On Solutions of q -Painlevé Equation of Type $A_7^{(1)}$

By

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Abstract. In this paper, we will study a property of solutions of q -Painlevé equation of type $A_7^{(1)}$. We propose the notion of decomposable extension and then prove the equation has no solution in any decomposable extension of $C(t)$.

Key Words and Phrases. q -Painlevé equations, q -difference equations, Difference algebra.

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1. Introduction

In this paper, we will study a property of solutions of q -Painlevé equation of type $A_7^{(1)}$,

$$q\text{-}P(A_7): \quad \bar{f}f^2\underline{f} = t(1 - f),$$

where $\bar{f} = f(qt)$ and $\underline{f} = f(t/q)$. q -Painlevé equations are q -difference equations which are discrete analogs of the Painlevé equations. $q\text{-}P(A_7)$ appears in the paper of A. Ramani and B. Grammaticos ([8]) as their equation (4.3).

B. Grammaticos, A. Ramani and V. Papageorgiou presented in their [3] a notion called singularity confinement, by which they obtained an integrability criterion for discrete-time systems that is a discrete counter part of the Painlevé property for systems of a continuous variable. A. Ramani, B. Grammaticos and J. Hietarinta discovered discrete Painlevé equations such as $d\text{-}P_{\text{III}}$, $d\text{-}P_{\text{IV}}$, and $d\text{-}P_{\text{V}}$ (or $q\text{-}P(A_3)$, $d\text{-}P(A_2^*)$ and $q\text{-}P(A_2)$ respectively for their rational surfaces), using the method of singularity confinement (see [7]). Since then further discrete Painlevé equations have been found.

In his [9], H. Sakai introduced a geometric approach to theory of the Painlevé equations, and showed both classifications of Painlevé equations and discrete Painlevé equations by rational surfaces. The notation $q\text{-}P(A_7)$ is determined by the type $A_7^{(1)}$ of the rational surface of the equation. The list of discrete Painlevé equations and their notations can be seen in the paper of H. Sakai ([10]). In addition $q\text{-}P(A_7)$ has symmetry $A_1^{(1)}$.

Irreducibility of ordinary differential equations has been deeply studied (for example see the paper [11] on the first Painlevé equation of H. Umemura). On the other hand, we do not have an appropriate definition of irreducibility of

q -difference equations, instead we propose the notion of decomposable extension, difference analogue of K. Nishioka's in [4].

The notations on difference algebra are referred to R. M. Cohn [2]. A difference ring (field) (R, τ) is a pair of a ring (field) R and an isomorphism τ of R onto a subring (subfield) of R . C_R denotes a ring (field) of constants $\{a \in R \mid \tau a = a\}$. A difference ring (field) (R', τ') is a difference overring (overfield) of a difference ring (field) (R, τ) if $R' \supset R$ and $\tau'|_R = \tau$. For difference field K and a subset φ of its difference overfield L , $K\{\varphi\}$ ($K\langle\varphi\rangle$) denotes the intersection of all difference subrings (subfields) of L which contain K and φ .

Let (R, τ) and (R', τ') be difference rings. A mapping ϕ is a difference homomorphism of (R, τ) into (onto) (R', τ') if ϕ is a homomorphism of R into (onto) R' and $\phi\tau = \tau'\phi$. If ϕ is also injective, it is a difference isomorphism of (R, τ) into (R', τ') .

Let (K, τ) be a difference field. A difference ring $R = K\{y\} = K[y_0 = y, y_1, y_2, \dots]$ is called a polynomial difference ring over K if y_i , $i = 0, 1, \dots$ are algebraically independent over K and the isomorphism of R sends y_i to y_{i+1} . Let f be an element of a polynomial difference ring R over K . We define a solution of the difference equation $f = 0$ as an element α of some difference overfield L of K which satisfies $\phi f = 0$, where ϕ is a difference homomorphism of R into L sending y_i to α_i , the i -th transform of α .

Throughout the paper, C denotes an algebraically closed field of characteristic 0, t an element transcendental over C and q an element of $C^\times$ which is not a root of unity. Let U be an algebraically closed difference field. Unless specifically state we assume that every difference field that appears in this paper is a difference subfield of U . We may take the field of Puiseux series as U for example.

We prove the following theorem.

Theorem. *Let $q \in C^\times$ be not a root of unity, (K, τ_K) be a difference overfield of $(C(t), t \mapsto qt)$ whose operator τ_K is an automorphism of K , D a decomposable extension of K and $f \in D$ a solution of $y_2 y_1^2 y = qt(1 - y_1)$. Then f is algebraic over K .*

A definition of decomposable extensions is the following.

Definition 1 (decomposable extension). Let L/K be a difference field extension of finite transcendence degree and $n = \text{tr.deg } L/K$. We define decomposable extensions inductively.

- (i) If $n = 0$ or 1 then L/K is decomposable.
- (ii) When $n > 1$, L/K is decomposable if there exists a difference overfield E of K such that $\text{tr.deg } E/K < \infty$, E is free from L over K and there exists a difference intermediate field M of LE/E such that

$\text{tr.deg } LE/M \geq 1$, $\text{tr.deg } M/E \geq 1$, LE/M is decomposable and M/E is decomposable.

The author proved in [5] that solutions of q - $P(A_7)$ are all transcendental over $C(t)$. Thus q - $P(A_7)$ has no solution in any decomposable extension of $C(t)$.

2. Preliminaries

In this section we make several preparations and see an example of decomposable extensions.

Lemma 1. *Let N/K be a difference field extension and L a difference intermediate field of N/K . If N/L and L/K are decomposable, then N/K is decomposable.*

Proof. Let $n = \text{tr.deg } N/K$. We prove this inductively. When $n = 0$ or 1, we find N/K is decomposable by the definition. Let $n > 1$.

(a) If $\text{tr.deg } N/L \geq 1$ and $\text{tr.deg } L/K \geq 1$, then K is free from N over L . Since we have $NK = N$ and $LK = L$, we conclude N/K is a decomposable extension.

(b) If $\text{tr.deg } N/L = 0$, we have $\text{tr.deg } L/K \geq 2$. By definition, there exists a difference overfield E of K such that $\text{tr.deg } E/K < \infty$, E is free from L over K and there exists a difference intermediate field M of LE/E such that LE/M and M/E are both decomposable extensions of $\text{tr.deg} \geq 1$. By induction hypothesis we find NE/M is decomposable. Considering $\text{tr.deg } NE/M \geq 1$ and $\text{tr.deg } M/E \geq 1$, we conclude N/K is a decomposable extension.

(c) If $\text{tr.deg } L/K = 0$, we have $\text{tr.deg } N/L \geq 2$. By definition, there exists a difference overfield E of L such that $\text{tr.deg } E/L < \infty$, E is free from N over L and there exists a difference intermediate field M of NE/E such that NE/M and M/E are both decomposable extensions of $\text{tr.deg} \geq 1$. Since E is free from N over K , N/K is a decomposable extension. $\square$

From here to the end of this section we do not assume that every difference field that appears in the followings is a difference subfield of U . The existence of the composite difference overfield LE of L and E in Definition 1 is additionally required for a difference field extension L/K to be decomposable.

A. Białynicki-Birula defined strongly normal extensions, by which he developed a Galois theory for fields with operators, where a field with operators means a field together with a family of “automorphisms” and derivations of the field containing the zero derivation and the identity automorphism (see [1]). We introduce the definition of strongly normal extensions with one operator.

Definition 2 (strongly normal extension). Let (K, τ_K) be a difference field whose operator τ_K is an automorphism of K and (L, τ_L) a difference overfield of K whose operator τ_L is an automorphism of L . Then we say that (L, τ_L) is a strongly normal extension of (K, τ_K) if

- (i) The field L is a regular extension of the field K .
- (ii) The field L is finitely generated over the field K .
- (iii) $C_L = C_K$ and C_K is algebraically closed.
- (iv) $\langle L \otimes_K L \rangle = \langle L \otimes_K 1 \rangle C_{\langle L \otimes_K L \rangle}$

This type of strongly normal extension is decomposable, for the proof of which we prepare two lemmas.

Lemma 2. *Let $N_0 \subset N_1 \subset \cdots \subset N_m$ be a chain of difference field extensions such that $\text{tr.deg } N_i/N_{i-1} \leq 1$ for all $1 \leq i \leq m$. Then N_m/N_0 is decomposable.*

Proof. Let $n = \text{tr.deg } N_m/N_0$. We prove this inductively. When $n = 0$ or 1, we find N_m/N_0 is decomposable by the definition. Let $n \geq 2$. There exists i such that $\text{tr.deg } N_m/N_i \geq 1$ and $\text{tr.deg } N_i/N_0 \geq 1$. From the induction hypothesis we obtain N_m/N_i and N_i/N_0 are decomposable. Hence we conclude that N_m/N_0 is decomposable. $\square$

Lemma 3. *Let L/K be a decomposable extension, N/J a difference field extension and $\phi : L \xrightarrow{\sim} N$ a surjective difference isomorphism satisfying $\phi(K) = J$. Then N/J is decomposable.*

Proof. We inductively prove this. Note that $\text{tr.deg } L/K = \text{tr.deg } N/J$. When $\text{tr.deg } L/K = 0$ or 1, we have $\text{tr.deg } N/J \leq 1$, and so N/J is decomposable. Let $\text{tr.deg } L/K \geq 2$.

By definition there exists a difference overfield E of K with finite transcendence degree such that there exists a composite difference overfield LE of L and E , E is free from L over K , and there exists a difference intermediate field M of LE/E such that LE/M and M/E are both decomposable extensions of $\text{tr.deg} \geq 1$.

Let $\{a_1, a_2, \dots, a_n\}$ be a transcendence basis of E/K . Since L and E are free over K , a_i ($1 \leq i \leq n$) are algebraically independent over L . Choose $b_1, b_2, \dots, b_n$ to be algebraically independent over N . We extend the surjective isomorphism $\phi : L \xrightarrow{\sim} N$ to a surjective isomorphism $\phi_1 : L(a_1, \dots, a_n) \xrightarrow{\sim} N(b_1, \dots, b_n)$ sending a_i to b_i . Then we extend ϕ_1 to a surjective isomorphism

$$\overline{\phi_1} : \overline{LE} = \overline{L(a_1, \dots, a_n)} \xrightarrow{\sim} \overline{N(b_1, \dots, b_n)},$$

where $\overline{LE}$ is an algebraic closure of LE , etc. $\tilde{\phi} = \overline{\phi_1}|_{LE}$ is an isomorphism of LE into $\overline{N(b_1, \dots, b_n)}$ and an extension of ϕ . From $\phi(L) = N$ we obtain $\tilde{\phi}(LE) = N\tilde{\phi}(E)$.

Let τ_{LE} be the operator of the difference field LE . We define an operator τ of $\tilde{\phi}(LE)$ as $\tau = \tilde{\phi} \circ \tau_{LE} \circ \tilde{\phi}^{-1}$. In fact τ is an isomorphism of $\tilde{\phi}(LE)$ into $\tilde{\phi}(LE)$ because $\tilde{\phi}$, τ_{LE} and $\tilde{\phi}^{-1}$ are injective isomorphisms. Then $\tilde{\phi}$ is a difference isomorphism of (LE, τ_{LE}) onto $(\tilde{\phi}(LE), \tau)$.

We will see $\tau|_N = \tau_N$, where τ_N is the original operator of the difference field N . Set $\tau_L = \tau_{LE}|_L$. Since ϕ is a difference isomorphism, we have $\phi\tau_L = \tau_N\phi$. Hence for any $x \in N$ we find

$$\begin{aligned}\tau x &= \tilde{\phi} \circ \tau_{LE} \circ \tilde{\phi}^{-1}(x) = \tilde{\phi} \circ \tau_{LE}(\phi^{-1}(x)) \\ &= \tilde{\phi}(\tau_L \circ \phi^{-1}(x)) = \phi \circ \tau_L \circ \phi^{-1}(x) \\ &= \tau_N x,\end{aligned}$$

which means $\tau|_N = \tau_N$.

Since $\tilde{\phi}$ is a difference isomorphism, we obtain that $\tilde{\phi}(LE)/\tilde{\phi}(M)$ and $\tilde{\phi}(M)/\tilde{\phi}(E)$ are decomposable extensions with $\text{tr.deg} \geq 1$ by the induction hypothesis. We also obtain $N = \phi(L)$ and $\tilde{\phi}(E)$ are free over $J = \phi(K)$. Note that $\tilde{\phi}(LE) = N\tilde{\phi}(E)$ and $\text{tr.deg } \tilde{\phi}(E)/J < \infty$. Then we conclude that N/J is decomposable. $\square$

Proposition 1. *Let L/K be a strongly normal extension. Then L/K is decomposable.*

Proof. We only have to consider the case $\text{tr.deg } L/K \geq 2$. Put $L_1 = L \otimes_K 1$ and $L_2 = 1 \otimes_K L$. L_1 and L_2 are linearly disjoint over K , so we have that L_1 is free from L_2 over K . By definition L_2/K is finitely generated, which implies that $L_1 L_2 / L_1$ is also finitely generated. Since $L_1 L_2 = L_1 C_{L_1 L_2}$, we obtain $C_{L_1 L_2} / C_{L_1}$ is finitely generated by using Corollary 1 in [1].

Let $C_{L_1 L_2} = C_{L_1}(x_1, \dots, x_n)$ and $N_i = L_1(x_1, \dots, x_i)$ for all i ($0 \leq i \leq n$). We find that

$$L_1 = N_0 \subset N_1 \subset \dots \subset N_n = L_1 C_{L_1 L_2} = L_1 L_2$$

is a finite chain of difference field extensions. Since $\text{tr.deg } N_j / N_{j-1} \leq 1$ for all j ($1 \leq j \leq n$), there exists an integer i such that $\text{tr.deg } L_1 L_2 / N_i \geq 1$ and $\text{tr.deg } N_i / L_1 \geq 1$. We find the extensions $L_1 L_2 / N_i$ and N_i / L_1 are decomposable by Lemma 2. Then we have L_2 / K is decomposable by definition. Using Lemma 3, we conclude that L/K is decomposable. $\square$

The following Lemma shows that some kind of algebraically closed difference fields have an automorphism operator.

Lemma 4. *Let (K, τ_K) be a difference field whose operator τ_K is an automorphism of K and L a difference overfield of K such that $\text{tr.deg } L/K < \infty$.*

Then algebraic closure $\bar{L}$ of L is a difference overfield of L with some automorphism of $\bar{L}$.

Proof. Let τ be the operator of L . Then there is an extension of τ ,

$$\bar{\tau} : \bar{L} \xrightarrow{\sim} \overline{\tau L} \subset \bar{L},$$

which is an isomorphism. Hence $(\bar{L}, \bar{\tau})$ is a difference overfield of (L, τ) . We prove that the image of $\bar{\tau}$ equals to $\bar{L}$. Let $x_1, x_2, \dots, x_n$ be a transcendence basis of L/K . Since τ_K is an automorphism, $\tau x_1, \tau x_2, \dots, \tau x_n$ are algebraically independent over K . Thus we obtain $\text{tr.deg } \tau L/K = n$, and so we find that $\text{tr.deg } \overline{\tau L}/\bar{K} = n$. From $\text{tr.deg } \bar{L}/\bar{K} = n$, we conclude that $\bar{L}/\overline{\tau L}$ is algebraic, which implies $\overline{\tau L} = \bar{L}$. $\square$

3. Proof of Theorem

In this section, we prove a lemma and a proposition, which imply the theorem.

Lemma 5. *Let (K, τ_K) be a difference field, D a decomposable extension of K and f an element of D . Suppose f satisfies the following; for any difference overfield L of K with finite transcendence degree satisfying $\text{tr.deg } L\langle f \rangle/L \leq 1$, f is algebraic over L . Then f is algebraic over K .*

Proof. Assume that f is transcendental over K . Choose (L, N) be an element of

$$\{(L, N) \mid K \subset L \subset N, N/L \text{ is decomposable, } \text{tr.deg } L/K < \infty, f \notin \bar{L}, f \in N\}$$

which has the minimal transcendence degree. (L, N) can be chosen because (K, D) satisfies the conditions. From $f \notin \bar{L}$ and the hypothesis, we find $\text{tr.deg } N/L \geq 2$. Since N/L is decomposable, there exists a difference overfield E of L such that $\text{tr.deg } E/L < \infty$, E is free from N over L and there exists a difference intermediate field M of NE/E such that NE/M and M/E are both decomposable extensions of $\text{tr.deg } \geq 1$.

Then we have $f \in NE$ and $\text{tr.deg } NE/M < \text{tr.deg } N/L$, which implies $f \in \bar{M}$. Thus we obtain $M\langle f \rangle/E$ is decomposable by Lemma 1. Hence we find $f \in \bar{E}$ from $\text{tr.deg } M\langle f \rangle/E < \text{tr.deg } N/L$, which contradicts the fact that N and E are free over L . Therefore f is algebraic over K . $\square$

Proposition 2. *Let $q \in C^\times$ be not a root of unity, (K, τ_K) be a difference overfield of $(C(t), t \mapsto qt)$ whose operator τ_K is an automorphism of K , L a difference overfield of K such that $\text{tr.deg } L/K < \infty$ and $f (\in U)$ a solution of $y_2 y_1^2 y = qt(1 - y_1)$. If $\text{tr.deg } L\langle f \rangle/L \leq 1$ then f is algebraic over L .*

Proof. It is enough to prove this for algebraically closed L . Then we find the operator of L is an automorphism of L by Lemma 4. Let τ be the operator of U . For a polynomial $F = \sum a_{ij} Y^i Y_1^j \in L[Y, Y_1]$, we define $F^* = \sum (\tau a_{ij}) Y^i Y_1^j$.

Assume that $f \notin L$. Then we have that $f_i = \tau^i f$ ($i = 0, 1, 2, \dots$) are all transcendental over L . From $\text{tr.deg } L\langle f \rangle / L \leq 1$, we find that f and f_1 are algebraically dependent over L . Thus there exists an irreducible polynomial $F \in L[Y, Y_1]$ such that $F(f, f_1) = 0$. Let F be the polynomial.

Put

$$F_0 = (Y_1 Y^2)^{n_0} F\left(\frac{qt(1-Y)}{Y_1 Y^2}, Y\right)$$

and

$$F_1 = (Y_1^2 Y)^{n_1} F^*\left(Y_1, \frac{qt(1-Y_1)}{Y_1^2 Y}\right),$$

where $n_0 = \deg_Y F$ and $n_1 = \deg_{Y_1} F$. We easily find that $F_0, F_1 \in L[Y, Y_1] \setminus \{0\}$. From

$$F_0(f_1, f_2) = (f_2 f_1^2)^{n_0} F\left(\frac{qt(1-f_1)}{f_2 f_1^2}, f_1\right) = (f_2 f_1^2)^{n_0} F(f, f_1) = 0$$

and

$$F_1(f, f_1) = (f_1^2 f)^{n_1} F^*\left(f_1, \frac{qt(1-f_1)}{f_1^2 f}\right) = (f_1^2 f)^{n_1} F^*(f_1, f_2) = 0,$$

we obtain $F^*|F_0$ and $F|F_1$, where we note that f_i s are all transcendental over L .

However we find the nonexistence of such a polynomial F from the subsequent Sublemma, a statement analogous to Theorem 1 in the paper of M. Noumi and K. Okamoto ([6]), where they defined an invariant divisor by a polynomial like F . Hence f is an element of L . $\square$

Sublemma. Let $q \in C^\times$ be not a root of unity, (L, τ) be a difference overfield of $(C(t), t \mapsto qt)$ whose operator τ is an automorphism of L , Y and Y_1 algebraically independent over L , ϕ_0 an isomorphism such that

$$\phi_0 : L(Y, Y_1) \rightarrow L(Y, Y_1)$$

$$Y \mapsto \frac{qt(1-Y)}{Y_1 Y^2}$$

$$Y_1 \mapsto Y$$

$$L \ni x \mapsto x \in L$$

and ϕ_1 an isomorphism such that

$$\phi_1 : L(Y, Y_1) \rightarrow L(Y, Y_1).$$

$$Y \mapsto Y_1$$

$$Y_1 \mapsto \frac{qt(1 - Y_1)}{Y_1^2 Y}$$

$$L \ni x \mapsto \tau x \in L$$

For a polynomial $F \in L[Y, Y_1]$ we define F^* as in the proof of Proposition 2. Then there is no irreducible polynomial $F \in L[Y, Y_1] \setminus (L[Y] \cup L[Y_1])$ such that $F^* \mid (Y_1 Y^2)^{n_0} \phi_0 F$ and $F \mid (Y_1^2 Y)^{n_1} \phi_1 F$, where $n_0 = \deg_Y F$ and $n_1 = \deg_{Y_1} F$.

Proof. Assume that there exists such F . Put $F_0 = (Y_1 Y^2)^{n_0} \phi_0 F$ and $F_1 = (Y_1^2 Y)^{n_1} \phi_1 F$. Then it follows that

$$n_1 = \deg_{Y_1} F^* \leq \deg_{Y_1} F_0 \leq n_0$$

and

$$n_0 = \deg_Y F \leq \deg_Y F_1 \leq n_1,$$

which imply $n_0 = n_1$. Put $n = n_0 = n_1$. We have $n \geq 1$.

From $F \mid F_1$, there exists a polynomial $P \in L[Y, Y_1] \setminus \{0\}$ such that $F_1 = PF$. Since $\deg_Y P = \deg_Y F_1 - \deg_Y F = 0$, we find $P \in L[Y_1]$. We express F as

$$F = \sum_{i,j} a_{ij} Y^i Y_1^j, \quad a_{ij} \in L.$$

Then we obtain

$$\begin{aligned} F_1 &= (Y_1^2 Y)^n \sum_{i,j} \tau a_{ij} Y_1^i \left(\frac{qt(1 - Y_1)}{Y_1^2 Y} \right)^j \\ &= \sum_{i,j} \tau a_{ij} Y_1^i (qt)^j (1 - Y_1)^j (Y_1^2 Y)^{n-j} \\ &= \sum_{i,j} \tau a_{i,n-j} Y_1^i (qt)^{n-j} (1 - Y_1)^{n-j} (Y_1^2 Y)^j \\ &= \sum_{j=0}^n (qt)^{n-j} (1 - Y_1)^{n-j} Y_1^{2j} \left(\sum_{i=0}^n \tau a_{i,n-j} Y_1^i \right) Y^j. \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} F_1 &= P(Y_1) \sum_{i,j} a_{ij} Y_1^i Y_1^j \\ &= \sum_{j=0}^n P(Y_1) \left(\sum_{i=0}^n a_{ji} Y_1^i \right) Y_1^j. \end{aligned}$$

Hence we have the following equations:

$$\begin{aligned} (*j) \quad (qt)^{n-j} (1 - Y_1)^{n-j} Y_1^{2j} \left(\sum_{i=0}^n \tau a_{i,n-j} Y_1^i \right) \\ = P(Y_1) \sum_{i=0}^n a_{ji} Y_1^i. \quad (0 \leq j \leq n) \end{aligned}$$

We prove that $P = p Y_1^n (1 - Y_1)^{n/2}$, $p \in L$. The equation $(*n)$ is

$$(*n) \quad Y_1^{2n} \sum_{i=0}^n \tau a_{i0} Y_1^i = P(Y_1) \sum_{i=0}^n a_{ni} Y_1^i.$$

Since F is irreducible, we find $\sum_{i=0}^n \tau a_{i0} Y_1^i \neq 0$. Thus Y_1^{2n} divides the right side of eq. $(*n)$, which implies $Y_1^n | P(Y_1)$. On the other hand, the equation $(*0)$ is

$$(*0) \quad (qt)^n (1 - Y_1)^n \sum_{i=0}^n \tau a_{in} Y_1^i = P(Y_1) \sum_{i=0}^n a_{0i} Y_1^i.$$

From the irreducibility of F , we find that $\sum_{i=0}^n a_{0i} Y_1^i \neq 0$. Thus Y_1^n divides $\sum_{i=0}^n \tau a_{in} Y_1^i$, which implies $\sum \tau a_{in} Y_1^i = \tau a_{nn} Y_1^n$, $\tau a_{nn} \neq 0$. We also find $Y_1^n \parallel P(Y_1)$, which means $Y_1^n | P$ and $Y_1^{n+1} \nmid P$. Then the equation $(*0)$,

$$(*0) \quad (qt)^n (1 - Y_1)^n \tau a_{nn} Y_1^n = P(Y_1) \sum_{i=0}^n a_{0i} Y_1^i,$$

implies $P(Y_1) = p Y_1^n (1 - Y_1)^k$ for some $p \in L^\times$, where $k \in \mathbf{Z}$ satisfies $0 \leq k \leq n$.

In order to prove $k = n/2$, we think about all of the equations

$$(*j) \quad (qt)^{n-j} (1 - Y_1)^{n-j} Y_1^{2j} \sum_{i=0}^n \tau a_{i,n-j} Y_1^i = p Y_1^n (1 - Y_1)^k \sum_{i=0}^n a_{ji} Y_1^i.$$

(i) If $2j < n$ then we obtain $Y_1^{n-2j} \mid \sum_{i=0}^n \tau a_{i,n-j} Y_1^i$. Thus we have $\tau a_{0,n-j} = 0$ because $n - 2j > 0$. (ii) If $2j > n$ then we obtain $Y_1^{2j-n} \mid \sum_{i=0}^n a_{ji} Y_1^i$, which implies $a_{j0} = 0$.

Hence we find that $a_{0j} = a_{j0} = 0$ for all $j > n/2$. Then the equation $(*0)$ yields

$$(qt)^n(1 - Y_1)^n \tau a_{nn} = p(1 - Y_1)^k \sum_{0 \leq i \leq n/2} a_{0i} Y_1^i.$$

Comparing degrees of both sides of the equation, we obtain $k \geq n - n/2 = n/2$. On the other hand, from the equation $(*n)$, we have

$$\sum_{0 \leq i \leq n/2} \tau a_{i0} Y_1^i = p(1 - Y_1)^k a_{nn},$$

which implies $k \leq n/2$. Therefore we find $k = n/2$.

We begin to compare coefficients. From the equation $(*0)$, we have $(qt)^n(1 - Y_1)^k \tau a_{nn} = p \sum a_{0i} Y_1^i$. If we look at the coefficients of Y_1^0 and Y_1^k , we obtain

$$(1) \quad (qt)^n \tau a_{nn} = p a_{00}$$

and

$$(2) \quad (qt)^n (-1)^k \tau a_{nn} = p a_{0k}$$

respectively. Similarly from the equation $(*n)$, we obtain

$$(3) \quad \tau a_{00} = p a_{nn}$$

and

$$(4) \quad \tau a_{k0} = p (-1)^k a_{nn}.$$

The equation $(*k)$ is

$$(qt)^k (1 - Y_1)^k Y_1^n \sum_{i=0}^n \tau a_{ik} Y_1^i = p Y_1^n (1 - Y_1)^k \sum_{i=0}^n a_{ki} Y_1^i,$$

so we have $(qt)^k \sum \tau a_{ik} Y_1^i = p \sum a_{ki} Y_1^i$. Thus we obtain

$$(5) \quad (qt)^k \tau a_{0k} = p a_{k0}.$$

From $a_{nn} \neq 0$, it follows that a_{00} , a_{0k} and a_{k0} are not equal to 0.

By (1) and (2), we find $a_{0k} = (-1)^k a_{00}$. By (3) and (4), we find $\tau a_{k0} = (-1)^k \tau a_{00}$. Combining them with (5), we obtain $p = (qt)^k \tau a_{00} / a_{00}$. Hence

$$\begin{aligned} (\tau p)p &= (q^2 t)^k \frac{\tau^2 a_{00}}{\tau a_{00}} \times (qt)^k \frac{\tau a_{00}}{a_{00}} \\ &= (q^3 t^2)^k \frac{\tau^2 a_{00}}{a_{00}} \end{aligned}$$

follows. On the other hand, from (1) and (3), we have

$$\tau^2 a_{00} = (\tau p)(\tau a_{nn}) = (\tau p)p \frac{a_{00}}{(qt)^n},$$

which implies

$$(\tau p)p = (qt)^n \frac{\tau^2 a_{00}}{a_{00}}.$$

From these two equations we find $q^k = 1$. We note that $k = n/2$ is a positive integer. Therefore q is a root of unity, which is a contradiction. $\square$

References

- [1] Bialynicki-Birula, A., On Galois theory of fields with operators, Amer. J. Math., **84** (1962), 89–109.
- [2] Cohn, R. M., *Difference Algebra*, Interscience Publ., 1965.
- [3] Grammaticos, B., Ramani, A. and Papageorgiou, V., Do Integrable Mappings Have the Painlevé Property?, Phys. Rev. Lett., **67** (1991), 1825–1828.
- [4] Nishioka, K., A note on the transcendency of Painlevé's first transcendent, Nagoya Math. J., **109** (1988), 63–67.
- [5] Nishioka, S., Transcendence of Solutions of q -Painlevé Equation of Type $A_7^{(1)}$, Preprint, 2007.
- [6] Noumi, M. and Okamoto, K., Irreducibility of the Second and the Fourth Painlevé Equations, Funkcial. Ekvac., **40** (1997), 139–163.
- [7] Ramani, A., Grammaticos, B. and Hietarinta, J., Discrete Versions of the Painlevé Equations, Phys. Rev. Lett., **67** (1991), 1829–1832.
- [8] Ramani, A. and Grammaticos, B., Discrete Painlevé equations: coalescences, limits and degeneracies, Physica A, **228** (1996), 160–171.
- [9] Sakai, H., Rational surfaces associated with affine root systems and geometry of the Painlevé equations, Comm. Math. Phys., **220** (2001), 165–229.
- [10] Sakai, H., Problem: Discrete Painlevé equations and their Lax forms, RIMS Kôkyûroku Bessatsu, **B2** (2007), 195–208.
- [11] Umemura, H., On the Irreducibility of the First Differential Equation of Painlevé, Algebraic Geometry and Commutative Algebra in Honor of Masayoshi NAGATA, pp. 771–789. Kinokuniya, Tokyo, 1987.

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