), \quad 0 \leq k_\delta(s) \leq 1 \quad \text{and} \quad 0 \leq \left\{ \frac{d}{ds} k_\delta \right\}^{1+\theta}(s) \leq C_\theta k_\delta(s)
\]

for any \( 0 < \theta \leq 1 \).

We fix \( 0 < \delta < 1 \) and set \( \phi(x) = k_\delta(|x|/R) \), \( R \gg 1 \) for \( x \in R^N \). Then we have

\[
|\nabla \phi(x)|^{1+\theta} \leq \frac{C_\theta}{R^{1+\theta}} \phi(x), \quad x \in R^N, \ 0 < \theta \leq 1.
\]

We also note that \( \phi(x) = 0 \) if \( |x| \leq R/2 \) and \( \phi(x) = 1 \) if \( |x| \geq 2R \).

Multiplying the equation by \( \phi(x)^2 |u|^{p-2}u \) we have

\[
\frac{1}{p} \frac{d}{dt} \int_{R^N} \phi(x)^2 |u(t)|^p \, dx + (p-1) k_0 \int_{R^N} |\nabla u|^{m+2} \phi(x)^2 |u|^{p-2} \, dx + \frac{\lambda}{2} \int_{R^N} \phi^2 |u|^p \, dx
\]

\[
\leq 2 \int_{R^N} |\nabla u|^m |\nabla \phi| |\phi| |u|^p \, dx + \int_{R^N} \phi^2 (L(x) + f(x)) |u|^{p-1} \, dx.
\]

Here,

\[
2 \int_{R^N} |\nabla u|^m |\nabla \phi| |\phi| |u|^p \, dx \leq \frac{k_0(p-1)}{2} \int_{R^N} |\nabla u|^{m+2} \phi^2 |u|^{p-2} \, dx
\]

\[+ C \int_{R^N} \phi^{-m} |\nabla \phi|^{m+2} |u|^{p+m} \, dx.
\]

Hence, we have

\[
\frac{1}{p} \frac{d}{dt} \int_{R^N} \phi(x)^2 |u(t)|^p \, dx + \frac{\lambda}{2} \int_{R^N} \phi^2 |u|^p \, dx
\]

\[
\leq C(\|f\|_{L^p(B(R/2)^c)} + \|L\|_{L^p(B(R/2)^c)}) + C \int_{R^N} |\phi|^{-m} |\nabla \phi|^{m+2} |u|^{p+m} \, dx
\]

\[
\leq C(\|f\|_{L^p(B(R/2)^c)} + \|L\|_{L^p(B(R/2)^c)}) + CR^{-m-2} \|u(t)\|_{p+m}^{p+m},
\]

where we have used the inequality (see (4.2))

\[
|\nabla \phi|^{m+2} |\phi|^{-m} \leq CR^{-m-2} \quad \text{if} \quad 0 \leq \theta m \leq 2, \quad \theta > 0.
\]

We know already

\[
\|u(t)\|_{p+m} \leq C(\|u_0\|_2, M_{p+m} + L_{p+m}) \quad \text{for} \quad t \geq 1.
\]

Thus we obtain from (4.3)
of the embedding $W^{1,2}(B(R))$, with $S$ continuous dependence on the initial data of solutions. For the asymptotic bounded sequence in $L^p$, let $n$ be a subsequence such that Proposition 4.1 is valid.

Proof of Theorem 2.3

Let us show that $S(t)B_0$ is asymptotically compact in $L^p$. Let $\{u_{0,n}\}$ be a bounded sequence in $L^2$ and $t_n \to \infty$. By Proposition 4.1 we see that for any $\varepsilon > 0$, there exist $N_\varepsilon > 0$ and $R = R_\varepsilon > 0$ such that

$$\|S(t_n)u_{0,n}\|_{L^p(B(R))} \leq \varepsilon$$

if $n \geq N_\varepsilon$. Since $S(t_n)u_{0,n}$ is bounded in $L^\infty \cap W^{1,m+2}$ for $t_n \geq 1$ and $L^\infty \cap W^{1,m+2}$ is compactly embedded in $L^p(B(R))$ for any $R$ we can take a subsequence $n'$ by a diagonal argument such that $S(t_{n'})u_{0,n'}$ is convergent in $L^p(B(R))$ for any $R > 0$. Hence, there exists $N_\varepsilon > 0$ such that if $n' > N_\varepsilon$,

$$\|S(t_{n'})u_{0,n'} - S(t_{n''})u_{0,n''}\|_p \leq \|S(t_{n'})u_{0,n'} - S(t_{n''})u_{0,n''}\|_{L^p(B(R))} + 2\varepsilon < 3\varepsilon.$$ This shows that $S(t_{n'})u_{0,n'}$ is convergent in $L^p(R^N)$.

Proof of the Corollaries 2.1 and 2.2.

Corollary 2.1 follows immediately from the last result in Theorem 2.1. Under the assumptions of Corollary 2.2 we have

$$\|u(t) - v(t)\|_2^2 \leq \|u(0) - v(0)\|_2^2 + C \int_0^t \|u(s) - v(s)\|_2^2 ds$$

for possible two solutions $u(\cdot), v(\cdot) \in X_2$, which shows the uniqueness and the continuous dependence on the initial data of solutions. For the asymptotic compactness of $S(t)$ in $L^p$ follows from Proposition 4.1 and the compactness of the embedding $W^{1,m+2}(B(R)) \cap L^{p+m}(B(R)) \subset L^p(B(R))$ (the case $m > 0$) or $W^{1,2}(B(R)) \cap L^{p+\delta}(B(R)) \subset L^p(B(R))$ (the case $m = 0$) for any $R > 0$.

5. Proof of Theorem 2.3

We continue the estimation of the gradient $\nabla u(t)$ of the solution $u(t)$. We carry out formal calculations, which can be justified through appropriately
smooth approximate solutions. The following $L^p$ estimate of $\nabla u(t)$ seems to be independently meaningful.

**Proposition 5.1.** For $p \geq m + 2$ we have the estimate

\[
\|\nabla u(t)\|_p \leq \begin{cases} 
C(p, \|u_0\|_2 + M + L, t_0^{-\mu_0}) (t - t_0)^{-\mu_p} & \text{if } 0 < t_0 < t \leq t_0 + 1, \\
C(p, M + L, M) < \infty & \text{if } t \geq 1
\end{cases}
\]

for any $0 < t_0 < t$ where we set

\[
\mu_0 = \max\{2\mu, \mu\} \quad \text{and} \quad \mu_p = \frac{N(p - m - 2)}{p(mN + 2m + 4)}.
\]

**Proof.** The proof is given again by a similar argument as in [11] and we sketch an outline. Multiplying the equation (1.1) by $-\text{div}(\nabla u)$, $p_b m + 2$, and integrating by parts, we have

\[
\frac{1}{p} \frac{d}{dt} \|\nabla u\|_p^p + \int_{\mathbb{R}^N} (\sigma u_i)(|\nabla u|^{p-2} u_j) dx + \lambda \|\nabla u\|_p^p = -\int_{\mathbb{R}^N} (g + g_{x_i} f_j) |\nabla u|^{p-2} u_j dx,
\]

where we use the notation $u_i$ for $\partial u/\partial x_i$. Here, again by integration by parts,

\[
\int_{\mathbb{R}^N} (\sigma u_i)(|\nabla u|^{p-2} u_j) dx
\]

\[
\geq \int_{\mathbb{R}^N} |\nabla u|^{p-2} \left( \sigma u_j^2 + 2\sigma' \sum_j \left( \sum_i u_i u_{ij} \right)^2 \right) dx
\]

\[
+ \frac{p - 2}{4} \int_{\mathbb{R}^N} |\nabla u|^{p-4} \left\{ \sigma \sum_i \left( \sum_j u_i u_{ij} \right)^2 + 2\sigma' \left( \sum_{j,k} u_k u_{jk} \right)^2 \right\} dx
\]

\[
\geq k_0 \left( \int_{\mathbb{R}^N} |\nabla u|^{m+p-2} |D^2 u|^2 + \frac{p - 2}{4} \int_{\mathbb{R}^N} |\nabla u|^{m+p-4} |\nabla (|\nabla u|^2)|^2 dx \right).
\]

Note that the approximate solutions $u_R$ are given as solutions of the problem in a ball $B(R)$, $R \gg 1$ with the boundary condition $u|_{\partial B(R)} = 0$. For these solutions we consider the integrals on $B(R)$ and the following boundary integral appears on the right-hand side of (5.3):

\[
(N - 1) \int_{\partial B(R)} H(x) \sigma(|\nabla u|^2)
\]
where $H(x)$ is the mean curvature of $\partial B(R)$. But, since $H(x) > 0$ we can drop this term and we arrive at (5.3) for smooth solutions in $\mathbb{R}^N$.

We see for $p \geq m + 2$,

\begin{equation}
\int_{\Omega} [g_u u_j + g_x + f_j] |\nabla u|^{p-2} |\nabla u| dx \leq C \gamma(t) \|\nabla u\|_p^p + (\tilde{M} + \gamma(t)) \|\nabla u\|_p^{p-1}
\end{equation}

where

\[\gamma(t) = \begin{cases} 
C(M) t^{-\mu} & \text{if } 0 < t \leq 1, \\
C(M + L) & \text{if } t \geq 1.
\end{cases}\]

Thus, we have

\begin{equation}
\frac{1}{p} \frac{d}{dt} \|\nabla u(t)\|_p^p + C_1 p^{-1} \|\nabla u\|_{(p+m)/2}^p \|u\|_{1,2} + \lambda \|\nabla u(t)\|_p^p \\
\leq C_1 \frac{1}{p} \int_{\Omega} |\nabla u|^{p+m} dx + \gamma(t) \|\nabla u(t)\|_p^p + (\tilde{M} + \gamma(t)) \|\nabla u\|_p^{p-1}.
\end{equation}

Here,

\[\|\nabla u(t)\|_{p+m}^{p+m} \leq C \|\nabla u(t)\|_{m+2}^{\theta_1(p+m)} \|\nabla u\|_{p+1}^{\theta_3(p+m)} \|\nabla u\|_{(p+m)/2}^{2\theta_3} \|\nabla u\|_{1,2}^2 \leq \frac{C_1}{4} \|\nabla u\|_{(p+m)/2}^{2\theta_3} + C p^4 \|\nabla u\|_{m+2}^m \|\nabla u\|_p^p\]

with certain $0 \leq \theta_1, \theta_2, \theta_3 \leq 1$. Also we have

\[\|\nabla u\|_p \leq C \|\nabla u\|_{m+2}^{1-\theta} \|\nabla u\|_{(p+m)/2}^{2\theta/(p+m)}\]

with

\[\theta = \frac{(p+m)N(p-m-2)}{p(N(p-2)+2m+4)}\]

Then (5.5) yields

\begin{equation}
\frac{d}{dt} \|\nabla u(t)\|_p + C \|u_0\|_2 + M + L^{-1} p^{-1} \tilde{\gamma}(t)^{(m+2)(m+N+2)/N(p-m-2)} \|\nabla u(t)\|_p^{1+p(mN+2m+4)/(p-m-2)} \\
\leq C (\|u_0\|_2 + M + L, t^{-\mu_0}) + C (\gamma(t) + \tilde{M}) \leq C (\|u_0\|_2 + M + L, t^{-\mu_0} + \tilde{M}), \quad t_0 \leq t < \infty,
\end{equation}

where we set \(\tilde{\gamma}(t) = t^{1/(m+2)}\) if $0 < t \leq 1$ and \(\tilde{\gamma}(t) = 1\) if $1 \leq t$. Applying Lemma 2.1 to (5.6) on $t_0 < t \leq t_0 + 1$ we obtain
(5.7) \[ \| \nabla u(t) \|_p \leq C(p, \| u_0 \|_2 M + \widetilde{M}, t^{-\mu_0}) (t - t_0)^{-\beta}, \quad t_0 < t \leq t_0 + 1, \]

for any \( 0 < t_0 < 1 \) with \( \mu_p = N(p - m - 2)/p(mN + 2m + 4) \). For \( t \geq 1 \) (5.6) implies

\[
(5.8) \quad \frac{d}{dt} \| \nabla u(t) \|_p + C(\| u_0 \|_2 + M + L)^{-1} p^{-1} \| \nabla u(t) \|_p^{1 + p(mN + 2m + 4)/N(p - m - 2)} 
\leq C(p, M + L, \widetilde{M})
\]

and we have, by Lemma 2.1,

\[ \| \nabla u(t) \|_p \leq C(p, M + L, \widetilde{M}), \quad 1 \leq t < \infty. \]

Let us proceed to the estimation of \( \| \nabla u(t) \|_\infty \).

Our result is the following.

**Proposition 5.2.** For \( 0 < t_0 < 1 \), we have the estimates:

\[
(5.9) \quad \| \nabla u(t) \|_\infty \leq \begin{cases} 
C(\| u_0 \|_2 + M + L, \widetilde{M}, t_0^{-\mu_0}) (t - t_0)^{-\mu} \quad & \text{if } t_0 < t \leq 1, \\
C(\| u_0 \|_2 + M + L, \widetilde{M}) < \infty \quad & \text{if } t \geq 1,
\end{cases}
\]

where \( \mu = N/(mN + 2m + 4) \).

**Proof.** We let \( p_1 = m + 2 \) and define a sequence \( \{p_n\} \) by

\[ p_n = 2p_{n-1} - m. \]

Then, by Lemma 2.4,

\[ \| \nabla u \|_{p_n} \leq C^{1/(p_n+m)} \| \nabla u_n \|_{p_{n-1}}^{1-\theta_n} \| \nabla u \|_{p_{n+1}}^{(p+n)/2} \| \nabla u \|_{1.2}^{2\theta_n/(p_n+m)} \]

with

\[ \theta_n = N(1 - m/p_n)/(N + 2). \]

From the inequality (5.5) a similar argument obtaining (3.6) gives

\[
(5.10) \quad \frac{d}{dt} \| \nabla u(t) \|_{p_n} + C_{p_n}^{-1} C^{-1/\theta_n} \| \nabla u \|_{p_{n-1}}^{m-\beta_n} \| \nabla u(t) \|_{p_n}^{1+\beta_n} 
\leq C_1 p_n^3 (t_0^{-\mu_0}) \| \nabla u(t) \|_{p_n} + \widetilde{M}, \quad t_0 < t < t_0 + 1,
\]

with \( 0 < t_0 < 1 \) where

\[ \beta_n = (p_n + m) \theta_n^{-1} = \frac{(N + 2)p_n(p_n + m)}{N(p_n - m)}. \]

In this situation we can show inductively that

\[
(5.11) \quad \| \nabla u(t) \|_{p_n} \leq \eta_n t^{-\beta_n}, \quad t_0 < t \leq t_0 + 1,
\]
where η₁ = C(∥u₀∥₂ + M + L, t₀⁻¹/(m+2)) < ∞ and λ₁ = 0 (note that

\[ \|Vu(t)\|_{m+2} \leq C \cdot (∥u₀∥₂ + M + L)^{2/(m+2)} t^{-1/(m+2)} \]

\[ \leq C \cdot (∥u₀∥₂ + M + L)^{1/(m+2)} t₀^{-1/(m+2)}, \]

t₀ < t < t₀ + 1) and

(5.12) \( \eta_n = (Cp_n)^{C/p_n} \eta_{n-1}^{1-m/p_n} + Cp_n^{-3}, \quad λ_n = (1 + λ_{n-1}(m - β_n)/β_n). \)

We can show the boundedness of \{ηₙ\} as in (3.8). Setting \( w_n = 2p_n + mN \) we have (see [1, 13])

\[ \dot{λ}_n - \frac{1}{m} = \frac{β_n - m}{β_n} \left( \dot{λ}_{n-1} - \frac{1}{m} \right) = \frac{p_{n-1}w_n}{p_n w_{n-1}} \left( λ_{n-1} - \frac{1}{m} \right) = \frac{w_n p_1}{p_n w_1} \left( λ_1 - \frac{1}{m} \right) \]

and

\[ \lim_{n→∞} \dot{λ}_n = \frac{2p_1 λ_1 + N}{2p_1 + mN} = \frac{N}{mN + 2m + 4}. \]

Finally we must consider the case \( t \geq 1 \). But, returning to a similar inequality as (5.10) and using the estimate of \( k_u(1) \) we obtain the estimate for \( ∥Vu(t)∥_∞, t \geq 1 \) (see [11]).

References


nuna adreso:
Mitsuhiro Nakao
Graduate School of Mathematics
Kyushu University
Fukuoka 810-8560
Japan
E-mail: mnakao@math.kyushu-u.ac.jp

Caisheng Chen
Department of Mathematics
Hohai University
Nanjing 210098
P.R. China
E-mail: cshengchen@mailsvr.hhu.edu.cn

(Ricevita la 17-an de agosto, 2006)
(Reviziita la 1-an de decembro, 2006)