On Positive Decaying Solutions of Quasilinear Ordinary Differential Equations with Singular Nonlinearities

By

Misako MOTAI and Hiroyuki USAMI*

(Hiroshima University, Japan)

Abstract. We consider singular boundary value problems to quasilinear ordinary differential equations with singular nonlinearities. We give sufficient conditions for such problems to have a positive solution. Moreover, for a typical case we can show also the uniqueness of the solutions.

Key Words and Phrases. Quasilinear ordinary differential equation, Positive solution.

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1. Introduction and main results

Let us consider the following singular boundary value problem:

$$(1) \quad (|x'|^{\alpha-1}x')' = q(t)x^{-\beta}, \quad t \geq t_0,$$

$$(2) \quad \lim_{t \rightarrow \infty} x(t) = 0.$$

For equation (1) we always assume that:

(A₁) $\alpha, \beta > 0$ are constants;

(A₂) $q \in C([t_0, \infty); \mathbf{R})$, and $\text{supp } q$, the support of $q(t)$, is unbounded;

(A₃) q is integrable on $[t_0, \infty)$, and it satisfies $\int_t^\infty q(s)ds \geq 0$, $t \geq t_0$.

By assumption (A₃) we can introduce the auxiliary function

$$Q(t) = \int_t^\infty q(s)ds \geq 0 \quad \text{for } t \geq t_0.$$

Since $\text{supp } q$ is unbounded, so is $\text{supp } Q$. Note that $q(t)$ is allowed to change the sign near $+\infty$. In the paper $Q(t)$ will play an important role.

A function $x : [t_0, \infty) \rightarrow (0, \infty)$ is said to be a positive solution of (1) if x and $|x'|^{\alpha-1}x'$ are of class $C^1[t_0, \infty)$ and satisfy equation (1) on $[t_0, \infty)$. A positive solution of problem (1)-(2) is often called positive decaying solution of equation (1).

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It is known [1, 2] that singular differential equations such as (1) arise, for example, in the boundary layer theory of viscous fluids. When $\alpha = 1$ and $q(t) \geq 0$, the study of this problem was initiated by [1, 3] as far as the authors are aware. In [1, 6] the existence of positive solutions of (1)-(2) with $\alpha = 1$ was discussed. One of the authors obtained in [7] systematic results concerning asymptotic behavior of positive solutions to equation (1) with $\alpha = 1$. The aim of the paper is to establish sufficient conditions for problem (1)-(2) to have at least one positive solution. More precisely, we intend not only to extend known results, for example, in [1, 6, 7] to the general case where $\alpha > 0$, but also to establish new existence theorems. In fact, one of our results presented here are new even though $\alpha = 1$. Related results for the general case $\alpha > 0$ are found in [4].

In the case of $q(t) \geq 0$, $t \geq t_0$, if either (A_2) or (A_3) is violated, then problem (1)-(2) never has positive solutions. Therefore it is natural to assume (A_2) and (A_3) in advance; see Remark 2 for details. Roughly speaking, problem (1)-(2) has a positive solution when $q(t)$ is sufficiently small near $+\infty$.

The main results of the paper are as follows:

Theorem 1. *Let the integral conditions*

$$(3) \quad \int_0^\infty Q(t)^{1/\alpha} dt < \infty,$$

$$(4) \quad \int_0^\infty \left(\int_t^\infty Q(s)^{(1+\alpha)/\alpha} ds \right)^{1/\alpha} dt < \infty,$$

and

$$\int_0^\infty \left\{ \int_t^\infty |q(s)| \left(\int_s^\infty Q(r)^{1/\alpha} dr \right)^{-\alpha\beta/(\alpha+\beta)} ds \right\}^{1/\alpha} dt < \infty$$

hold. Then, problem (1)-(2) has a positive solution.

When $0 < \beta < 1$, we can obtain the results of other types, which may be more convenient to apply.

Theorem 2. *Let $0 < \beta < 1$, and (3) and (4) hold. Suppose moreover that there is a nonincreasing function $\hat{q} \in C([t_0, \infty); (0, \infty))$ satisfying*

$$|q(t)| \leq \hat{q}(t), \quad t \geq t_0,$$

and

$$\int_0^\infty \hat{q}(t)^{1/(\alpha+1)} dt < \infty.$$

Then, problem (1)-(2) has a positive solution.

Theorem 3. *Let $0 < \beta < 1$, and (3) and (4) hold. Suppose moreover that there is a constant $\delta \in (\beta, 1)$ satisfying*

$$\int^{\infty} \left(\int_t^{\infty} |q(s)|^{1/(1-\delta)} ds \right)^{(1-\delta)/(\alpha+\delta)} dt < \infty.$$

Then, problem (1)-(2) has a positive solution.

Remark 1. (i) When $\alpha = 1$, Theorems 1 and 2 reduce to [6, Theorems 1 and 2], respectively.

(ii) Since, for each $p > 0$, there are nonnegative and unbounded continuous functions belonging to $L^p[t_0, \infty)$, Theorems 2 and 3 are independent.

An immediate consequence of Theorem 2 (or Theorem 3) follows:

Corollary 4. *Let $0 < \beta < 1$. If $q(t) = O(t^{-\nu})$ as $t \rightarrow \infty$ for some $\nu > \alpha + 1$, then problem (1)-(2) has a positive solution.*

Remark 2. Suppose that $q(t) > 0$ and problem (1)-(2) has a positive solution $x(t)$. Then we can show that $q(t)$ is integrable, and (3) and (4) hold. To see this we firstly notice that $x'(t) < 0$ and $\lim_{t \rightarrow \infty} x'(t) = 0$. Let us rewrite equation (1) as $-\{[-x'(t)]^{\alpha}\}' x(t)^{\beta} = q(t)$. Integrating on $[t, \tau]$, we obtain

$$[-x'(t)]^{\alpha} x(t)^{\beta} \geq \int_t^{\tau} q(s) ds, \quad \tau \geq t \geq t_0,$$

where integral by part and the decreasing nature of $x(t)$ are employed. Letting $\tau \rightarrow \infty$, we obtain $[-x'(t)]^{\alpha} x(t)^{\beta} \geq \int_t^{\infty} q(s) ds$; that is $-x'(t)x(t)^{\beta/\alpha} \geq Q(t)^{1/\alpha}$. Hence one more integration gives $\int^{\infty} Q(t)^{1/\alpha} ds < \infty$. Since $Q(t)$ is decreasing, we can obtain

$$\begin{aligned} \int_{t_0}^{\infty} \left(\int_t^{\infty} Q(s)^{(1+\alpha)/\alpha} ds \right)^{1/\alpha} dt &\leq \int_{t_0}^{\infty} Q(t)^{1/\alpha} \left(\int_t^{\infty} Q(s)^{1/\alpha} ds \right)^{1/\alpha} dt \\ &\leq \left(\int_{t_0}^{\infty} Q(t)^{1/\alpha} dt \right)^{1/\alpha+1} < \infty. \end{aligned}$$

We do not know whether or not under conditions (3) and (4), without any additional conditions, problem (1)-(2) has a positive solutions. This is an unsettled problem.

Remark 3. Let $\liminf_{t \rightarrow \infty} t^{\nu} q(t) > 0$ for some $\nu > 0$. As is seen from Remark 1, if problem (1)-(2) has a positive solutions, then $\nu > \alpha + 1$. This fact shows that the decaying order of $q(t)$ required in the assumption of Corollary 4 is the best possible in some sense. We conjecture that Corollary 4 is still valid even though $\beta \geq 1$.

The paper is organized as follows. In Section 2 we give proofs of the main results. For this purpose we prepare an important lemma Lemma 5. At the end of Section 2 we give illustrative examples. In Section 3 we consider problem (1)-(2) in the restricted case where $q(t) = t^{-\nu}$, $\nu > 0$. We then prove that, in this case, this problem has exactly one solution given explicitly if $\nu > \alpha + 1$.

2. Proofs of main results

To establish our results we need the following lemma.

Lemma 5. *Let integral conditions (3) and (4) hold. Then, for any constant $\ell > 0$, equation (1) has a positive solution x satisfying $x'(t) < 0$, $t \geq t_0$, $\lim_{t \rightarrow \infty} x'(t) = 0$, and $\lim_{t \rightarrow \infty} x(t) = \ell$.*

Remark 4. The following facts are employed in proving Lemma 5.

(i) There is a constant $\hat{C} = \hat{C}(\alpha) > 0$ such that

$$(u + v)^{1/\alpha} \leq \hat{C}(u^{1/\alpha} + v^{1/\alpha}) \quad \text{for all } u, v \geq 0.$$

(ii) When (3) and (4) hold, we have the estimate

$$\begin{aligned} & \int_t^\infty Q(s) \left(\int_s^\infty Q(r)^{(1+\alpha)/\alpha} dr \right)^{1/\alpha} ds \\ & \leq \left(\int_t^\infty Q(s)^{(1+\alpha)/\alpha} ds \right) \left\{ \int_t^\infty \left(\int_s^\infty Q(r)^{(1+\alpha)/\alpha} dr \right)^{1/\alpha} ds \right\}^{1/(\alpha+1)}. \end{aligned}$$

Actually, since $\alpha/(\alpha+1) + 1/(\alpha+1) = 1$, Hölder's inequality implies that

$$\begin{aligned} & \int_t^\infty Q(s) \left(\int_s^\infty Q(r)^{(1+\alpha)/\alpha} dr \right)^{1/\alpha} ds \\ & \leq \left(\int_t^\infty Q(s)^{(1+\alpha)/\alpha} ds \right)^{\alpha/(\alpha+1)} \left\{ \int_t^\infty \left(\int_s^\infty Q(r)^{(1+\alpha)/\alpha} dr \right)^{(\alpha+1)/\alpha} ds \right\}^{1/(\alpha+1)} \\ & = \left(\int_t^\infty Q(s)^{(1+\alpha)/\alpha} ds \right)^{\alpha/(\alpha+1)} \\ & \quad \times \left\{ \int_t^\infty \left(\int_s^\infty Q(r)^{(1+\alpha)/\alpha} dr \right) \left(\int_s^\infty Q(r)^{(1+\alpha)/\alpha} dr \right)^{1/\alpha} ds \right\}^{1/(\alpha+1)} \\ & \leq \left(\int_t^\infty Q(s)^{(1+\alpha)/\alpha} ds \right) \left\{ \int_t^\infty \left(\int_s^\infty Q(r)^{(1+\alpha)/\alpha} dr \right)^{1/\alpha} ds \right\}^{1/(\alpha+1)}. \end{aligned}$$

Proof of Lemma 5. We notice that a positive solution x of equation (1) satisfies the conditions $\lim_{t \rightarrow \infty} x'(t) = 0$ and $\lim_{t \rightarrow \infty} x(t) = \ell$ if and only if it satisfies the integro-differential equation

$$(5) \quad x(t) = \ell + \int_t^\infty \left\{ Q(s)x(s)^{-\beta} - \beta \int_s^\infty Q(r)x(r)^{-\beta-1}x'(r)dr \right\}^{(1/\alpha)*} ds, \quad t \geq t_0.$$

Here we have employed the definition of $Q(t)$ and integral by part, and we use the notation $c^{p*} = |c|^{p-1}c$ for $c \in \mathbf{R}$ and $p > 0$. We try to find a positive solution of this equation. The proof is divided into several steps.

Step 1: We firstly construct a positive solution of (5) near $+\infty$ in a suitable function space by Schauder-Tychonoff's fixed point theorem ([8, Theorems 2.3.8 and 4.5.1]). Let $k, m > 0$, and $T \geq t_0$ be numbers such that

$$\begin{aligned} \hat{C}\ell^{-\beta/\alpha} &\leq m; \\ \hat{C}^2\beta^{1/\alpha}\ell^{-(\beta+1)/\alpha} &\left[m^{1/\alpha} + k^{1/\alpha} \left\{ \int_T^\infty \left(\int_s^\infty Q(r)^{(\alpha+1)/\alpha} dr \right)^{1/\alpha} ds \right\}^{1/(\alpha(\alpha+1))} \right] \leq k; \end{aligned}$$

and

$$\int_T^\infty \left[mQ(t)^{1/\alpha} + k \left(\int_t^\infty Q(s)^{(\alpha+1)/\alpha} ds \right)^{1/\alpha} \right] dt \leq \ell.$$

Here $\hat{C}$ denotes the number introduced in (i) of Remark 4. Consider the linear space $C^1[T, \infty)$ consisting of all continuously differentiable functions on $[T, \infty)$. We regard it as a Fréchet space equipped with the usual topology; that is, a sequence $\{x_n\} \subset C^1[T, \infty)$ converges to $x \in C^1[T, \infty)$ if and only if $\lim_{n \rightarrow \infty} x_n(t) = x(t)$ and $\lim_{n \rightarrow \infty} x'_n(t) = x'(t)$ uniformly on each compact sub-interval in $[T, \infty)$. Let $X \subset C^1[T, \infty)$ be the closed non-empty subset of $C^1[T, \infty)$ consisting of all functions $x(t)$ satisfying inequalities

$$\ell \leq x(t) \leq 2\ell, \quad t \geq T;$$

and

$$(6) \quad 0 \leq -x'(t) \leq mQ(t)^{1/\alpha} + k \left(\int_t^\infty Q(s)^{(1+\alpha)/\alpha} ds \right)^{1/\alpha}, \quad t \geq T.$$

For $x \in X$ we define $\mathcal{F}x(t)$ by the right hand side of (5). Below we will show that $\mathcal{F}$ is a continuous mapping such that $\mathcal{F}X \subset X$ and $\mathcal{F}X$ is relatively compact.

Let $x \in X$. We then have

$$\begin{aligned}
0 &\leq \left(\int_t^\infty Q(s)x(s)^{-\beta-1}[-x'(s)]ds \right)^{1/\alpha} \\
&\leq \ell^{-(\beta+1)/\alpha} \left[m \int_t^\infty Q(s)^{(\alpha+1)/\alpha} ds + k \int_t^\infty Q(s) \left(\int_s^\infty Q(r)^{(\alpha+1)/\alpha} dr \right)^{1/\alpha} ds \right]^{1/\alpha} \\
&\leq \hat{C} \ell^{-(\beta+1)/\alpha} \left[\left(m \int_t^\infty Q(s)^{(\alpha+1)/\alpha} ds \right)^{1/\alpha} \right. \\
&\quad \left. + \left\{ k \int_t^\infty Q(s) \left(\int_s^\infty Q(r)^{(\alpha+1)/\alpha} dr \right)^{1/\alpha} ds \right\}^{1/\alpha} \right] \\
&\leq \hat{C} \ell^{-(\beta+1)/\alpha} \left[\left(m \int_t^\infty Q(s)^{(\alpha+1)/\alpha} ds \right)^{1/\alpha} \right. \\
&\quad \left. + \left(k \int_t^\infty Q(s)^{(\alpha+1)/\alpha} ds \right)^{1/\alpha} \left\{ \int_t^\infty \left(\int_s^\infty Q(r)^{(1+\alpha)/\alpha} dr \right)^{1/\alpha} ds \right\}^{1/(\alpha(\alpha+1))} \right] \\
&\leq \hat{C} \ell^{-(\beta+1)/\alpha} \left[m^{1/\alpha} + k^{1/\alpha} \left\{ \int_T^\infty \left(\int_s^\infty Q(r)^{(1+\alpha)/\alpha} dr \right)^{1/\alpha} ds \right\}^{1/(\alpha(\alpha+1))} \right] \\
&\quad \times \left(\int_t^\infty Q(s)^{(\alpha+1)/\alpha} ds \right)^{1/\alpha} \\
&\leq \hat{C}^{-1} \beta^{-1/\alpha} k \left(\int_t^\infty Q(s)^{(\alpha+1)/\alpha} ds \right)^{1/\alpha}.
\end{aligned}$$

Here we have used (ii) of Remark 4. It follows therefore that

$$\begin{aligned}
0 &\leq -(\mathcal{F}x)'(t) \leq \hat{C} \left\{ Q(t)^{1/\alpha} x(t)^{-\beta/\alpha} + \beta^{1/\alpha} \left(\int_t^\infty Q(s)x(s)^{-\beta-1}[-x'(s)]ds \right)^{1/\alpha} \right\} \\
&\leq \hat{C} \ell^{-\beta/\alpha} Q(t)^{1/\alpha} + k \left(\int_t^\infty Q(s)^{(\alpha+1)/\alpha} ds \right)^{1/\alpha} \\
&\leq m Q(t)^{1/\alpha} + k \left(\int_t^\infty Q(s)^{(\alpha+1)/\alpha} ds \right)^{1/\alpha}.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
0 \leq \mathcal{F}x(t) - \ell &= - \int_t^\infty (\mathcal{F}x)'(s) ds \\
&\leq \int_t^\infty \left[mQ(t)^{1/\alpha} + k \left(\int_t^\infty Q(s)^{(\alpha+1)/\alpha} ds \right)^{1/\alpha} \right] dt \leq \ell,
\end{aligned}$$

implying that $\mathcal{F}x \in X$. Consequently we find that $\mathcal{F}X \subset X$.

To prove the continuity of $\mathcal{F}$, let a sequence $\{x_n\} \subset X$ converge to $x \in X$. We put for $x \in X$

$$I[x](t) = \int_t^\infty Q(s)x(s)^{-\beta-1}[-x'(s)]ds.$$

As seen before we have by (6)

$$(7) \quad 0 \leq Q(s)x(s)^{-\beta-1}[-x'(s)] \leq c_1 Q(s)^{(\alpha+1)/\alpha} + c_2 Q(s) \left(\int_s^\infty Q(r)^{(1+\alpha)/\alpha} dr \right)^{1/\alpha}$$

for some positive constants c_1 and c_2 not depending on x . Since the right hand side is integrable on $[T, \infty)$, the dominated convergence theorem implies that

$$\begin{aligned}
&|I[x_n](t) - I[x](t)| \\
&\leq \int_T^\infty |Q(s)x_n(s)^{-\beta-1}[-x'_n(s)] - Q(s)x(s)^{-\beta-1}[-x'(s)]| ds \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Hence we find that $\lim_{n \rightarrow \infty} I[x_n](t) = I[x](t)$ uniformly on $[T, \infty)$. Since

$$-(\mathcal{F}x)'(t) = \{Q(t)x(t)^{-\beta} + \beta I[x](t)\}^{1/\alpha},$$

we find that $\lim_{n \rightarrow \infty} (\mathcal{F}x_n)'(t) = (\mathcal{F}x)'(t)$ uniformly on each compact subinterval in $[T, \infty)$.

Next, we recall by the definition of X that $\mathcal{F}x$ satisfies

$$|(\mathcal{F}x)'(t)| \leq mQ(t)^{1/\alpha} + k \left(\int_t^\infty Q(s)^{(1+\alpha)/\alpha} ds \right)^{1/\alpha}, \quad t \geq T.$$

Notice that the function on the right hand side of this inequality is integrable on $[T, \infty)$. Since

$$|\mathcal{F}x(t) - \mathcal{F}x_n(t)| \leq \int_t^\infty |(\mathcal{F}x)'(s) - (\mathcal{F}x_n)'(s)| ds$$

and we have just shown that $\lim_{n \rightarrow \infty} (\mathcal{F}x_n)'(t) = (\mathcal{F}x)'(t)$, the dominated convergence theorem implies that $\lim_{n \rightarrow \infty} \mathcal{F}x_n(t) = \mathcal{F}x(t)$ uniformly on each compact subinterval in $[T, \infty)$. Hence $\mathcal{F}$ is continuous.

Finally, we show the relative compactness of the set $\mathcal{F}X$. It suffices, by Ascoli-Arzelà's theorem, to show that the set $\{(\mathcal{F}x)' : x \in X\}$ is equicontinuous

on each compact subinterval in $[T, \infty)$. Let $0 < \alpha \leq 1$. Since each $\mathcal{F}x$, $x \in X$, is of class $C^2[T, \infty)$ and

$$(\mathcal{F}x)''(t) = -\frac{1}{\alpha}\{Q(t)x(t)^{-\beta} + \beta I[x](t)\}^{(1-\alpha)/\alpha} q(t)x(t)^{-\beta},$$

the set $\{(\mathcal{F}x)'' : x \in X\}$ is uniformly bounded on each compact subinterval in $[T, \infty)$. Next let $\alpha > 1$ and $t_1, t_2 \in [T, \infty)$. Noting the inequality $|u^{1/\alpha} - v^{1/\alpha}| \leq |u - v|^{1/\alpha}$ for $u, v \geq 0$, we have

$$\begin{aligned} |(\mathcal{F}x)'(t_1) - (\mathcal{F}x)'(t_2)| &\leq |\{Q(t_1)x(t_1)^{-\beta} - Q(t_2)x(t_2)^{-\beta}\} \\ &\quad + \beta\{I[x](t_1) - I[x](t_2)\}|^{1/\alpha} \\ &\leq |Q(t_1)x(t_1)^{-\beta} - Q(t_2)x(t_2)^{-\beta}|^{1/\alpha} \\ &\quad + \beta^{1/\alpha}|I[x](t_1) - I[x](t_2)|^{1/\alpha} \\ &= |Q(t_1)\{x(t_1)^{-\beta} - x(t_2)^{-\beta}\} + \{Q(t_1) - Q(t_2)\}x(t_2)^{-\beta}|^{1/\alpha} \\ &\quad + \beta^{1/\alpha}\left|\int_{t_1}^{t_2} Q(s)x(s)^{-\beta-1}[-x'(s)]ds\right|^{1/\alpha}. \end{aligned}$$

Since the integrand of the last integral is dominated by an integrable function on $[T, \infty)$ by (7), the set $\{(\mathcal{F}x)' : x \in X\}$ is equicontinuous on each compact subinterval of $[T, \infty)$.

From Schauder-Tychonoff's fixed point theorem there is a fixed point $x \in X$ of $\mathcal{F} : \mathcal{F}x = x$.

Step 2: We will show that $x'(t) < 0$, $t \geq T$, where x is the solution of (5) just constructed above. To see this suppose the contrary that $x'(\tilde{t}) = 0$ for some $\tilde{t} \geq T$. Since

$$[-x'(t)]^\alpha = Q(t)x(t)^{-\beta} + \beta I[x](t) \geq 0,$$

we have $Q(\tilde{t}) = I[x](\tilde{t}) = 0$. Therefore by the definition of $I[x](t)$ we find that $Q(t)[-x'(t)] \equiv 0$ on $[\tilde{t}, \infty)$. It follows that

$$Q(t)^\alpha [-x'(t)]^\alpha = Q(t)^{1+\alpha} x(t)^{-\beta} + \beta Q(t)^\alpha I[x](t) \equiv 0, \quad t \geq \tilde{t}.$$

Consequently, we have $Q(t)^{1+\alpha} x(t)^{-\beta} \equiv 0$, that is $Q(t) \equiv 0$, $t \geq \tilde{t}$. This contradicts the noncompactness of $\text{supp } q$. Hence $x'(t) < 0$ on $[T, \infty)$.

Step 3: This is the final step. Since the exponent $-\beta$ of the nonlinear term of equation (1) is negative, and $x(t)$ is strictly decreasing on $[T, \infty)$, we can easily show that $x(t)$ can be extended to the left as a positive solution of

equation (1) so that $x(t)$ exists on the whole interval $[t_0, \infty)$, and $x'(t) < 0$ on $[t_0, \infty)$. This completes the proof of Lemma 5.

Proof of Theorem 1. By Lemma 5, we can find a sequence of functions $\{x_n\} \subset C^1[t_0, \infty)$ satisfying $x'_n(t) < 0$ on $[t_0, \infty)$, $\lim_{t \rightarrow \infty} x_n(t) = 1/n$, and $\lim_{t \rightarrow \infty} x'_n(t) = 0$, and

$$(8) \quad x_n(t) = \frac{1}{n} + \int_t^\infty \left(\int_s^\infty q(r) x_n(r)^{-\beta} dr \right)^{(1/\alpha)^*} ds$$

$$= \frac{1}{n} + \int_t^\infty \left\{ Q(s) x_n(s)^{-\beta} + \beta \int_s^\infty Q(r) x_n(r)^{-\beta-1} [-x'_n(r)] dr \right\}^{1/\alpha} ds, \quad t \geq t_0.$$

Below we show that the limit function of a suitable subsequence of $\{x_n\}$ gives rise to a desired positive solution of (1)-(2).

The decreasing nature of x_n and (8) imply for $n \in \mathbb{N}$ and $t \geq t_0$ that

$$x_n(t) \geq x_n(t)^{-\beta/\alpha} \int_t^\infty Q(s)^{1/\alpha} ds;$$

that is,

$$(9) \quad x_n(t) \geq \left(\int_t^\infty Q(s)^{1/\alpha} ds \right)^{\alpha/(\alpha+\beta)}.$$

Thus from (8) again we obtain

$$(10) \quad x_n(t) \leq 1 + \int_t^\infty \left\{ \int_s^\infty |q(r)| \left(\int_r^\infty Q(u)^{1/\alpha} du \right)^{-\alpha\beta/(\alpha+\beta)} dr \right\}^{1/\alpha} ds.$$

Since

$$-x'_n(t) = \left(\int_t^\infty q(s) x_n(s)^{-\beta} ds \right)^{(1/\alpha)^*},$$

by (9) we have

$$(11) \quad |x'_n(t)| \leq \left\{ \int_t^\infty |q(s)| \left(\int_s^\infty Q(r)^{1/\alpha} dr \right)^{-\alpha\beta/(\alpha+\beta)} ds \right\}^{1/\alpha}.$$

We find from (10) and (11) that the sequence $\{x_n\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. Therefore Ascoli-Arzelà's theorem and a diagonal argument imply that a subsequence $\{x_{n_i}\} \subset \{x_n\}$ converges to a function $\tilde{x} \in C[t_0, \infty)$ uniformly on each compact subset in $[t_0, \infty)$. We notice by (9) that $\tilde{x}(t) > 0$, $t \geq t_0$. Since x_{n_i} satisfies

$$x_{n_i}(t) = \frac{1}{n_i} + \int_t^\infty \left(\int_s^\infty q(r) x_{n_i}(r)^{-\beta} dr \right)^{1/\alpha} ds,$$

we have

$$\tilde{x}(t) = \int_t^\infty \left(\int_s^\infty q(r) \tilde{x}(r)^{-\beta} dr \right)^{1/\alpha} ds,$$

by letting $n_i \rightarrow \infty$. Here we have used the dominated convergence theorem. This formula shows obviously that $\tilde{x}$ is a positive solution of (1)-(2). The proof is completed.

Proof of Theorem 2. Let $\{x_n\}$ be the sequence introduced in the proof of Theorem 1. As before, we will show that $\{x_n\}$ contains a subsequence converging to a positive decaying solution by a method different from that of Theorem 1. To this end we establish for $\{x_n\}$ the following estimates for $n \in N$ and $t \geq t_0$:

$$(12) \quad x_n(t)^{(\alpha+\beta)/(\alpha+1)} \leq n^{-(\alpha+\beta)/(\alpha+1)} + \frac{\alpha+\beta}{\alpha+1} \left(\frac{\alpha+1}{\alpha(1-\beta)} \right)^{1/(\alpha+1)} \int_t^\infty \hat{q}(s)^{1/(\alpha+1)} ds;$$

$$(13) \quad [-x'_n(t)]^{\alpha+1} \leq \frac{\alpha+1}{\alpha(1-\beta)} \hat{q}(t) x_n(t)^{1-\beta};$$

and

$$(14) \quad [-x'_n(t)]^\alpha \geq \beta \int_t^\infty Q(s)^{(\alpha+1)/\alpha} x_n(s)^{-\beta-1-\beta/\alpha} ds.$$

In fact, rewriting equation (1) as

$$(15) \quad \alpha |x'_n|^{\alpha-1} x''_n = q(t) x_n^{-\beta}, \quad t \geq t_0,$$

and multiplying both sides by $-x'_n(>0)$, we have

$$-\left(\frac{\alpha}{\alpha+1} |x'_n|^{\alpha+1} \right)' = -q(t) x_n^{-\beta} x'_n.$$

In view of the decreasing nature of $\hat{q}$, by integrating this formula on $[t, \infty)$ we have

$$\begin{aligned} \frac{\alpha}{\alpha+1} |x'_n(t)|^{\alpha+1} &\leq \int_t^\infty |q(s) x_n(s)^{-\beta} [-x'_n(s)]| ds \\ &\leq \hat{q}(t) \int_t^\infty x_n(s)^{-\beta} [-x'_n(s)] ds \\ &= \frac{\hat{q}(t)}{1-\beta} x_n(t)^{1-\beta}. \end{aligned}$$

Thus (13) is proved. Since (13) is rewritten as

$$-(x_n(t)^{(\alpha+\beta)/(\alpha+1)})' \leq \frac{\alpha+\beta}{\alpha+1} \left(\frac{\alpha+1}{\alpha(1-\beta)} \right)^{1/(\alpha+1)} \hat{q}(s)^{1/(\alpha+1)},$$

an integration on $[t, \infty)$ of this inequality gives (12). Finally, since

$$[-x'_n(t)]^\alpha = Q(t)x_n(t)^{-\beta} + \beta \int_t^\infty Q(s)x_n(s)^{-\beta-1}[-x'_n(s)]ds,$$

by (8), we have

$$-x'_n(t) \geq Q(t)^{1/\alpha} x_n(t)^{-\beta/\alpha},$$

and

$$[-x'_n(t)]^\alpha \geq \beta \int_t^\infty Q(s)x_n(s)^{-\beta-1}[-x'_n(s)]ds.$$

Combining these inequalities we can obtain (14).

From (12) and (13) we find that $\{x_n\}$ and $\{x'_n\}$ are uniformly bounded on $[t_0, \infty)$. Moreover (9), (13), (14) and equation (15) imply that $\{x''_n\}$ is uniformly bounded on each compact subinterval in $[t_0, \infty)$. Hence, by Ascoli-Arzelà's theorem and the diagonal argument it follows that $\{x_n\}$ contains a subsequence $\{x_{n_i}\}$ satisfying $\lim_{n_i \rightarrow \infty} x_{n_i}(t) = \tilde{x}(t)$ and $\lim_{n_i \rightarrow \infty} x'_{n_i}(t) = \tilde{x}'(t)$ for some $\tilde{x} \in C^1[t_0, \infty)$ uniformly on each compact subinterval in $[t_0, \infty)$. Letting $n_i \rightarrow \infty$ in the formulas

$$[-x'_{n_i}(t)]^\alpha = [-x'_{n_i}(t_0)]^\alpha - \int_{t_0}^t q(s)x_{n_i}(s)^{-\beta}ds, \quad t \geq t_0,$$

and

$$x_{n_i}(t)^{(\alpha+\beta)/(\alpha+1)} \leq n_i^{-(\alpha+\beta)/(\alpha+1)} + \frac{\alpha+\beta}{\alpha+1} \left(\frac{\alpha+1}{\alpha(1-\beta)} \right)^{1/(\alpha+1)} \int_t^\infty \hat{q}(s)^{1/(\alpha+1)}ds,$$

we find that $\tilde{x}$ is a desired positive solution of (1)-(2). (Note that the positivity of $\tilde{x}$ is guaranteed by (9).) The proof is finished.

Proof of Theorem 3. Let $\{x_n\}$ be the sequence introduced in the proof of Theorem 1. As before, we will show that $\{x_n\}$ contains a subsequence converging to a positive decaying solution. We establish for $\{x_n\}$ the following estimates for $n \in \mathbb{N}$ and $t \geq t_0$:

$$(16) \quad x_n(t)^{(\alpha+\beta)/(\alpha+\delta)} \leq c_1 n^{-(\alpha+\beta)/(\alpha+\delta)} + c_2 \int_t^\infty \left(\int_s^\infty |q(r)|^{1/(1-\delta)} dr \right)^{(1-\delta)/(\alpha+\delta)} ds;$$

$$(17) \quad 0 < -x'_n(t) \leq c_3 \left(\int_t^\infty |q(s)|^{1/(1-\delta)} ds \right)^{(1-\delta)/(\alpha+\delta)} x_n(t)^{(\delta-\beta)/(\alpha+\delta)}.$$

Here c_1 , c_2 and c_3 are positive constants independent of n and t . To see (16), we multiply equation (1) by $(-x'_n)^\delta$:

$$-\left(\frac{\alpha}{\alpha+\delta}(-x'_n)^{\alpha+\delta}\right)' = q(t)x_n^{-\beta}(-x'_n)^\delta.$$

We integrate this on $[t, \infty)$ with noting the fact that $x'_n(t) < 0$ and $(1-\delta) + \delta = 1$:

$$\begin{aligned} \frac{\alpha}{\alpha+\delta}[-x'_n(t)]^{\alpha+\delta} &= \int_t^\infty q(s)x_n(s)^{-\beta}[-x'_n(s)]^\delta ds \\ &\leq \int_t^\infty |q(s)|x_n(s)^{-\beta}[-x'_n(s)]^\delta ds \\ &\leq \left(\int_t^\infty |q(s)|^{1/(1-\delta)} ds\right)^{1-\delta} \left(\int_t^\infty x_n(s)^{-\beta/\delta}[-x'_n(s)]^\delta ds\right)^\delta \\ &\leq \left(\frac{\delta}{\delta-\beta}\right)^\delta \left(\int_t^\infty |q(s)|^{1/(1-\delta)} ds\right)^{1-\delta} x_n(t)^{\delta-\beta}. \end{aligned}$$

This gives (17). The inequality (16) follows from (17) by an integration. As in the proof of Theorem 2, we can show that $\{x_n\}$ contains a subsequence converging to a desired positive solution of problem (1)-(2). This completes the proof.

Example 1. Let us consider problem (1)-(2) under the condition that q satisfies

$$c_1 t^{-v_1} \leq q(t) \leq c_2 t^{-v_2} \quad \text{near } +\infty$$

for some constants $c_1, c_2 > 0$ and $v_1, v_2 \in \mathbf{R}$. Theorem 1 shows that, if

$$\alpha + 1 < v_2 \leq v_1 < \frac{\alpha}{\beta}(v_2 - \alpha - 1) + v_2,$$

then problem (1)-(2) has a positive decaying solution.

In fact, the typical equation

$$(|x'|^{\alpha-1}x')' = t^{-v}x^{-\beta}, \quad v > \alpha + 1, t \geq 1,$$

has a positive decaying solution x_0 explicitly given by

$$(18) \quad x_0(t) = c_0 t^{-\mu}, \quad t \geq 1,$$

where

$$(19) \quad \mu = \frac{v - \alpha - 1}{\alpha + \beta} \quad \text{and} \quad c_0^{\alpha+\beta} = \frac{1}{\alpha\mu^\alpha(\mu+1)} = \frac{(\alpha+\beta)^{\alpha+1}}{\alpha(v-\alpha-1)^\alpha(v+\beta-1)}.$$

Example 2. Let $0 < \beta < 1$ and $v > \alpha + 1$. Then the problem

$$(|x'|^{\alpha-1}x')' = t^{-v-1}\{v(1 + \sin t) - t \cos t\}x^{-\beta}, \quad t \geq 1,$$

$$\lim_{t \rightarrow \infty} x(t) = 0$$

has a positive solution. To show this by Corollary 4, we observe that the coefficient function $q(t)$ of this equation is $O(t^{-v})$, and $\int_t^\infty q(t)dt = t^{-v}(1 + \sin t) \geq 0$. Note that the coefficient function $q(t)$ oscillates, and hence known existence results in [4] are not applicable.

3. A remark for uniqueness of positive decaying solutions

As is stated in Example 1 in Section 2, the typical equation

$$(20) \quad (|x'|^{\alpha-1}x')' = t^{-v}x^{-\beta}, \quad v > \alpha + 1, t \geq 1,$$

has a positive decaying solution $x = x_0$ explicitly given by (18) and (19). If $\alpha = 1$, we proved that x_0 is the only positive decaying solution of (20); see [7] for the details. We here show that, for this typical equation, such a uniqueness theorem is still valid.

Theorem 6. *Equation (20) has a unique positive decaying solution explicitly given by (18) and (19).*

This theorem is proved via asymptotic theory of planar dynamical systems. We need the following lemmas:

Lemma 7. *Let x be a positive decaying solution of (20). Then we have*

$$(21) \quad 0 < \liminf_{t \rightarrow \infty} \frac{x(t)}{x_0(t)} \leq \limsup_{t \rightarrow \infty} \frac{x(t)}{x_0(t)} < \infty,$$

and

$$(22) \quad 0 < \liminf_{t \rightarrow \infty} \frac{-x'(t)}{-x'_0(t)} \leq \limsup_{t \rightarrow \infty} \frac{-x'(t)}{-x'_0(t)} < \infty,$$

where x_0 is given by (18) and (19).

Proof. Computing as in the proof of (9), we can prove the validity of the first inequality in (21). Once we get an estimate of $x(t)$ from below, we can get an estimate of $x(t)$ from above by equation (1). Since

$$-x'(t) = \left(\int_t^\infty s^{-v} x(s)^{-\beta} ds \right)^{1/\alpha},$$

we see from the just verified inequality that

$$\begin{aligned} -x'(t) &\leq c_1 \left(\int_t^\infty s^{-v} (s^{-(v-\alpha-1)/(\alpha+\beta)})^{-\beta} ds \right)^{1/\alpha} \\ &= c_2 t^{-(v+\beta-1)/(\alpha+\beta)} \equiv \tilde{c}_2 [-x'_0(t)], \quad t \geq 1 \end{aligned}$$

for some constants $c_1, c_2, \tilde{c}_2 > 0$. This proves the validity of the last inequality in (22). The first inequality in (22) can be shown similarly. This completes the proof.

Lemma 8. *Let*

$$t = e^s \quad \text{and} \quad x/x_0(t) = z, \quad (\dot{z} - \mu z)^{\alpha*} = w,$$

where $\cdot = d/ds$ and $\mu = (v - \alpha - 1)/(\alpha + 1)$. Then, equation (20) is transformed into the dynamical system

$$(23) \quad \begin{cases} \dot{z} = \mu z + w^{(1/\alpha)*} \\ \dot{w} = \alpha(\mu + 1)w + \alpha\mu^\alpha(\mu + 1)z^{-\beta}. \end{cases}$$

Proof of Theorem 6. The proof is done by contradiction. Let x be a positive decaying solution of (20) which does not coincide with x_0 . Let us perform the change of variables introduced in Lemma 8. We denote $(z(s), w(s)) = \varphi(s)$ below. The only critical point of the dynamical system (23) lying in the region $z > 0$ is $P_0 = (1, -\mu^\alpha)$. Notice that the particular solution x_0 of equation (20) corresponds to the critical point P_0 .

Let $J(z, w)$ be the Jacobi matrix of the right hand side of system (23). It is easily seen that

$$J(1, -\mu^\alpha) = \begin{pmatrix} \mu & \alpha^{-1}\mu^{1-\alpha} \\ -\alpha\beta\mu^\alpha(\mu + 1) & \alpha(\mu + 1) \end{pmatrix}.$$

Therefore the real parts of the eigenvalues of $J(1, -\mu^\alpha)$ are all positive. We can find an open ball B centered at P_0 with a sufficiently small radius such that $\varphi(s) \in B^c$ for $s \geq 0$; see [5, Section 26]. By Lemma 7 we can find a compact set K lying in the region $z > 0$ such that $\varphi(s) \in K$ for $s \geq 0$. We may suppose that K contains B entirely. Hence, Poincare-Bendixson's theorem asserts that system (23) has a periodic solution contained in K .

On the other hand, since

$$\frac{\partial}{\partial z}(\mu z + w^{(1/\alpha)*}) + \frac{\partial}{\partial w}\{\alpha(\mu + 1)w + \alpha\mu^\alpha(\mu + 1)z^{-\beta}\} \equiv \mu + \alpha(\mu + 1) > 0,$$

Bendixson's theorem implies that system (23) does not have periodic solutions. This is an obvious contradiction. This completes the proof.

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Misako Motai
Kanzaki Seimei High School
Kanzaki, Saga 842-0012
Japan

Hiroyuki Usami
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8521
Japan
E-mail: usami@mis.hiroshima-u.ac.jp

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