On the Garnier System of Half-Integer Type in Two Variables

By

Hiroyuki KAWAMUKO

(Mie University, Japan)

Abstract. In [5], Kimura proposed degenerate Garnier systems in two variables. In this paper, we propose Hamiltonian systems which are considered as degenerate cases of Kimura's degenerate Garnier systems.

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1. Introduction

We consider a second order linear differential equation

$$(1) \quad \frac{d^2 y}{dx^2} = R(x)y$$

with regular or irregular singular points at $p_1, p_2, \dots, p_m$ and apparent singular points at $\lambda_1, \lambda_2, \dots, \lambda_g$. We denote (1) by $L(r_1, r_2, \dots, r_m)$ if (1) satisfies the following conditions.

- 1) The Poincaré rank at p_i ($i = 1, 2, \dots, m$) is r_i .
- 2) The characteristic exponents at λ_i ($i = 1, 2, \dots, g$) are $-1/2$ and $3/2$.
- 3) The number g (which is the number of apparent singular points) is given by

$$g = [r_1 + (1/2)] + [r_2 + (1/2)] + \dots + [r_m + (1/2)] - 3.$$

Here the symbol $[x]$ denotes the greatest integer not exceeding x .

If some r_i ($i = 1, 2, \dots, g$) is a half-integer, we say that $L(r_1, r_2, \dots, r_m)$ is a half-integer type equation. An equation $L(r_1, r_2, \dots, r_m)$ which is not a half-integer type is called an integer type.

For $g = 1$ and $g = 2$, the equations which satisfies 1) and 2) are given by

$$\begin{aligned} g = 1 &\Rightarrow L(4), L(2, 2), L(3, 1), L(2, 1, 1), L(1, 1, 1, 1), \\ &\quad L(7/2), L(5/2, 1), L(3/2, 1, 1), L(3/2, 2), L(3/2, 3/2), \\ g = 2 &\Rightarrow L(5), L(4, 1), L(3, 2), L(3, 1, 1), L(2, 2, 1), L(2, 1, 1, 1), \\ &\quad L(1, 1, 1, 1, 1), L(9/2), L(7/2, 1), L(5/2, 2), L(3/2, 3), \end{aligned}$$

$$L(5/2, 3/2), L(5/2, 1, 1), L(3/2, 2, 1), L(3/2, 3/2, 1), \\ L(3/2, 1, 1, 1).$$

In this paper, we consider the monodromy preserving deformations of half-integer type equations $L(\sigma)$ ($\sigma \in \{(7/2, 1), (5/2, 2), (3/2, 3), (5/2, 3/2), (5/2, 1, 1), (3/2, 2, 1), (3/2, 3/2, 1), (3/2, 1, 1, 1)\}$).

Here, we review some results on monodromy preserving deformation equation (or MPD equation for short). Let

$$S'_1 = \{(4), (2, 2), (3, 1), (2, 1, 1), (1, 1, 1, 1), (7/2)\}, \\ S''_1 = \{(5/2, 1), (3/2, 1, 1), (3/2, 2), (3/2, 3/2)\}, \\ S'_2 = \{(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1), (9/2)\}, \\ S''_2 = \{(7/2, 1), (5/2, 2), (3/2, 3), (5/2, 3/2), (5/2, 1, 1), (3/2, 2, 1), \\ (3/2, 3/2, 1), (3/2, 1, 1, 1)\}.$$

It is well known that the MPD equation of $L(\sigma)$ ($\sigma \in S'_1$) can be described by one of the Painlevé equations $P_I, P_{II}, P_{III}^{D_6}, P_{IV}, P_V, P_{VI}$. (See [1], [2], [8].) Y. Ohya and S. Okumura [6] considered half-integer type linear differential equation $L(\sigma)$ ($\sigma \in S''_1$) and they showed that the MPD equation could be described by one of the Painlevé equations $P_{II}, P_{III}^{D_6}, P_{III}^{D_7}, P_{III}^{D_8}$. H. Kimura [5] considered degeneration of $L(1, 1, 1, 1, 1)$. He showed that $L(\sigma)$ ($\sigma \in S'_2$) could be obtained from $L(1, 1, 1, 1, 1)$ by means of degeneration, and the MPD equations could be written as Hamiltonian systems

$$\frac{\partial q_i}{\partial s_j} = \frac{\partial H_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial s_j} = -\frac{\partial H_j}{\partial q_i} \quad (i, j \in \{1, 2\}),$$

whose Hamiltonians H_1, H_2 satisfy

$$(2) \quad \sum_{k=1}^2 \left(\frac{\partial H_1}{\partial p_k} \frac{\partial H_2}{\partial q_k} - \frac{\partial H_1}{\partial q_k} \frac{\partial H_2}{\partial p_k} \right) = 0.$$

The purpose of this paper is to show that the MPD equations of $L(\sigma)$ ($\sigma \in S'_2$) can be written as Hamiltonian systems

$$\frac{\partial u_i}{\partial t_j} = \frac{\partial K_j}{\partial v_i}, \quad \frac{\partial v_i}{\partial t_j} = -\frac{\partial K_j}{\partial u_i} \quad (i, j \in \{1, 2\}),$$

and give transformations $(u, v, K, t) \rightarrow (q, p, H, s)$ such that the Hamiltonians H_1, H_2 satisfy (2). (Since the differential equations $L(\sigma)$ ($\sigma \in S'_2$) can be ob-

tained by degeneration of integer type equations, these Hamiltonian systems can be considered as degenerate cases of Kimura's Hamiltonian systems.)

Remark. For convenience, we denote the MPD equation of $L(\sigma)$ by $G(\sigma)$. Looking at the Hamiltonians of $G(\sigma)$ ($\sigma \in S'_2 \cup S''_2$), we see that the MPD equations which have two parameters are $G(4, 1)$, $G(3, 2)$ and $G(5/2, 1, 1)$. For these equations, the following facts hold. (See [3] and [4].)

- 1) The equations $G(4, 1)$, $G(3, 2)$ and $G(5/2, 1, 1)$ admit a special case which can be reducible to the second Painlevé equation P_{II} .
- 2) The equation $G(3, 2)$ has no rational solution.
- 3) If parameters κ_0 and κ_∞ of $G(4, 1)$ satisfy the conditions
 - κ_0 and κ_∞ are integers,
 - $\kappa_0 \geq 0$, $\kappa_\infty \leq \kappa_0 - 1$ or $\kappa_\infty \leq -1$, $\kappa_\infty \geq \kappa_0$,
 then $G(4, 1)$ has a unique rational solution.
- 4) If parameters κ_0 and κ_∞ of $G(5/2, 1, 1)$ are integers, then $G(5/2, 1, 1)$ has two rational solutions.

These facts imply that there is no birational transformation which maps solutions of $G(5/2, 1, 1)$ to solutions of $G(\sigma)$ ($\sigma \in S'_2$). Hence $G(5/2, 1, 1)$ can be considered as another extension of the second Painlevé equation. In this sense, we expect that some of $G(\sigma)$ ($\sigma \in S''_2$) define new transcendental functions. The author was motivated to study the monodromy preserving deformations of $L(\sigma)$ ($\sigma \in S''_2$) by this fact.

This paper is organized as follows. In Section 2, we give explicit form of differential equations $L(\sigma)$ ($\sigma \in S''_2$), and derive Hamiltonian systems (u, v, K, t) by considering monodromy preserving deformation of $L(\sigma)$. In Section 3, we define transformations $(u, v, K, t) \rightarrow (q, p, H, s)$ so as to Hamiltonians H_1, H_2 satisfy (2). Moreover we show that the Hamiltonian systems (q, p, H, s) are completely integrable.

2. Monodromy preserving deformation

Let σ be an element of S''_2 . By suitable change of variables we may assume that the Riemann scheme of $L(\sigma)$ is

$$\sigma = (7/2, 1) : \left\{ \begin{array}{ccccccc} x = 0 & x = \lambda_1 & x = \lambda_2 & \overbrace{x = \infty(1/2)}^{8 \text{ points}} \\ \kappa_0 & -1/2 & -1/2 & -1/5 & 0 & -t_1/3 & 0 & -t_2 & 3/2 \\ 1 - \kappa_0 & 3/2 & 3/2 & 1/5 & 0 & t_1/3 & 0 & t_2 & 3/2 \end{array} \right\},$$

$$\sigma = (5/2, 2) : \left\{ \begin{array}{ccccccc} \overbrace{x = 0}^{2 \text{ points}} & x = \lambda_1 & x = \lambda_2 & \overbrace{x = \infty(1/2)}^{4 \text{ points}} \\ -t_2 & \kappa_\infty & -1/2 & -1/2 & -1/3 & 0 & -t_1 & 1/2 \\ t_2 & -\kappa_\infty + 2 & 3/2 & 3/2 & 1/3 & 0 & t_1 & 1/2 \end{array} \right\},$$

$$\begin{aligned}
\sigma = (3/2, 3) : & \left\{ \begin{array}{cccccc} x=0(1/2) & x=\lambda_1 & x=\lambda_2 & \overbrace{x=\infty} & & \\ \underbrace{-1 \quad 3/2} & -1/2 & -1/2 & \underbrace{-t_1/2 \quad -t_2} & \kappa_\infty & \\ 1 & 3/2 & 3/2 & t_1/2 & t_2 & -\kappa_\infty + 1 \end{array} \right\}, \\
\sigma = (5/2, 3/2) : & \left\{ \begin{array}{cccccc} x=0(1/2) & x=\lambda_1 & x=\lambda_2 & \overbrace{x=\infty(1/2)} & & \\ \underbrace{-t_2 \quad 3/2} & -1/2 & -1/2 & \underbrace{-1/3 \quad 0 \quad -t_1} & 1/2 & \\ t_2 & 3/2 & 3/2 & 1/3 & 0 & t_1 \quad 1/2 \end{array} \right\}, \\
\sigma = (5/2, 1, 1) : & \left\{ \begin{array}{cccccc} x=0 & x=1 & x=\lambda_1 & x=\lambda_2 & \overbrace{x=\infty(1/2)} & \\ \kappa_0 & \kappa_1 & -1/2 & -1/2 & \underbrace{-t_1/3 \quad 0 \quad -t_2} & 1/2 \\ 1-\kappa_0 & 1-\kappa_1 & 3/2 & 3/2 & t_1/3 & 0 \quad t_2 \quad 1/2 \end{array} \right\}, \\
\sigma = (3/2, 2, 1) : & \left\{ \begin{array}{cccccc} x=0(1/2) & x=1 & x=\lambda_1 & x=\lambda_2 & \overbrace{x=\infty} & \\ \underbrace{-t_1 \quad 3/2} & \kappa_1 & -1/2 & -1/2 & \underbrace{-t_2 \quad \kappa_\infty} & \\ t_1 & 3/2 & 1-\kappa_1 & 3/2 & 3/2 & t_2 \quad -\kappa_\infty \end{array} \right\}, \\
\sigma = (3/2, 3/2, 1) : & \left\{ \begin{array}{cccccc} x=0(1/2) & x=1 & x=\lambda_1 & x=\lambda_2 & \overbrace{x=\infty(1/2)} & \\ \underbrace{-t_2 \quad 3/2} & \kappa_1 & -1/2 & -1/2 & \underbrace{-t_1 \quad -1/2} & \\ t_2 & 3/2 & 1-\kappa_1 & 3/2 & 3/2 & t_1 \quad -1/2 \end{array} \right\},
\end{aligned}$$

where $\kappa_0, \kappa_1, \kappa_t$ and κ_∞ are complex parameters. Hence $R(x)$ of $L(\sigma)$ is given by

$$\begin{aligned}
R(x) &= \frac{1}{4}x^3 + \frac{t_1}{2}x^2 + \frac{1}{4}(t_1^2 + 2t_2)x + h_2^{01} + \frac{h_1^{01}}{x} - \frac{\kappa_0(1-\kappa_0)}{x^2} \\
&\quad + \frac{3}{4(x-\lambda_1)^2} + \frac{3}{4(x-\lambda_2)^2} - \frac{v_1}{x-\lambda_1} - \frac{v_2}{x-\lambda_2} \quad (\sigma = (7/2, 1)), \\
R(x) &= \frac{t_2^2}{x^4} + \frac{2(\kappa_\infty - 1)t_2}{x^3} + \frac{h_1^{02}}{x^2} + \frac{h_2^{02}}{x} + \frac{t_1}{2} + \frac{1}{4}x \\
&\quad + \frac{3}{4(x-\lambda_1)^2} + \frac{3}{4(x-\lambda_2)^2} - \frac{v_1}{x-\lambda_1} - \frac{v_2}{x-\lambda_2} \quad (\sigma = (5/2, 2)), \\
R(x) &= \frac{1}{4} \frac{1}{x^3} + \frac{h_1^{03}}{x^2} + \frac{h_2^{03}}{x} + (2\kappa_\infty - 1)t_1 + t_2^2 + 2t_1t_2x + t_1^2x^2 \\
&\quad + \frac{3}{4(x-\lambda_1)^2} + \frac{3}{4(x-\lambda_2)^2} - \frac{v_1}{x-\lambda_1} - \frac{v_2}{x-\lambda_2} \quad (\sigma = (3/2, 3)), \\
R(x) &= \frac{1}{4} \frac{t_2^2}{x^3} + \frac{h_1^{04}}{x^2} + \frac{h_2^{04}}{x} + \frac{t_1}{2} + \frac{1}{4}x \\
&\quad + \frac{3}{4(x-\lambda_1)^2} + \frac{3}{4(x-\lambda_2)^2} - \frac{v_1}{x-\lambda_1} - \frac{v_2}{x-\lambda_2} \quad (\sigma = (5/2, 3/2)),
\end{aligned}$$

$$\begin{aligned}
R(x) &= -\frac{\kappa_0(1-\kappa_0)}{x^2} + \frac{h_1^{05}}{x} - \frac{\kappa_1(1-\kappa_1)}{(x-1)^2} + \frac{h_2^{05}}{x-1} + \frac{t_1 t_2}{2} + \frac{t_1^2}{4} x \\
&\quad + \frac{3}{4(x-\lambda_1)^2} + \frac{3}{4(x-\lambda_2)^2} - \frac{v_1}{x-\lambda_1} - \frac{v_2}{x-\lambda_2} \quad (\sigma = (5/2, 1, 1)), \\
R(x) &= \frac{1}{4} \frac{t_1^2}{x^3} + \frac{h_1^{06}}{x^2} + \frac{h_2^{06}}{x} + t_2^2 - \frac{\kappa_1(1-\kappa_1)}{(x-1)^2} + \frac{v_1 + v_2 + 2\kappa_\infty t_2 - h_2^{06}}{x-1} \\
&\quad + \frac{3}{4(x-\lambda_1)^2} + \frac{3}{4(x-\lambda_2)^2} - \frac{v_1}{x-\lambda_1} - \frac{v_2}{x-\lambda_2} \quad (\sigma = (3/2, 2, 1)), \\
R(x) &= \frac{1}{4} \frac{t_2^2}{x^3} + \frac{h_1^{07}}{x^2} + \frac{h_2^{07}}{x} - \frac{\kappa_1(1-\kappa_1)}{(x-1)^2} + \frac{-h_2^{07} + v_1 + v_2 + t_1^2/4}{x-1} \\
&\quad + \frac{3}{4(x-\lambda_1)^2} + \frac{3}{4(x-\lambda_2)^2} - \frac{v_1}{x-\lambda_1} - \frac{v_2}{x-\lambda_2} \quad (\sigma = (3/2, 3/2, 1)), \\
R(x) &= -\frac{\kappa_0(1-\kappa_0)}{x^2} + \frac{h_1^{08}}{x} - \frac{\kappa_1(1-\kappa_1)}{(x-1)^2} + \frac{h_2^{08}}{x-1} - \frac{\kappa_t(1-\kappa_t)}{(x-t_1)^2} \\
&\quad + \frac{-h_1^{08} - h_2^{08} + v_1 + v_2 + t_2^2/4}{x-t_1} + \frac{3}{4(x-\lambda_1)^2} + \frac{3}{4(x-\lambda_2)^2} \\
&\quad - \frac{v_1}{x-\lambda_1} - \frac{v_2}{x-\lambda_2} \quad (\sigma = (3/2, 1, 1, 1)).
\end{aligned}$$

Here v_1 and v_2 are complex parameters and $h_1^{(*)}$, $h_2^{(*)}$ ($(*) \in \{01, 02, 03, 04, 05, 06, 07, 08\}$) are rational functions of λ_1 , λ_2 , v_1 , v_2 , t_1 and t_2 which are determined by the condition 2) for the equation (1) as mentioned in Introduction. One can show that $h_1^{(*)}$ and $h_2^{(*)}$ are given by

$$\begin{aligned}
h_1^{01} &= -\frac{\lambda_1 \lambda_2 v_1^2}{\lambda_1 - \lambda_2} + \frac{\lambda_1 \lambda_2 v_2^2}{\lambda_1 - \lambda_2} - \frac{\lambda_1 \lambda_2 v_1}{(\lambda_1 - \lambda_2)^2} - \frac{\lambda_1 \lambda_2 v_2}{(\lambda_1 - \lambda_2)^2} \\
&\quad + \frac{1}{4}(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) \lambda_1 \lambda_2 - \frac{\kappa_0(\kappa_0 - 1)(\lambda_1 + \lambda_2)}{\lambda_1 \lambda_2} \\
&\quad + \frac{1}{4} t_1^2 \lambda_1 \lambda_2 + \frac{1}{2} t_1 (\lambda_1 + \lambda_2) \lambda_1 \lambda_2 + \frac{1}{2} t_2 \lambda_1 \lambda_2, \\
h_2^{01} &= \frac{\lambda_1 v_1^2}{\lambda_1 - \lambda_2} - \frac{\lambda_2 v_2^2}{\lambda_1 - \lambda_2} + \frac{\lambda_2 v_1}{(\lambda_1 - \lambda_2)^2} + \frac{\lambda_1 v_2}{(\lambda_1 - \lambda_2)^2} \\
&\quad - \frac{1}{4}(\lambda_1 + \lambda_2)(\lambda_1^2 + \lambda_2^2) + \frac{\kappa_0(\kappa_0 - 1)}{\lambda_1 \lambda_2} - \frac{3}{4} \frac{1}{(\lambda_1 - \lambda_2)^2} \\
&\quad - \frac{1}{4} t_1^2 (\lambda_1 + \lambda_2) - \frac{1}{2} t_1 (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) - \frac{1}{2} t_2 (\lambda_1 + \lambda_2),
\end{aligned}$$

$$\begin{aligned}
h_1^{02} &= -\frac{\lambda_1^2 \lambda_2 v_1^2}{\lambda_1 - \lambda_2} + \frac{\lambda_1 \lambda_2^2 v_2^2}{\lambda_1 - \lambda_2} - \frac{\lambda_1 \lambda_2^2 v_1}{(\lambda_1 - \lambda_2)^2} - \frac{\lambda_1^2 \lambda_2 v_2}{(\lambda_1 - \lambda_2)^2} + \frac{1}{4} \lambda_1^2 \lambda_2 \\
&\quad + \frac{1}{4} \lambda_1 \lambda_2^2 + \frac{3}{4} \frac{\lambda_1^2}{(\lambda_1 - \lambda_2)^2} - \frac{3}{4} \frac{\lambda_1}{\lambda_1 - \lambda_2} - \frac{t_2^2 (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2)}{\lambda_1^2 \lambda_2^2} \\
&\quad + \frac{1}{2} t_1 \lambda_1 \lambda_2 - \frac{2(\kappa_\infty - 1)t_2(\lambda_1 + \lambda_2)}{\lambda_1 \lambda_2}, \\
h_2^{02} &= \frac{\lambda_1^2 v_1^2}{\lambda_1 - \lambda_2} - \frac{\lambda_2^2 v_2^2}{\lambda_1 - \lambda_2} + \frac{\lambda_2^2 v_1}{(\lambda_1 - \lambda_2)^2} + \frac{\lambda_1^2 v_2}{(\lambda_1 - \lambda_2)^2} \\
&\quad - \frac{1}{4} (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) - \frac{3}{4} \frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_2)^2} + \frac{(\lambda_1 + \lambda_2)t_2^2}{\lambda_1^2 \lambda_2^2} \\
&\quad - \frac{1}{2} t_1 (\lambda_1 + \lambda_2) + \frac{2(\kappa_\infty - 1)t_2}{\lambda_1 \lambda_2}, \\
h_1^{03} &= -\frac{\lambda_1^2 \lambda_2 v_1^2}{\lambda_1 - \lambda_2} + \frac{\lambda_1 \lambda_2^2 v_2^2}{\lambda_1 - \lambda_2} - \frac{\lambda_1 \lambda_2^2 v_1}{(\lambda_1 - \lambda_2)^2} - \frac{\lambda_1^2 \lambda_2 v_2}{(\lambda_1 - \lambda_2)^2} + \frac{3}{4} \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^2} \\
&\quad - \frac{1}{4} \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} + t_1^2 (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) \lambda_1 \lambda_2 + 2t_1 t_2 (\lambda_1 + \lambda_2) \lambda_1 \lambda_2 \\
&\quad + t_2^2 \lambda_1 \lambda_2 + (2\kappa_\infty - 1)t_1 \lambda_1 \lambda_2, \\
h_2^{03} &= \frac{\lambda_1^2 v_1^2}{\lambda_1 - \lambda_2} - \frac{\lambda_2^2 v_2^2}{\lambda_1 - \lambda_2} + \frac{\lambda_2^2 v_1}{(\lambda_1 - \lambda_2)^2} + \frac{\lambda_1^2 v_2}{(\lambda_1 - \lambda_2)^2} - \frac{3}{4} \frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_2)^2} \\
&\quad + \frac{1}{4} \frac{1}{\lambda_1 \lambda_2} - t_1^2 (\lambda_1 + \lambda_2) (\lambda_1^2 + \lambda_2^2) - 2t_1 t_2 (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) \\
&\quad - t_2^2 (\lambda_1 + \lambda_2) - (2\kappa_\infty - 1)t_1 (\lambda_1 + \lambda_2), \\
h_1^{04} &= -\frac{\lambda_1^2 \lambda_2 v_1^2}{\lambda_1 - \lambda_2} + \frac{\lambda_1 \lambda_2^2 v_2^2}{\lambda_1 - \lambda_2} - \frac{\lambda_1 \lambda_2^2 v_1}{(\lambda_1 - \lambda_2)^2} - \frac{\lambda_1^2 \lambda_2 v_2}{(\lambda_1 - \lambda_2)^2} + \frac{1}{4} \lambda_1 \lambda_2 (\lambda_1 + \lambda_2) \\
&\quad + \frac{3}{4} \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^2} - \frac{1}{4} \frac{t_2^2 (\lambda_1 + \lambda_2)}{\lambda_1 \lambda_2} + \frac{1}{2} t_1 \lambda_1 \lambda_2, \\
h_2^{04} &= \frac{\lambda_1^2 v_1^2}{\lambda_1 - \lambda_2} - \frac{\lambda_2^2 v_2^2}{\lambda_1 - \lambda_2} + \frac{\lambda_2^2 v_1}{(\lambda_1 - \lambda_2)^2} + \frac{\lambda_1^2 v_2}{(\lambda_1 - \lambda_2)^2} \\
&\quad - \frac{1}{4} (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) - \frac{3}{4} \frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_2)^2} + \frac{1}{4} \frac{t_2^2}{\lambda_1 \lambda_2} - \frac{1}{2} t_1 (\lambda_1 + \lambda_2),
\end{aligned}$$

$$\begin{aligned}
h_1^{05} &= \frac{(\lambda_1 - 1)\lambda_1^2\lambda_2v_1^2}{\lambda_1 - \lambda_2} - \frac{(\lambda_2 - 1)\lambda_1\lambda_2^2v_2^2}{\lambda_1 - \lambda_2} - \frac{(\lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2)\lambda_1\lambda_2v_1}{(\lambda_1 - \lambda_2)^2} \\
&\quad + \frac{(2\lambda_1\lambda_2 - \lambda_2^2 - \lambda_1)\lambda_1\lambda_2v_2}{(\lambda_1 - \lambda_2)^2} - \frac{3}{4}\lambda_1 - \kappa_1(1 - \kappa_1) \\
&\quad + \frac{3}{4} \frac{\lambda_1(\lambda_1^2 - 3\lambda_1\lambda_2 + \lambda_2)}{(\lambda_1 - \lambda_2)^2} - \frac{\kappa_1(1 - \kappa_1)(\lambda_1 + \lambda_2 - 1)}{(\lambda_1 - 1)(\lambda_2 - 1)} \\
&\quad + \frac{1}{4} \frac{t_1^2(\lambda_1\lambda_2 - \lambda_1 - \lambda_2)}{\lambda_1\lambda_2} - t_2^2(\lambda_1 + \lambda_2 - 1)\lambda_1\lambda_2 - 2\kappa_\infty t_2\lambda_1\lambda_2, \\
h_2^{05} &= \frac{(\lambda_1 - 1)(\lambda_2 - 1)\lambda_1^2v_1^2}{\lambda_1 - \lambda_2} - \frac{(\lambda_1 - 1)(\lambda_2 - 1)\lambda_2^2v_2^2}{\lambda_1 - \lambda_2} \\
&\quad - \frac{(\lambda_1^3\lambda_2 - 2\lambda_1^2\lambda_2^2 - \lambda_1^3 + \lambda_1^2\lambda_2 + 2\lambda_1\lambda_2^2 - \lambda_2^2)v_1}{(\lambda_1 - \lambda_2)^2} \\
&\quad + \frac{(2\lambda_1^2\lambda_2^2 - \lambda_1\lambda_2^3 - 2\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2 + \lambda_2^3 + \lambda_1^2)v_2}{(\lambda_1 - \lambda_2)^2} - \frac{3}{4}\lambda_1 \\
&\quad - 2\kappa_1(1 - \kappa_1) + \frac{3}{4} + \frac{3}{4} \frac{(\lambda_1 - 1)(\lambda_1^2 - 3\lambda_1\lambda_2 + \lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)^2} \\
&\quad - \frac{\kappa_1(1 - \kappa_1)(\lambda_1 + \lambda_2 - 2)}{(\lambda_1 - 1)(\lambda_2 - 1)} + \frac{1}{4} \frac{t_1^2(\lambda_1 - 1)(\lambda_2 - 1)}{\lambda_1\lambda_2} \\
&\quad - t_2^2(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 + \lambda_2) - 2\kappa_\infty t_2(\lambda_1\lambda_2 - \lambda_1 - \lambda_2), \\
h_1^{06} &= \frac{(\lambda_1 - 1)\lambda_1\lambda_2v_1^2}{\lambda_1 - \lambda_2} - \frac{(\lambda_2 - 1)\lambda_1\lambda_2v_2^2}{\lambda_1 - \lambda_2} + \frac{(\lambda_2 - 1)\lambda_1\lambda_2v_1}{(\lambda_1 - \lambda_2)^2} \\
&\quad + \frac{(\lambda_1 - 1)\lambda_1\lambda_2v_2}{(\lambda_1 - \lambda_2)^2} + \kappa_0^2 + \kappa_1^2 - \kappa_0 - \kappa_1 - \frac{3}{4} \frac{\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)^2} \\
&\quad + \frac{\kappa_0(1 - \kappa_0)(\lambda_1 + \lambda_2)}{\lambda_1\lambda_2} - \frac{\kappa_1(1 - \kappa_1)(\lambda_1 + \lambda_2 - 1)}{(\lambda_1 - 1)(\lambda_2 - 1)} \\
&\quad - \frac{1}{4} t_1^2(\lambda_1 + \lambda_2 - 1)\lambda_1\lambda_2 - \frac{1}{2} t_1 t_2 \lambda_1 \lambda_2,
\end{aligned}$$

$$\begin{aligned}
h_2^{06} = & -\frac{(\lambda_1 - 1)(\lambda_2 - 1)\lambda_1 v_1^2}{\lambda_1 - \lambda_2} + \frac{(\lambda_1 - 1)(\lambda_2 - 1)\lambda_2 v_2^2}{\lambda_1 - \lambda_2} \\
& - \frac{(\lambda_1 - 1)(\lambda_2 - 1)\lambda_2 v_1}{(\lambda_1 - \lambda_2)^2} - \frac{(\lambda_1 - 1)(\lambda_2 - 1)\lambda_1 v_2}{(\lambda_1 - \lambda_2)^2} \\
& - (\kappa_0^2 + \kappa_1^2 - \kappa_0 - \kappa_1) - \frac{\kappa_0(1 - \kappa_0)(\lambda_1 + \lambda_2 - 1)}{\lambda_1 \lambda_2} \\
& + \frac{3}{4} \frac{(\lambda_1 - 1)(\lambda_2 - 1)}{(\lambda_1 - \lambda_2)^2} + \frac{\kappa_1(1 - \kappa_1)(\lambda_1 + \lambda_2 - 2)}{(\lambda_1 - 1)(\lambda_2 - 1)} \\
& + \frac{1}{4} t_1^2 (\lambda_2 - 1)(\lambda_1 - 1)(\lambda_1 + \lambda_2) + \frac{1}{2} t_1 t_2 (\lambda_1 - 1)(\lambda_2 - 1), \\
h_1^{07} = & \frac{(\lambda_1 - 1)\lambda_1^2 \lambda_2 v_1^2}{\lambda_1 - \lambda_2} - \frac{\lambda_1(\lambda_2 - 1)\lambda_2^2 v_2^2}{\lambda_1 - \lambda_2} - \frac{(\lambda_1^2 - 2\lambda_1 \lambda_2 + \lambda_2)\lambda_1 \lambda_2 v_1}{(\lambda_1 - \lambda_2)^2} \\
& + \frac{(2\lambda_1 \lambda_2 - \lambda_2^2 - \lambda_1)\lambda_1 \lambda_2 v_2}{(\lambda_1 - \lambda_2)^2} - \frac{1}{4} \{3\lambda_1 + 4\kappa_1(1 - \kappa_1)\} \\
& + \frac{3}{4} \frac{\lambda_1(\lambda_1^2 - 3\lambda_1 \lambda_2 + \lambda_2)}{(\lambda_1 - \lambda_2)^2} - \frac{\kappa_1(1 - \kappa_1)(\lambda_1 + \lambda_2 - 1)}{(\lambda_1 - 1)(\lambda_2 - 1)} \\
& - \frac{1}{4} t_1^2 \lambda_1 \lambda_2 + \frac{1}{4} \frac{t_2^2(\lambda_1 \lambda_2 - \lambda_1 - \lambda_2)}{\lambda_1 \lambda_2}, \\
h_2^{07} = & \frac{(\lambda_1 - 1)(\lambda_2 - 1)\lambda_1^2 v_1^2}{\lambda_1 - \lambda_2} - \frac{(\lambda_1 - 1)(\lambda_2 - 1)\lambda_2^2 v_2^2}{\lambda_1 - \lambda_2} \\
& - \frac{(\lambda_1^3 \lambda_2 - 2\lambda_1^2 \lambda_2^2 - \lambda_1^3 + \lambda_1^2 \lambda_2 + 2\lambda_1 \lambda_2^2 - \lambda_2^2)v_1}{(\lambda_1 - \lambda_2)^2} \\
& + \frac{(2\lambda_1^2 \lambda_2^2 - \lambda_1 \lambda_2^3 - 2\lambda_1^2 \lambda_2 - \lambda_1 \lambda_2^2 + \lambda_2^3 + \lambda_1^2)v_2}{(\lambda_1 - \lambda_2)^2} \\
& - \frac{1}{4} (3\lambda_1 - 8\kappa_1^2 + 8\kappa_1 - 3) \\
& + \frac{3}{4} \frac{(\lambda_1 - 1)(\lambda_1^2 - 3\lambda_1 \lambda_2 + \lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)^2} - \frac{\kappa_1(1 - \kappa_1)(\lambda_1 + \lambda_2 - 2)}{(\lambda_1 - 1)(\lambda_2 - 1)} \\
& - \frac{1}{4} t_1^2 (\lambda_1 \lambda_2 - \lambda_1 - \lambda_2) + \frac{1}{4} \frac{t_2^2(\lambda_2 - 1)(\lambda_1 - 1)}{\lambda_1 \lambda_2},
\end{aligned}$$

$$\begin{aligned}
t_1 h_1^{08} = & -\frac{\lambda_1(\lambda_1-1)(\lambda_1-t_1)\lambda_2 v_1^2}{\lambda_1-\lambda_2} + \frac{\lambda_1\lambda_2(\lambda_2-1)(\lambda_2-t_1)v_2^2}{\lambda_1-\lambda_2} \\
& + \frac{\{\lambda_1^2-2\lambda_1\lambda_2+(t_1+1)\lambda_2-t_1\}\lambda_1\lambda_2 v_1}{(\lambda_1-\lambda_2)^2} \\
& - \frac{\{2\lambda_1\lambda_2-\lambda_2^2-(t_1+1)\lambda_1+t_1\}\lambda_1\lambda_2 v_2}{(\lambda_1-\lambda_2)^2} \\
& + \frac{1}{4}t_2^2\lambda_1\lambda_2 + \frac{3}{4}\lambda_1 + (\kappa_0^2 + \kappa_1^2 - \kappa_t^2 - \kappa_0 - \kappa_1 + \kappa_t)t_1 \\
& + \kappa_0^2 - \kappa_1^2 + \kappa_t^2 - \kappa_0 + \kappa_1 - \kappa_t - \frac{3}{4}\frac{\lambda_1\{\lambda_1^2-3\lambda_1\lambda_2+(t_1+1)\lambda_2\}}{(\lambda_1-\lambda_2)^2} \\
& + \frac{\kappa_0(1-\kappa_0)t_1(\lambda_1+\lambda_2)}{\lambda_1\lambda_2} - \frac{\kappa_1(1-\kappa_1)(t_1-1)(\lambda_1+\lambda_2-1)}{(\lambda_1-1)(\lambda_2-1)} \\
& + \frac{\kappa_t(1-\kappa_t)(t_1-1)t_1(\lambda_1+\lambda_2-t_1)}{(\lambda_1-t_1)(\lambda_2-t_1)}, \\
h_2^{08} = & \frac{\lambda_1(\lambda_1-1)(\lambda_1-t_1)(\lambda_2-1)v_1^2}{(t_1-1)(\lambda_1-\lambda_2)} - \frac{(\lambda_1-1)\lambda_2(\lambda_2-1)(\lambda_2-t_1)v_2^2}{(t_1-1)(\lambda_1-\lambda_2)} \\
& - \frac{(\lambda_1-1)(\lambda_2-1)(\lambda_1^2-2\lambda_1\lambda_2+t_1\lambda_2)v_1}{(t_1-1)(\lambda_1-\lambda_2)^2} \\
& + \frac{(\lambda_1-1)(\lambda_2-1)(2\lambda_1\lambda_2-\lambda_2^2-t_1\lambda_1)v_2}{(t_1-1)(\lambda_1-\lambda_2)^2} + \frac{3}{4}\lambda_1 + 2\kappa_1(1-\kappa_1) - \frac{3}{4} \\
& - \frac{1}{4}\frac{1}{t_1-1}\left[t_2^2(\lambda_1\lambda_2-\lambda_1-\lambda_2) + 3t_1\lambda_2 + t_2^2\right. \\
& \quad \left.+ 4\left(\kappa_0^2 - \kappa_1^2 - \kappa_t^2 - \kappa_0 + \kappa_1 + \kappa_t - \frac{3}{4}\right)t_1\right] \\
& - \frac{3}{4}\frac{(\lambda_1-1)(\lambda_1^2-3\lambda_1\lambda_2+\lambda_1+\lambda_2)}{(\lambda_1-\lambda_2)^2} \\
& - \frac{3}{4}\frac{t_1(\lambda_2-1)(3\lambda_1\lambda_2-\lambda_2^2-2\lambda_1-\lambda_2+1)}{(t_1-1)(\lambda_1-\lambda_2)^2} \\
& - \frac{\kappa_0(1-\kappa_0)t_1(\lambda_1+\lambda_2-1)}{(t_1-1)\lambda_1\lambda_2} + \frac{\kappa_1(1-\kappa_1)(\lambda_1+\lambda_2-2)}{(\lambda_1-1)(\lambda_2-1)} \\
& - \frac{\kappa_t(1-\kappa_t)t_1(\lambda_1+\lambda_2-t_1-1)}{(\lambda_1-t_1)(\lambda_2-t_1)}.
\end{aligned}$$

Theorem 1. Suppose that $\lambda_1, \lambda_2, v_1$ and v_2 are functions in t_1 and t_2 . Then the equation $L(\sigma)$ ($\sigma \in S_2''$) has a fundamental system of solutions whose monodromy matrices and Stokes multipliers are independent of t_1 and t_2 , if and only if the following system is completely integrable:

$$(3) \quad \frac{\partial^3 A_i}{\partial x^3} - 4R \frac{\partial A_i}{\partial x} - 2A_i \frac{\partial R}{\partial x} + 2 \frac{\partial R}{\partial t_i} = 0 \quad (i = 1, 2),$$

where $A_1(x, t)$ and $A_2(x, t)$ are given by

$$\sigma = (7/2, 1) \Rightarrow$$

$$A_1 = \frac{2x(x - t_1 - \lambda_1 - \lambda_2)}{3(x - \lambda_1)(x - \lambda_2)}, \quad A_2 = \frac{2x}{(x - \lambda_1)(x - \lambda_2)},$$

$$\sigma = (5/2, 2) \Rightarrow$$

$$A_1 = \frac{2x^2}{(x - \lambda_1)(x - \lambda_2)}, \quad A_2 = -\frac{\lambda_1 \lambda_2 x}{t_2(x - \lambda_1)(x - \lambda_2)},$$

$$\sigma = (3/2, 3) \Rightarrow$$

$$A_1 = \frac{1}{2} \frac{x^2 \{t_1 x - t_1(\lambda_1 + \lambda_2) - t_2\}}{t_1^2(x - \lambda_1)(x - \lambda_2)}, \quad A_2 = \frac{x^2}{t_1(x - \lambda_1)(x - \lambda_2)},$$

$$\sigma = (5/2, 3/2) \Rightarrow$$

$$A_1 = \frac{2x^2}{(x - \lambda_1)(x - \lambda_2)}, \quad A_2 = -\frac{2\lambda_1 \lambda_2 x}{t_2(x - \lambda_1)(x - \lambda_2)},$$

$$\sigma = (5/2, 1, 1) \Rightarrow$$

$$A_1 = \frac{2}{3} \frac{x(x-1) \{t_1 x - t_1(\lambda_1 + \lambda_2) + t_1 - t_2\}}{t_1^2(x - \lambda_1)(x - \lambda_2)},$$

$$A_2 = \frac{2x(x-1)}{t_1(x - \lambda_1)(x - \lambda_2)},$$

$$\sigma = (3/2, 2, 1) \Rightarrow$$

$$A_1 = \frac{2\lambda_1 \lambda_2 x(x-1)}{t_1(x - \lambda_1)(x - \lambda_2)}, \quad A_2 = \frac{x^2(x-1)}{t_2(x - \lambda_1)(x - \lambda_2)},$$

$$\sigma = (3/2, 3/2, 1) \Rightarrow$$

$$A_1 = \frac{2x^2(x-1)}{t_1(x - \lambda_1)(x - \lambda_2)}, \quad A_2 = \frac{2\lambda_1 \lambda_2 x(x-1)}{t_2(x - \lambda_1)(x - \lambda_2)},$$

$$\sigma = (3/2, 1, 1, 1) \Rightarrow$$

$$A_1 = -\frac{(\lambda_1\lambda_2 - t_1\lambda_1 - t_1\lambda_2 + t_1^2)x(x-1)}{t_1(t_1-1)(x-\lambda_1)(x-\lambda_2)}, \quad A_2 = \frac{2x(x-1)(x-t_1)}{t_2(x-\lambda_1)(x-\lambda_2)}.$$

Theorem 2. *The functions $\lambda_1(t_1, t_2)$, $\lambda_2(t_1, t_2)$, $v_1(t_1, t_2)$ and $v_2(t_1, t_2)$ satisfy (3) if and only if*

$$\frac{\partial u_i}{\partial t_j} = \frac{\partial K_j}{\partial v_i}, \quad \frac{\partial v_i}{\partial t_j} = -\frac{\partial K_j}{\partial u_i},$$

where $u_1, u_2, v_1, v_2, K_1, K_2$ are given by

$$u_1 = \lambda_1 + \lambda_2, \quad u_2 = \lambda_1\lambda_2,$$

$$v_1 = \frac{1}{2} \frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_2)^2} + \frac{\lambda_1 v_1 - \lambda_2 v_2}{\lambda_1 - \lambda_2}, \quad v_2 = -\frac{1}{(\lambda_1 - \lambda_2)^2} - \frac{v_1 - v_2}{\lambda_1 - \lambda_2},$$

$$\sigma = (7/2, 1) \Rightarrow$$

$$\begin{aligned} K_1 = & -\frac{2}{3}t_1v_1^2 + \frac{4}{3}u_2v_1v_2 + \frac{2}{3}(u_1 + t_1)u_2v_2^2 - \frac{4}{3}v_1 - \frac{2}{3}(u_1 + t_1)v_2 \\ & + \frac{1}{6}t_1u_1^3 + \frac{1}{6}u_1^2u_2 + \frac{1}{3}t_1^2u_1^2 - \frac{1}{6}u_2^2 + \frac{1}{6}t_1^3u_1 + \frac{1}{3}t_1t_2u_1 - \frac{1}{6}t_1^2u_2 \\ & + \frac{1}{3}t_2u_2 - \frac{2}{3} \frac{\kappa_0(\kappa_0 - 1)(u_1 + t_1)}{u_2}, \end{aligned}$$

$$\begin{aligned} K_2 = & 2v_1^2 - 2u_2v_2^2 + 2v_2 - \frac{1}{2}u_1^3 - t_1u_1^2 + u_1u_2 - \frac{1}{2}t_1^2u_1 - t_2u_1 + t_1u_2 \\ & + \frac{2\kappa_0(\kappa_0 - 1)}{u_2}, \end{aligned}$$

$$\sigma = (5/2, 2) \Rightarrow$$

$$\begin{aligned} K_1 = & 2u_1v_1^2 + 4u_2v_1v_2 - 4v_1 - \frac{1}{2}u_1^2 - t_1u_1 + \frac{1}{2}u_2 + \frac{2t_2^2u_1}{u_2^2} \\ & + \frac{4(\kappa_\infty - 1)t_2}{u_2}, \end{aligned}$$

$$\begin{aligned} t_2K_2 = & -u_2v_1^2 + u_2^2v_2^2 - u_2v_2 + \frac{1}{4}(u_1 + 2t_1)u_2 - \frac{t_2^2u_1^2}{u_2^2} \\ & - \frac{t_2\{2(\kappa_\infty - 1)u_1 - t_2\}}{u_2}, \end{aligned}$$

$$\sigma = (3/2, 3) \Rightarrow$$

$$\begin{aligned} t_1^2 K_1 = & -\frac{1}{2}(t_2 u_1 + t_1 u_2) v_1^2 - t_2 u_2 v_1 v_2 + \frac{1}{2} t_1 u_2^2 v_2^2 - \frac{1}{2} (t_1 u_1 - 2t_2) v_1 \\ & - \frac{3}{2} t_1 u_2 v_2 + \frac{1}{2} t_1^2 u_1^2 (t_2 u_1 + t_1 u_2) + t_1 t_2^2 u_1^2 - \frac{1}{2} t_1^3 u_2^2 \\ & + \frac{1}{2} \{ (2\kappa_\infty - 1) t_1 + t_2^2 \} t_2 u_1 + \frac{1}{2} t_1 \{ (2\kappa_\infty - 1) t_1 - t_2^2 \} u_2 \\ & - \frac{1}{8} \frac{t_1 u_1 + t_2}{u_2}, \end{aligned}$$

$$\begin{aligned} t_1 K_2 = & u_1 v_1^2 + 2u_2 v_1 v_2 - 2v_1 - t_1^2 u_1^3 - 2t_1 t_2 u_1^2 + 2t_1^2 u_1 u_2 \\ & - \{ (2\kappa_\infty - 1) t_1 + t_2^2 \} u_1 + 2t_1 t_2 u_2 + \frac{1}{4} \frac{1}{u_2}, \end{aligned}$$

$$\sigma = (5/2, 3/2) \Rightarrow$$

$$\begin{aligned} K_1 = & 2u_1 v_1^2 + 4u_2 v_1 v_2 - 4v_1 - \frac{1}{2} u_1^2 - t_1 u_1 + \frac{1}{2} u_2 + \frac{1}{2} \frac{t_2^2}{u_2}, \\ t_2 K_2 = & -2u_2 v_1^2 + 2u_2^2 v_2^2 - 2v_2 u_2 + \frac{1}{2} u_1 u_2 + t_1 u_2 - \frac{1}{2} \frac{t_2^2 u_1}{u_2}, \end{aligned}$$

$$\sigma = (5/2, 1, 1) \Rightarrow$$

$$\begin{aligned} t_1^2 K_1 = & \frac{2}{3} (t_1 u_1 - t_2 u_1 - t_1 u_2 - t_1 + t_2) v_1^2 - \frac{4}{3} t_2 u_2 v_1 v_2 \\ & - \frac{2}{3} (t_1 u_1 - t_1 u_2 - t_1 + t_2) u_2 v_2^2 - \frac{2}{3} (t_1 u_1 - 2t_2) v_1 \\ & + \frac{2}{3} (t_1 u_1 - 3t_1 u_2 - t_1 + t_2) v_2 - \frac{1}{6} t_1^3 (u_1 - u_2 - 1) u_1 \\ & + \frac{1}{6} t_1^2 t_2 (u_1^2 - 3u_1 + u_2) + \frac{1}{3} t_1 t_2^2 u_1 \\ & - \frac{2}{3} t_1 \left\{ \frac{\kappa_0(1 - \kappa_0)(u_1 - 1)}{u_2} + \frac{\kappa_1(1 - \kappa_1)(u_1 - 2)}{u_1 - u_2 - 1} - \kappa_1(1 - \kappa_1) \right\} \\ & - \frac{2}{3} t_2 \left\{ \frac{\kappa_0(1 - \kappa_0)}{u_2} + \frac{\kappa_1(1 - \kappa_1)}{u_1 - u_2 - 1} \right\}, \\ t_1 K_2 = & 2(u_1 - 1) v_1^2 + 4u_2 v_1 v_2 + 2u_2 v_2^2 - 4v_1 - 2v_2 + \frac{2\kappa_0(1 - \kappa_0)}{u_2} \\ & + \frac{2\kappa_1(1 - \kappa_1)}{u_1 - u_2 - 1} - \frac{1}{2} t_1^2 (u_1^2 - u_1 - u_2) - t_1 t_2 u_1, \end{aligned}$$

$$\sigma = (3/2, 2, 1) \Rightarrow$$

$$\begin{aligned} t_1 K_1 &= 2(u_1 - 1)u_2 v_1^2 + 4u_2^2 v_1 v_2 + 2u_2^2 v_2^2 - 4u_2 v_1 - 2u_2 v_2 \\ &\quad + \frac{2\kappa_1(1 - \kappa_1)u_2}{u_1 - u_2 - 1} - \frac{1}{2} \frac{t_1^2 u_1}{u_2} - 2t_2^2(u_1 - 1)u_2 - 4\kappa_\infty t_2 u_2, \\ t_2 K_2 &= (u_1^2 - u_1 - u_2)v_1^2 + 2u_2(u_1 - 1)v_1 v_2 + u_2^2 v_2^2 - (3u_1 - 2)v_1 \\ &\quad - 3u_2 v_2 + \frac{\kappa_1(1 - \kappa_1)}{u_1 - u_2 - 1} - \frac{1}{4} \frac{t_1^2}{u_2} - t_2^2(u_1^2 - u_1 - u_2) - 2\kappa_\infty t_2 u_1, \end{aligned}$$

$$\sigma = (3/2, 3/2, 1) \Rightarrow$$

$$\begin{aligned} t_1 K_1 &= (2u_1^2 - 2u_1 - 2u_2)v_1^2 + 4(u_1 - 1)u_2 v_1 v_2 + 2u_2^2 v_2^2 - 2(3u_1 - 2)v_1 \\ &\quad - 6u_2 v_2 + \frac{2\kappa_1(1 - \kappa_1)}{u_1 - u_2 - 1} - \frac{1}{2} t_1^2 u_1 - \frac{1}{2} \frac{t_2^2}{u_2}, \\ t_2 K_2 &= 2(u_1 - 1)u_2 v_1^2 + 4u_2^2 v_1 v_2 + 2u_2^2 v_2^2 - 4u_2 v_1 - 2u_2 v_2 \\ &\quad + \frac{2\kappa_1(1 - \kappa_1)u_2}{u_1 - u_2 - 1} - \frac{1}{2} t_1^2 u_2 - \frac{1}{2} \frac{t_2^2 u_1}{u_2}, \end{aligned}$$

$$\sigma = (3/2, 1, 1, 1) \Rightarrow$$

$$\begin{aligned} t_1(t_1 - 1)K_1 &= (t_1 u_1 - u_2 - t_1^2)(u_1 - 1)v_1^2 + 2(t_1 u_1 - u_2 - t_1^2)u_2 v_1 v_2 \\ &\quad + (t_1 u_1 - u_2 - t_1^2)u_2 v_2^2 - (2t_1 u_1 - 2u_2 - 2t_1^2)v_1 \\ &\quad - (t_1 u_1 - u_2 - t_1^2)v_2 - \frac{1}{4} t_2^2(t_1 u_1 - u_2) \\ &\quad + (\kappa_1 - \kappa_t)(\kappa_1 + \kappa_t - 1)(t_1 - 1) + \frac{\kappa_0(1 - \kappa_0)t_1(u_1 - t_1)}{u_2} \\ &\quad + \frac{\kappa_1(1 - \kappa_1)(t_1 - 1)(u_1 - t_1 - 1)}{u_1 - u_2 - 1} \\ &\quad - \frac{\kappa_t(1 - \kappa_t)t_1(t_1 - 1)(u_1 - 2t_1)}{t_1 u_1 - u_2 - t_1^2}, \\ t_2 K_2 &= \{2u_1^2 - 2(t_1 + 1)u_1 - 2u_2 + 2t_1\}v_1^2 + 4(u_1 - t_1 - 1)u_2 v_1 v_2 \\ &\quad + 2(u_2 - t_1)u_2 v_2^2 - 2(3u_1 - 2t_1 - 2)v_1 - 2(3u_2 - t_1)v_2 \\ &\quad - \frac{1}{2} t_2^2 u_1 - \frac{2\kappa_0(1 - \kappa_0)t_1}{u_2} - \frac{2\kappa_1(1 - \kappa_1)(t_1 - 1)}{u_1 - u_2 - 1} \\ &\quad + \frac{2\kappa_t(1 - \kappa_t)t_1(t_1 - 1)}{t_1 u_1 - u_2 - t_1^2}. \end{aligned}$$

Proof of Theorem 1 and 2. These Theorems are proved in a similar way to [8]. $\square$

3. Canonical transformation

Let φ_σ be a function of u_1, u_2, v_1, v_2, t_1 and t_2 . Then a change of variables

$$\begin{aligned}\bar{u}_1 &= u_1, & \bar{u}_2 &= u_2, & \bar{v}_1 &= v_1 - \frac{\partial \varphi_\sigma}{\partial u_1}, & \bar{v}_2 &= v_2 - \frac{\partial \varphi_\sigma}{\partial u_2}, \\ \bar{K}_1 &= K_1 + \frac{\partial \varphi_\sigma}{\partial t_1}, & \bar{K}_2 &= K_2 + \frac{\partial \varphi_\sigma}{\partial t_2}\end{aligned}$$

defines a canonical transformation $(u, v, K, t) \rightarrow (\bar{u}, \bar{v}, \bar{K}, t)$. If we put

$$\varphi_\sigma = \kappa_0 \log(u_2) \quad (\sigma = (7/2, 1)), \quad \varphi_\sigma = (1/2) \log(u_2) \quad (\sigma = (5/2, 2)),$$

$$\varphi_\sigma = \log(u_2) \quad (\sigma = (3/2, 3), (5/2, 3/2)),$$

$$\varphi_\sigma = (-\kappa_1 + 1) \log(u_2) + \kappa_1 \log(u_1 - u_2 - 1) \quad (\sigma = (3/2, 2, 1)),$$

$$\varphi_\sigma = \kappa_0 \log(u_2) + \kappa_1 \log(u_1 - u_2 - 1) \quad (\sigma = (5/2, 1, 1)),$$

$$\varphi_\sigma = (-\kappa_1 + (3/2)) \log(u_2) + \kappa_1 \log(u_1 - u_2 - 1) \quad (\sigma = (3/2, 3/2, 1)),$$

$$\varphi_\sigma = \kappa_0 \log(u_2) + \kappa_1 \log(u_1 - u_2 - 1) + \kappa_t \log(t_1 u_1 - u_2 - t_1^2) \quad (\sigma = (3/2, 1, 1, 1)),$$

then we can see that the number of terms of $\bar{K}_i$ is less than the number of terms of K_i . Especially, if $\sigma \in \{(7/2, 1), (5/2, 1, 1), (3/2, 1, 1, 1)\}$, we find that $\bar{K}_1$ and $\bar{K}_2$ are polynomials in $\bar{u}_1, \bar{u}_2, \bar{v}_1$ and $\bar{v}_2$. Moreover the following Theorem holds.

Theorem 3. *Let $(\bar{u}, \bar{v}, \bar{K}, t) \rightarrow (q, p, \bar{H}, s)$ be a canonical transformation defined by*

$$\sigma = (7/2, 1) \Rightarrow \bar{u}_1 = q_1 - 2s_1, \bar{u}_2 = q_2,$$

$$\bar{v}_1 = p_1, \bar{v}_2 = p_2, t_1 = 3s_1, t_2 = s_2,$$

$$\bar{K}_1 = (1/3)(\bar{H}_1 - 2p_1), \bar{K}_2 = \bar{H}_2,$$

$$\sigma = (5/2, 2) \Rightarrow \bar{u}_1 = q_1 - 2s_1, \bar{u}_2 = q_2,$$

$$\bar{v}_1 = p_1, \bar{v}_2 = p_2, t_1 = s_1, t_2 = s_2,$$

$$\bar{K}_1 = \bar{H}_1 - 2p_1, \bar{K}_2 = \bar{H}_2,$$

$$\sigma = (3/2, 3) \Rightarrow \bar{u}_1 = (s_1 q_1 + s_1 + s_2)/s_1^2, \bar{u}_2 = q_2,$$

$$\bar{v}_1 = s_1 p_1, \bar{v}_2 = p_2, t_1 = s_1^2, t_2 = s_2,$$

$$\bar{K}_1 = \{s_1^2 \bar{H}_1 - (s_1 q_1 + s_1 + 2s_2)p_1\}/(2s_1^3),$$

$$\bar{K}_2 = \bar{H}_2 + (p_1/s_1),$$

$$\sigma = (5/2, 3/2) \Rightarrow \bar{u}_1 = q_1 - 2s_1, \bar{u}_2 = q_2,$$

$$\bar{v}_1 = p_1, \bar{v}_2 = p_2, t_1 = s_1, t_2 = s_2,$$

$$\bar{K}_1 = \bar{H}_1 - 2p_1, \bar{K}_2 = \bar{H}_2,$$

$$\sigma = (3/2, 2, 1) \Rightarrow \bar{u}_1 = q_1 + s_1^2 q_2, \bar{u}_2 = s_1^2 q_2,$$

$$\bar{v}_1 = p_1, \bar{v}_2 = -p_1 + (p_2/s_1^2)t_1 = s_1, t_2 = s_2,$$

$$\bar{K}_1 = \bar{H}_1 + (2q_2 p_2/s_1), \bar{K}_2 = \bar{H}_2,$$

$$\sigma = (5/2, 1, 1) \Rightarrow \bar{u}_1 = (q_1 + s_1^2 + 1)/s_1^2, \bar{u}_2 = (q_1 + q_2 + 1)/s_1^2,$$

$$\bar{v}_1 = s_1^2(p_1 - p_2), \bar{v}_2 = s_1^2 p_2, t_1 = s_1^3, t_2 = s_2,$$

$$\bar{K}_1 = [s_1 \bar{H}_1 - 2\{(q_1 + 1)p_1 + q_2 p_2\}]/(3s_1^3),$$

$$\bar{K}_2 = \bar{H}_2,$$

$$\sigma = (3/2, 3/2, 1) \Rightarrow \bar{u}_1 = q_1/s_1^2, \bar{u}_2 = (s_2/s_1)^2 q_2,$$

$$\bar{v}_1 = s_1^2 p_1, \bar{v}_2 = (s_1/s_2)^2 p_2, t_1 = s_1, t_2 = s_2,$$

$$\bar{K}_1 = \bar{H}_1 - 2(q_1 p_1 + q_2 p_2)/s_1,$$

$$\bar{K}_2 = \bar{H}_2 + (2q_2 p_2/s_2),$$

$$\sigma = (3/2, 1, 1, 1) \Rightarrow \bar{u}_1 = q_1, \bar{u}_2 = q_2,$$

$$\bar{v}_1 = p_1, \bar{v}_2 = p_2, t_1 = s_1, t_2 = s_2,$$

$$\bar{K}_1 = \bar{H}_1, \bar{K}_2 = \bar{H}_2,$$

then Hamiltonians $\bar{H}_1, \bar{H}_2$ satisfy

$$(4) \quad \sum_{k=1}^2 \left(\frac{\partial \bar{H}_1}{\partial p_k} \frac{\partial \bar{H}_2}{\partial q_k} - \frac{\partial \bar{H}_1}{\partial q_k} \frac{\partial \bar{H}_2}{\partial p_k} \right) = 0.$$

Proof. This Theorem can be checked by a direct calculation. $\square$

One can see that the Hamiltonians $\bar{H}_1, \bar{H}_2$ of Theorem 3 are given by

$$\sigma = (7/2, 1) \Rightarrow$$

$$\begin{aligned} \bar{H}_1 = & -6s_1p_1^2 + 4q_2p_1p_2 + 2(q_1 + s_1)q_2p_2^2 + 4\kappa'_0p_1 + 4\kappa'_0(q_1 + s_1)p_2 \\ & + \frac{3}{2}s_1q_1^3 + \frac{1}{2}q_1^2q_2 - 2s_1q_1q_2 - \frac{1}{2}q_2^2 - \frac{3}{2}s_1(3s_1^2 - 2s_2)q_1 \\ & - \frac{1}{2}(5s_1^2 - 2s_2)q_2 - 3s_1^4 - 6s_1^2s_2, \end{aligned}$$

$$\begin{aligned} \bar{H}_2 = & 2p_1^2 - 2q_2p_2^2 - 4\kappa'_0p_2 - \frac{1}{2}q_1^3 + q_1q_2 + \frac{1}{2}(3s_1^2 - 2s_2)q_1 + s_1q_2 \\ & + s_1^3 + 2s_1s_2, \end{aligned}$$

$$\sigma = (5/2, 2) \Rightarrow$$

$$\begin{aligned} \bar{H}_1 = & (2q_1 - 4s_1)p_1^2 + 4q_2p_1p_2 - \frac{1}{2}q_1^2 + s_1q_1 + \frac{1}{2}q_2 + \frac{4\kappa''_\infty s_2}{q_2} \\ & + \frac{2s_2^2(q_1 - 2s_1)}{q_2^2}, \end{aligned}$$

$$\begin{aligned} s_2\bar{H}_2 = & -q_2p_1^2 + q_2^2p_2^2 + \frac{1}{4}q_1q_2 - \frac{s_2(2\kappa''_\infty q_1 - 4\kappa''_\infty s_1 - s_2)}{q_2} \\ & - \frac{s_2^2(q_1 - 2s_1)^2}{q_2^2} - \frac{1}{4}, \end{aligned}$$

$$\sigma = (3/2, 3) \Rightarrow$$

$$\begin{aligned} s_1^4\bar{H}_1 = & -s_1(s_1s_2q_1 + s_1^4q_2 + s_1s_2 + s_2^2)p_1^2 - 2s_1^2s_2q_2p_1p_2 + s_1^3q_2^2p_2^2 \\ & + s_1^2s_2p_1 - s_1^3q_2p_2 + s_1^2s_2q_1^3 + s_1^5q_1^2q_2 + s_1s_2(3s_1 + 5s_2)q_1^2 \\ & + 2s_1^4(s_1 + s_2)q_1q_2 - s_1^7q_2^2 + s_2\{(2\kappa'_\infty + 3)s_1^2 + 10s_1s_2 + 8s_2^2\}q_1 \\ & + s_1^4\{(2\kappa'_\infty + 1)s_1 + 2s_2\}q_2 - \frac{1}{4}\frac{s_1(s_1q_1 + s_1 + 2s_2)}{q_2} \\ & - 2s_1^3 + (2\kappa'_\infty + 1)s_1^2s_2 + (2\kappa'_\infty + 5)s_1s_2^2 + 8s_2^3 + \frac{4s_2^4}{s_1}, \end{aligned}$$

$$\begin{aligned} s_1^3\bar{H}_2 = & s_1(s_1q_1 + s_1 + s_2)p_1^2 + 2s_1^2q_2p_1p_2 - s_1^2p_1 - s_1^2q_1^3 \\ & - s_1(3s_1 + 5s_2)q_1^2 + 2s_1^4q_1q_2 - \{(2\kappa'_\infty + 3)s_1^2 + 10s_1s_2 + 8s_2^2\}q_1 \\ & + 2s_1^3(s_1 + 2s_2)q_2 + \frac{1}{4}\frac{s_1}{q_2} - (2\kappa'_\infty + 1)s_1^2 - (2\kappa'_\infty + 5)s_1s_2 - 8s_2^2 \\ & - 4s_2^3\frac{1}{s_1}, \end{aligned}$$

$$\sigma = (5/2, 3/2) \Rightarrow$$

$$\bar{H}_1 = 2(q_1 - 2s_1)p_1^2 + 4q_2p_1p_2 + 2p_1 - \frac{1}{2}q_1^2 + s_1q_1 + \frac{1}{2}q_2 + \frac{1}{2}\frac{s_2^2}{q_2},$$

$$s_2\bar{H}_2 = -2q_2p_1^2 + 2q_2^2p_2^2 + 2q_2p_2 + \frac{1}{2}q_1q_2 - \frac{1}{2}\frac{s_2^2(q_1 - 2s_1)}{q_2},$$

$$\sigma = (3/2, 2, 1) \Rightarrow$$

$$\begin{aligned} s_1\bar{H}_1 &= 2s_1^2(q_1 - 1)q_2p_1^2 + 2q_2^2p_2^2 + 2(2\kappa_1 - 1)s_1^2q_2p_1 - 4\kappa_1q_2p_2 \\ &\quad - 2s_1^2s_2\{s_2q_1 + s_1^2s_2q_2 - (s_2 - 2\kappa_\infty)\}q_2 - \frac{1}{2}\frac{q_1}{q_2} \\ &\quad - \frac{1}{2}s_1^2 + 2\kappa_1^2 - 2\kappa_1, \end{aligned}$$

$$\begin{aligned} s_2\bar{H}_2 &= q_1(q_1 - 1)p_1^2 + 2(q_1 - 1)q_2p_1p_2 + q_2^2p_2^2 - (q_1 - 2\kappa_1)p_1 - q_2p_2 \\ &\quad - s_2^2q_1^2 - 2s_1^2s_2^2q_1q_2 - s_1^4s_2^2q_2^2 + s_2(s_2 - 2\kappa_\infty)q_1 \\ &\quad + 2s_1^2s_2(s_2 - \kappa_\infty)q_2 - \frac{1}{4}\frac{1}{q_2} - 2, \end{aligned}$$

$$\sigma = (5/2, 1, 1) \Rightarrow$$

$$\begin{aligned} s_1^3\bar{H}_1 &= -2s_1(s_2q_1 + s_1^3q_2 + s_2)p_1^2 + 4s_1(s_1^3 - s_2)q_2p_1p_2 \\ &\quad + 2s_1(s_1q_1 + s_1q_2 - s_1^3 + s_1 + s_2)q_2p_2^2 \\ &\quad + 2s_1\{(2\kappa_1 - 1)s_1^3 - 2(\kappa_0 + \kappa_1 - 1)s_2\}p_1 \\ &\quad + 2s_1\{s_1(2\kappa_1 - 1)q_1 + 2s_1(\kappa_0 + \kappa_1 - 1)q_2 \\ &\quad - (2\kappa_1 - 1)(s_1^3 - s_1 - s_2)\}p_2 + \frac{1}{2}s_1s_2q_1^2 \\ &\quad + \frac{1}{2}s_1^4q_1q_2 + (s_1 + s_2)s_2q_1 + \frac{1}{2}s_1^3(s_1^3 + s_1 + s_2)q_2 \\ &\quad + s_1^2s_2^2 + 2(\kappa_0^2 + 2\kappa_0\kappa_1 - 3\kappa_0 - 2\kappa_1)s_1^2 + \frac{1}{2}s_1s_2 + s_2^2 - s_1^5s_2, \\ s_1^2\bar{H}_2 &= 2s_1(q_1 + 1)p_1^2 + 4s_1q_2p_1p_2 - 2s_1q_2p_2^2 + 4(\kappa_0 + \kappa_1 - 1)s_1p_1 \\ &\quad - 2(2\kappa_1 - 1)s_1p_2 - \frac{1}{2}s_1q_1^2 - (s_1 + s_2)q_1 + \frac{1}{2}s_1^3q_2 \\ &\quad - s_1^2s_2 - \frac{1}{2}s_1 - s_2, \end{aligned}$$

$$\sigma = (3/2, 3/2, 1) \Rightarrow$$

$$\begin{aligned} s_1 \bar{H}_1 &= (2q_1^2 - 2s_1^2 q_1 - 2s_1^2 s_2^2 q_2) p_1^2 + 4(q_1 - s_1^2) q_2 p_1 p_2 + 2q_2^2 p_2^2 \\ &\quad + (2q_1 + 4\kappa'_1 s_1^2) p_1 + 2q_2 p_2 - \frac{1}{2} q_1 - \frac{1}{2} \frac{s_1^2}{q_2} - \frac{9}{2}, \\ s_2 \bar{H}_2 &= 2s_2^2 (q_1 - s_1^2) q_2 p_1^2 + 4s_2^2 q_2^2 p_1 p_2 + 2q_2^2 p_2^2 + 2s_2^2 q_2 p_1 \\ &\quad - 4\kappa'_1 q_2 p_2 - \frac{1}{2} s_2^2 q_2 - \frac{1}{2} \frac{q_1}{q_2} + 2\kappa'_1 (\kappa'_1 - 1), \end{aligned}$$

$$\sigma = (3/2, 1, 1, 1) \Rightarrow$$

$$\begin{aligned} s_1 (s_1 - 1) \bar{H}_1 &= (q_1 - 1)(s_1 q_1 - q_2 - s_1^2) p_1^2 + 2(s_1 q_1 - q_2 - s_1^2) q_2 p_1 p_2 \\ &\quad + (s_1 q_1 - q_2 - s_1^2) q_2 p_2^2 + 2\{(\kappa_0 + \kappa_1 + \kappa_t - 1) s_1 q_1 \\ &\quad - (\kappa_0 + \kappa_1 + \kappa_t - 1) q_2 - (\kappa_0 + \kappa_1 - 1) s_1^2 - \kappa_t s_1\} p_1 \\ &\quad + \{(2\kappa_0 - 1) s_1 q_1 + (2\kappa_t s_1 - 2\kappa_0 - 2\kappa_t + 1) q_2 \\ &\quad - (2\kappa_0 - 1) s_1^2\} p_2 - \frac{1}{4} s_1 s_2^2 q_1 + \frac{1}{4} s_2^2 q_2 \\ &\quad + (2\kappa_0 \kappa_t + \kappa_1^2 + 2\kappa_1 \kappa_t - \kappa_1 + \kappa_t^2 - \kappa_t) s_1 - \kappa_0^2 - 2\kappa_0 \kappa_t + \kappa_0, \\ s_2 \bar{H}_2 &= 2\{(q_1 - 1)(q_1 - s_1) - q_2\} p_1^2 + 4(q_1 - s_1 - 1) q_2 p_1 p_2 \\ &\quad + 2(q_2 - s_1) q_2 p_2^2 + 2\{(2\kappa_0 + 2\kappa_1 + 2\kappa_t - 3) q_1 - 2(\kappa_0 + \kappa_1 - 1) s_1 \\ &\quad - 2(\kappa_0 + \kappa_t - 1)\} p_1 + 2\{(2\kappa_0 + 2\kappa_1 + 2\kappa_t - 3) q_2 \\ &\quad - (2\kappa_0 - 1) s_1\} p_2 - \frac{1}{2} s_2^2 q_1 + 2(\kappa_0 + \kappa_t + \kappa_1)(\kappa_0 + \kappa_1 + \kappa_t - 3), \end{aligned}$$

where $\kappa' = \kappa_0 - (1/2)$ and $\kappa'' = \kappa_0 - 1$.

By these expressions, we see that the Hamiltonians H_1, H_2 given by

$$\sigma = (7/2, 1) \Rightarrow H_1 = \bar{H}_1 + 3s_1^4 + 6s_1^2 s_2, \quad H_2 = \bar{H}_2 - s_1^3 - 2s_1 s_2,$$

$$\sigma = (5/2, 2) \Rightarrow H_1 = \bar{H}_1, \quad H_2 = \bar{H}_2 + \frac{1}{4s_2},$$

$$\sigma = (3/2, 3) \Rightarrow$$

$$H_1 = \bar{H}_1 - \frac{1}{s_1^4} \left\{ -2s_1^3 + (2\kappa'_\infty + 1) s_1^2 s_2 + (2\kappa'_\infty + 5) s_1 s_2^2 + 8s_2^3 + \frac{4s_2^4}{s_1} \right\},$$

$$H_2 = \bar{H}_2 + \frac{1}{s_1^3} \left\{ (2\kappa'_\infty + 1) s_1^2 + (2\kappa'_\infty + 5) s_1 s_2 + 8s_2^2 + 4s_2^3 \frac{1}{s_1} \right\},$$

$$\sigma = (5/2, 3/2) \Rightarrow H_1 = \bar{H}_1, H_2 = \bar{H}_2,$$

$$\sigma = (3/2, 2, 1) \Rightarrow H_1 = \bar{H}_1 - \frac{1}{s_1} \left(-\frac{1}{2} s_1^2 + 2\kappa_1^2 - 2\kappa_1 \right), H_2 = \bar{H}_2 + \frac{2}{s_2},$$

$$\sigma = (5/2, 1, 1) \Rightarrow$$

$$H_1 = \bar{H}_1 - \frac{1}{s_1^3} \left\{ s_1^2 s_2^2 + 2(\kappa_0^2 + 2\kappa_0 \kappa_1 - 3\kappa_0 - 2\kappa_1) s_1^2 + \frac{1}{2} s_1 s_2 + s_2^2 - s_1^5 s_2 \right\},$$

$$H_2 = \bar{H}_2 + \frac{1}{s_1^2} \left(s_1^2 s_2 + \frac{1}{2} s_1 + s_2 \right),$$

$$\sigma = (3/2, 3/2, 1) \Rightarrow H_1 = \bar{H}_1 + \frac{9}{2s_1}, H_2 = \bar{H}_2 - \frac{2\kappa_1'(\kappa_1' - 1)}{s_2},$$

$$\sigma = (3/2, 1, 1, 1) \Rightarrow$$

$$H_1 = \bar{H}_1 - \frac{(2\kappa_0 \kappa_t + \kappa_1^2 + 2\kappa_1 \kappa_t - \kappa_1 + \kappa_t^2 - \kappa_t) s_1 - \kappa_0^2 - 2\kappa_0 \kappa_t + \kappa_0}{s_1(s_1 - 1)},$$

$$H_2 = \bar{H}_2 - \frac{2(\kappa_0 + \kappa_t + \kappa_1)(\kappa_0 + \kappa_1 + \kappa_t - 3)}{s_2},$$

are simpler than $\bar{H}_1, \bar{H}_2$ as rational functions of q_1, q_2, p_1, p_2, s_1 and s_2 . Moreover these Hamiltonians satisfy

$$(5) \quad \sum_{k=1}^2 \frac{\partial H_1}{\partial p_k} \frac{\partial H_2}{\partial q_k} - \frac{\partial H_1}{\partial q_k} \frac{\partial H_2}{\partial p_k} = 0,$$

$$(6) \quad \left(\frac{\partial}{\partial s_1} \right) H_2 = \left(\frac{\partial}{\partial s_2} \right) H_1.$$

Here $\left(\frac{\partial}{\partial s_i} \right)$ ($i = 1, 2$) are the differentiations with respect to s_i regarding q_1, q_2, p_1, p_2 as constants. (In fact, the first equation is obtained by (4) because the difference $H_i - \bar{H}_i$ is rational function of s_1 and s_2 . The second equation can be obtained by direct calculation.) Note that the transformation $(q, p, \bar{H}, s) \rightarrow (q, p, H, s)$ is not a canonical transformation, but the Hamiltonian system (q, p, H, s) is equivalent to $(q, p, \bar{H}, s)$.

The equations (5) and (6) yield that the following Theorem.

Theorem 4. *The Hamiltonian system*

$$(7) \quad \frac{\partial q_i}{\partial s_j} = \frac{\partial H_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial s_j} = -\frac{\partial H_j}{\partial q_i} \quad (i, j \in \{1, 2\}),$$

is completely integrable.

Proof. By using (5), the following equality holds:

$$(8) \quad \frac{\partial}{\partial s_i} H_j = \left(\frac{\partial}{\partial s_i} \right) H_j + \sum_{k=1}^2 \frac{\partial H_j}{\partial q_k} \frac{\partial q_k}{\partial s_i} + \sum_{k=1}^2 \frac{\partial H_j}{\partial p_k} \frac{\partial p_k}{\partial s_i} = \left(\frac{\partial}{\partial s_i} \right) H_j.$$

Therefore

$$\frac{\partial}{\partial s_1} H_2 - \frac{\partial}{\partial s_2} H_1 + \sum_{k=1}^2 \frac{\partial H_1}{\partial p_k} \frac{\partial H_2}{\partial q_k} - \frac{\partial H_1}{\partial q_k} \frac{\partial H_2}{\partial p_k}$$

is equal to 0 because of (5), (6) and (8). This implies that (7) is completely integrable. $\square$

Remark. The equation (5) implies that

$$H_1 ds_1 + H_2 ds_2$$

is closed 1-form. Hence we see that there exists a function τ of s_1 and s_2 which satisfies

$$d \log \tau = H_1 ds_1 + H_2 ds_2.$$

This function is called τ function. We expect that this function will play an important role in study of $G(\sigma)$ ($\sigma \in S_2''$) such as the τ function of Painlevé equation.

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References

- [1] Fuchs, R., Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegene wesentlich singulären Stellen, *Math. Ann.*, **63** (1907), 301–321.
- [2] Garnier, R., Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, *Ann. Sci. École. Norm. Sup.*, **29** (1912), 1–126.
- [3] Kawamuko, H., Rational solutions of the fourth Painlevé equation in two variables, *Funkcial. Ekvac.*, **46** (2003), 1–21.
- [4] Kawamuko, H., On Special solutions of the degenerate Garnier system (3,2) and $G(5/2, 1, 1)$, in preparation.
- [5] Kimura, H., The degeneration of the two dimensional Garnier system and the polynomial Hamiltonian structure, *Ann. Mat. Pura Appl.*, **155** (1989), 25–74.
- [6] Ohyama, Y. and Okumura, S., Complete coalescent diagram of the Painlevé equations, *math.CA/0601614*.

- [7] Ohyama, Y., Kawamuko, H., Sakai, H. and Okamoto, K., Studies on the Painlevé Equations V, J. Math. Sci. Univ. Tokyo, **13** (2006), 145–204.
- [8] Okamoto, K., Isomonodromic deformation and painlevé equations, and the Garnier system, J. Fac. Sci. Univ. of Tokyo, Sect. I-A, Math., **33** (1986), 575–618.

nuna adreso:

Department of Mathematics

Faculty of Education

Mie University

Tsu, Mie 514-8507

Japan

E-mail: kawam@edu.mie-u.ac.jp

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