Structure Theorems for Positive Radial Solutions of the Generalized Scalar Curvature Equation

By

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Abstract. In this paper we analyze radial solutions for the generalized scalar curvature equation. In particular we prove the existence of ground states and singular ground states when the curvature $K(r)$ is monotone as $r \rightarrow 0$ and as $r \rightarrow \infty$. The results are new even when $p = 2$, that is when we consider the usual Laplacian.

The proofs use a new Fowler transform which allow us to consider a 2-dimensional dynamical system thus giving a geometrical point of view on the problem. A key role in the analysis is played by an energy function which is a dynamical interpretation of the Pohozaev function used in [21] and [22].

Key Words and Phrases. p -Laplace equations, Radial solution, Regular/singular ground state, Fowler inversion, Invariant manifold.

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1. Introduction

The aim of this paper is to discuss the existence and the asymptotic behavior of radial solutions of the following equation

$$(1.1) \quad \Delta_p u + K(|x|)u|u|^{\sigma-2} = 0$$

where $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$, $p > 1$, denotes the p -Laplace operator, $x \in \mathbf{R}^n$, $2n/(n+2) \leq p \leq 2$, $\sigma = np/(n-p)$ is the Sobolev critical exponent, and K is a radially symmetric, positive and continuous function. We are particularly interested in ground states, singular ground states and crossing solutions. By ground state (G.S.) we mean a positive solution $u(x)$ defined in the whole space $\mathbf{R}^n$ such that $\lim_{|x| \rightarrow \infty} u(x) = 0$, by singular ground state (S.G.S.) we mean a G.S. which is not defined at the origin and satisfies $\lim_{|x| \rightarrow 0} u(x) = +\infty$. By crossing solution we mean a solution $u(x)$ such that $u(x) > 0$ if $|x| < R$ and $u(x) = 0$ if $|x| = R$, therefore such a solution can also be regarded as a Dirichlet solution in a ball of radius R . We are interested also in Dirichlet solutions in exterior domains; that is solutions $u(r)$ for which there is $R > 0$ such that $u(R) = 0$, $u(r) > 0$ for $r > 0$ and $u(r)$ has fast decay.

We will only deal with radial solutions, so we shall consider the following O.D.E.

$$(1.2) \quad (u'|u'|^{p-2})' + \frac{n-1}{r} u'|u'|^{p-2} + K(r)u|u|^{\sigma-2} = 0$$

where $r = |x|$ and we commit the following abuse of notation: we write $u(r)$ for $u(x)$ where $|x| = r$; here and later $'$ denotes derivation with respect to r .

We introduce now some notation that will be in force throughout all the paper. We use the term “regular solution” to refer to a solution $u(x)$ of Eq. (1.2) satisfying $u(0) = u_0 > 0$ and $u'(0) = 0$. We use the term “singular solution” to refer to a solution $v(x)$ of Eq. (1.2) such that $\lim_{|x| \rightarrow 0} v(x) = +\infty$. Furthermore when we write $u(r) \sim r^{-\alpha}$ as $r \rightarrow c$ we mean that the limits $\liminf_{r \rightarrow c} u(r)r^\alpha$ and $\limsup_{r \rightarrow c} u(r)r^\alpha$ are both finite and positive. We say that $u(r)$ has fast decay if the limit $u(r)r^{(n-p)/(p-1)}$ exists, is positive and finite, and that it has slow decay if $\lim_{r \rightarrow \infty} u(r)r^{(n-p)/(p-1)} = +\infty$. We will see that, when $K(r)$ is monotone for r small and for r large, “usually” a singular solution is such that $u(r) \sim r^{-(n-p)/p}$ as $r \rightarrow 0$ and a solution with slow decay is such that $u(r) \sim r^{-(n-p)/p}$ as $r \rightarrow +\infty$, see Propositions 2.5 and Remark 2.7.

Equations with Δ and Δ_p have been extensively studied by many authors in recent years and nowadays the autonomous Eq. (1.2) is completely understood; see [13] for a survey on the topic. However there are still some unsolved problem concerning the spatial dependent case. Important papers on the case $p \neq 2$ are [21], and [22] where the authors, in particular, give a structure result for regular positive solutions when $K(r)$ is monotone or admits exactly one critical point and it is a maximum. In [9] we have been able to complete this result in the monotone case, in the sense that we give a better estimate on the asymptotic behavior, we prove the existence of positive singular solutions and we classify all such solutions. Putting together the results explained in [22] and in [9] and restricting our attention to the case $2n/(n+2) \leq p \leq 2$, we can state the following classification result:

Proposition 1.1. *Consider Eq. (1.2) and assume that $K(r) \in C^1$ is monotone decreasing. Then we can classify all the positive solutions as follows*

- *All the regular solutions $u(r)$ are monotone decreasing G.S. with slow decay, and $u(r) \sim r^{-(n-p)/p}$ as $r \rightarrow +\infty$.*
- *There exists at least one monotone decreasing S.G.S. with slow decay $v(r)$ such that $v(r) \sim r^{-(n-p)/p}$, both as $r \rightarrow 0$ and as $r \rightarrow +\infty$.*
- *There exist uncountably many Dirichlet solutions $u(r)$ in exterior domains. Assume that $K(r)$ is monotone increasing, then*
 - *All the regular solutions are crossing solutions.*
 - *There exists at least one monotone decreasing S.G.S. with slow decay $v(r)$ such that $v(r) \sim r^{-(n-p)/p}$, both as $r \rightarrow 0$ and as $r \rightarrow +\infty$.*
 - *There exist uncountably many monotone decreasing S.G.S. with fast decay $w(r)$, such that $w(r) \sim r^{-(n-p)/p}$ as $r \rightarrow 0$.*

The hypotheses of the Proposition may be weakened: if we let p take values in $(1, \infty)$, we still have the result concerning regular solutions, see [21]. We think in fact that the hypothesis $2n/(n+2) \leq p \leq 2$ is just technical, and that, if we remove it, also the result concerning singular solutions should hold.

When $K(r)$ is not monotone the structure of positive solutions becomes richer and less understood. Bianchi in [3] proved that, when $p = 2$ and there is $r = R$ such that $K(r)$ is monotone decreasing for $r \leq R$ and monotone increasing for $r \geq R$, all the solution of (1.1) are radial. Bianchi and Egnell in [4] and Bianchi in [2] give several types of conditions on $K(r)$ which are sufficient for the existence and for the non-existence of G.S. with fast decay. In particular they proved the existence of such solutions assuming that $(\int_0^\infty K'(s)s^n ds)(\int_0^\infty K'(s)s^{-n} ds) < 0$ and that the limits $\lim_{r \rightarrow 0} K(r)$ and $\lim_{r \rightarrow \infty} K(r)$ exist, are positive and bounded and $K(r)$ is flat enough as $r \rightarrow 0$ and as $r \rightarrow \infty$. Yanagida and Yotsutani in [24] consider eq. (1.2) assuming $p = 2$ and $\sigma > 2$. When σ is the Sobolev critical exponent (that is the case considered in this paper) they proved the existence of open sets of crossing solutions and G.S. with slow decay disconnected by a non-empty set of G.S. with fast decay, when $K(r)$ is increasing for r small and decreasing for r large. In fact, even if it was not explicitly stated in Corollary of [24], $K(r)$ must satisfy conditions analogous to α^+ and Ω^- stated below, that is $K(r)$ should not be “too flat” for r small and for r large.

The case $p \neq 2$ is less understood. In [22] the authors consider the function $J^+(r) = \int_0^r K'(s)s^n$ and they prove a structure result assuming that there is $R > 0$ such that $J^+(r) \geq 0$ for $r \in (0, R)$ and $K'(r) \leq 0$ for $r > R$. They state that positive regular solutions could have either the structure of the monotone increasing case either the structure of the monotone decreasing case, or a third richer structure, in which we have the coexistence of all the three different families of regular solutions. The aim of this paper is to investigate this latter case. In this direction goes also [19], where the authors proved a result concerning nodal solutions. If we restrict to eq. (1.2) with σ critical they proved the existence of ground states with fast decay, both positive for any $r > 0$ or with a prescribed finite number of 0, assuming that $K(r)$ is increasing for r small and decreasing for r large and satisfy some further mild assumptions. In particular they assume that $\lim_{r \rightarrow 0} K(r) = 0 = \lim_{r \rightarrow \infty} K(r)$.

We will consider hypotheses similar to the ones used in [24], but we extend the results to the case $p \neq 2$ and we also consider singular solutions. In the whole paper we assume without mentioning further that the following Hypotheses are satisfied:

H1 The function $K(r) \in C^1((0, \infty))$, is positive for any $r \geq 0$ and the limits $\lim_{r \rightarrow 0} K(r) = K(0)$ and $\lim_{r \rightarrow \infty} K(r) = K(\infty)$, exist are positive and finite.

H2 $2n/(n+2) \leq p \leq 2$

Note that hypothesis H1 is complementary to the ones discussed in [19]. We stress that we need to assume the technical condition H2, while the methods used in [19] and in [22] do not need such requirements. Under suitable hypotheses, we will find one of the following structures for positive solutions:

A • There exist uncountably many monotone decreasing G.S. with slow decay $u(r)$, such that $u(r) \sim r^{-(n-p)/p}$ as $r \rightarrow \infty$.

• There exist uncountably many crossing solutions.
 • There exists a non empty set of monotone decreasing G.S. with fast decay disconnecting the first two sets.

• There exist uncountably many solutions $u(r)$ of the Dirichlet problem in exterior domains.

• There exist uncountably many monotone decreasing S.G.S. with fast decay $w(r)$, such that $w(r) \sim r^{-(n-p)/p}$ as $r \rightarrow 0$.

• There exist uncountably many monotone decreasing S.G.S. with slow decay.

B • There exist uncountably many crossing solutions.

• There exist uncountably many solutions $u(r)$ of the Dirichlet problem in exterior domains.

• There exists a non empty set of monotone decreasing G.S. with fast decay.

Here we enumerate the hypotheses and the main results of this paper.

Hypotheses.

α^-) There is $\rho > 0$ such that $K(r)$ is nonincreasing for $r \in [0, \rho]$ and $K'(r)r^{-n/(p-1)} \notin L^1(0, \rho]$.

α^+) There is $\rho > 0$ such that $K(r)$ is nondecreasing for $r \in [0, \rho]$ and $K'(r)r^{-n/(p-1)} \notin L^1(0, \rho]$.

Ω^-) There is $R > 0$ such that $K(r)$ is nonincreasing for $r > R$ and $K'(r)r^n \notin L^1[R, +\infty)$.

Ω^+) There is $R > 0$ such that $K(r)$ is nondecreasing for $r > R$ and $K'(r)r^n \notin L^1[R, +\infty)$.

Theorem 1.2. *Assume that hypotheses α^+ and Ω^- are satisfied, then positive solutions have a structure of type A.*

Theorem 1.3. *Assume that hypotheses α^- and Ω^+ are satisfied, then positive solutions have a structure of type B.*

Thus we manage to complete the analysis of positive solutions of (1.2) by giving a classification of singular positive solutions. Moreover we give a condition which is sufficient to prove the existence of G.S. with fast decay

in the case $p \neq 2$ and also when $K(r)$ is decreasing for r small and increasing for r large, that is the case in which there are less results, see [2]. Furthermore we can refine the estimates on the asymptotic behavior of the solutions.

We make use of the techniques developed by Johnson, Pan, Yi, and Battelli in [5, 6, 17] for similar problems involving the Laplacian and in [9] for the p -Laplacian. We also take some of the basic ideas of [11], developed for the problem where $K(r)$ is a perturbation of a constant, and we manage to reapply them here. Thus we can extend some of the results obtained for functions $K(r)$ which are regular or singular perturbations of a constant, to “generic” uniformly positive $K(r)$ exhibiting the same oscillatory behavior.

We exploit the transform of Fowler type introduced in [10], which changes Eq. (1.2) into a dynamical system and enables us to give a geometrical interpretation to the problem. The refining of the estimates on the asymptotic behavior follows by an application to the problem of invariant manifold theory, extended to non-autonomous systems. A crucial role will be played by a function H which enables us to give a dynamical interpretation of the Pohozaev identity.

We finish this introduction by giving some terminology. Recall that given a system of the form

$$\dot{x} = f(x, t)$$

and a solution $x(t)$, the α -limit set of $x(t)$ is the set

$$\left\{ P : \exists t_n \rightarrow -\infty \text{ such that } \lim_{n \rightarrow \infty} x(t_n) = P \right\},$$

while the ω -limit set is the set

$$\left\{ P : \exists t_n \rightarrow +\infty \text{ such that } \lim_{n \rightarrow \infty} x(t_n) = P \right\}.$$

One can show that, if $x(t)$ is bounded on $\mathbf{R}$, then these sets are compact. Moreover, if the system is autonomous, these sets are invariant for the flow generated by the system.

2. Preliminaries

We begin by introducing a transform which generalizes to the p -Laplacian the well known Fowler transform which works for the classic Laplacian.

$$(2.1) \quad x_1 = u(r)r^\alpha \quad x_2 = u'(r)|u'(r)|^{p-2}r^\beta \quad r = e^t \quad \phi(t) = K(e^t)$$

$$\text{where } \alpha = (n - p)/p \text{ and } \beta = n(p - 1)/p$$

This change of variables allows us to transform the singular O.D.E. (1.2) into the following dynamical system:

$$(2.2) \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_2 |x_2|^{(2-p)/(p-1)} \\ -\phi(t)x_1 |x_1|^{\sigma-2} \end{pmatrix}$$

Here and later “.” denotes derivation with respect to t . Note that the preceding equation is C^1 if and only if $2n/(2+n) \leq p \leq 2$, thus we will restrict our analysis only to this case. We have a close relationship between trajectories $x(t)$ of our system and solutions $u(r)$ of our problem.

Remark 2.1. The solutions $u(r)$ of Eq. (1.2) correspond to the trajectories $x(t)$ of system (2.2) having the origin as α -limit point. Moreover if $u(r) > 0$ then $x_1(t) > 0$ and $u'(r) < 0$ implies $x_2(t) < 0$.

Since we are mainly interested in positive solutions $u(r)$ we will focus our attention on the halfplane $x_1 \geq 0$, which will be denoted by $\mathbf{R}_+^2$.

It will be useful to embed system (2.2) in the following one parameter family of systems:

$$(2.3) \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_2 |x_2|^{(2-p)/(p-1)} \\ -\phi(t+\tau)x_1 |x_1|^{\sigma-2} \end{pmatrix}$$

Let $x(\tilde{\tau}, x_0; t)$ be the trajectory of system (2.3) departing at $t = 0$ from $x = x_0$, for $\tau = \tilde{\tau}$. Assume that ϕ is uniformly continuous, then using the theory developed in [15] and in [16], we can state the existence of C^1 leaves defined as follows:

$$W_{loc}^u(\tau) = \left\{ x_0 \left| \lim_{t \rightarrow -\infty} x(\tau, x_0; t) \right. \right\} \quad \text{and} \quad W_{loc}^s(\tau) = \left\{ x_0 \left| \lim_{t \rightarrow +\infty} x(\tau, x_0; t) \right. \right\},$$

defined in a neighborhood of the origin. These leaves are C^1 graphs respectively on the unstable and stable manifold of the autonomous system obtained setting respectively $\phi(t) \equiv \lim_{t \rightarrow -\infty} \phi(t)$ and $\phi(t) \equiv \lim_{t \rightarrow +\infty} \phi(t)$, linearized in the origin (that in this case are respectively the x_1 and x_2 axis). The uniform continuity of ϕ ensures that we can choose $W_{loc}^u(\tau)$ and $W_{loc}^s(\tau)$ in such a way that their diameter is greater than a positive small constant which is independent from τ .

Then, following the techniques explained in [17] and in [5], we can construct global stable and unstable manifolds as follows:

$$W^u(\tau) = \bigcup_{t \in \mathbf{R}} \{x_0 \mid x(\tau - t, x_0; t) \in W_{loc}^u(\tau)\}$$

$$W^s(\tau) = \bigcup_{t \in \mathbf{R}} \{x_0 \mid x(\tau - t, x_0; t) \in W_{loc}^s(\tau)\}.$$

It can be proved that these manifolds are C^1 , and vary C^1 smoothly in τ , see Theorem 2.1 in [17] (page 1051), and [16]. We denote by $\Phi_\tau(t, x_0)$ the diffeomorphism which associates to a point x_0 its image through the flow of system (2.3) at time t , that is $x(\tau, x_0; t)$. Note that $\Phi_\tau(t, W^{u,s}(\tau)) = W^{u,s}(\tau + t)$.

We need also the following extended autonomous system obtained from system (2.4) by adding the extra variable $x_3 = \tau + t$:

$$(2.4) \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} x_2 |x_2|^{(2-p)/(p-1)} \\ -\phi(x_3)x_1 |x_1|^{\sigma-2} \\ 1 \end{pmatrix}$$

We will make use also of the following system, where we set $x_3 = e^{\xi t}$.

$$(2.5) \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} x_2 |x_2|^{(2-p)/(p-1)} \\ -\phi(\xi; x_3)x_1 |x_1|^{\sigma-2} \\ 0 \end{pmatrix}$$

where $\phi(\xi; x_3) = \phi(\log(x_3)/\xi)$. If we set $\xi > 0$, (2.5) is useful to investigate the asymptotic behaviour of the trajectories as $t \rightarrow -\infty$, while if we set $\xi < 0$ it can be used to analyze the asymptotic behaviour as $t \rightarrow +\infty$. In fact when $\xi > 0$ the α -limit set of any bounded trajectory of system (2.5) is contained in the plane $x_3 = 0$, while when $\xi < 0$ this plane contains the ω -limit set of any bounded trajectory. Note that when $\xi < 0$ the origin admits a two dimensional stable manifold W^s . Intersecting W^s with the plane $x_3 = e^{\xi\tau}$ we obtain $W^s(\tau) \times \{e^{\xi\tau}\}$ where $W^s(\tau)$ is the manifold already described. Analogously when $\xi > 0$ the origin admits a two dimensional unstable manifold W^u and its intersection with the plane $x_3 = e^{\xi\tau}$ is the manifold $W^u(\tau) \times \{e^{\xi\tau}\}$.

Furthermore if we pass from (2.5) to (2.4) through the change of variables $x_3 \rightarrow \log(x_3)/\xi$, we have that W^u and W^s are transformed into two dimensional manifolds which will be denoted with the same name. Again, intersecting these manifolds with the plane $x_3 = \tau$, we obtain respectively $W^u(\tau) \times \{\tau\}$ and $W^s(\tau) \times \{\tau\}$.

Let us give some further notation

$$\begin{aligned} U^+ &:= \{(x_1, x_2) \mid \alpha x_1 + x_2 |x_2|^{p/(p-1)} > 0\}, \\ U^- &:= \{(x_1, x_2) \mid \alpha x_1 + x_2 |x_2|^{p/(p-1)} < 0\} \\ U^0 &:= \{(x_1, x_2) \mid \alpha x_1 + x_2 |x_2|^{p/(p-1)} = 0\}. \end{aligned}$$

We introduce a function closely related to the Pohozaev identity, and already used in a more general setting in [9], which will play a key role in the following analysis:

$$H(x; t) := \alpha x_1 x_2 + \frac{p-1}{p} |x_2|^{p/(p-1)} + \phi(\tau + t) \frac{|x_1|^\sigma}{\sigma}.$$

This function is in fact the transposition in this dynamical context of the Pohozaev function

$$(2.6) \quad P(u, u', r) := \frac{n-p}{p} r^{n-1} u u' |u'|^{p-2} + \frac{p-1}{p} |u'|^p r^n + K(r) r^n \frac{|u|^\sigma}{\sigma}$$

of [22]. Observe that by differentiating we get

$$(2.7) \quad \frac{d}{dt} H(x(t); t) = \dot{\phi}(\tau + t) \frac{|x_1|^\sigma(t)}{\sigma}.$$

Note that (2.7) is a way of rewriting the Pohozaev identity. We now begin our analysis of Eq. (2.3). To do this we need to recall some results about the autonomous system (2.2) where $\phi \equiv \text{const} > 0$, see [9]. First of all observe that such a system admits exactly three critical points which are the origin, $P = (P_1, P_2)$, where $P_2 < 0 < P_1$ and $-P$. Note that P depends on ϕ . We need now the following result concerning the shape of the level sets of the function $H(\cdot, \cdot; t)$, for t fixed, see Figure 1. Let us define

$$H_b := \{(x_1, x_2) \mid H(x_1, x_2; T) = b\}.$$

For any fixed T such that $\phi(T)$ is positive and finite, the level sets H_b of the function $H(\cdot, \cdot; T)$ are as sketched in Figure 1. This claim follows easily from Lemma 2.9 in [9] and Lemma 2.6 in [10], therefore it will be skipped. Recalling that H is a first integral for the autonomous system we easily deduce the following.

Proposition 2.2. *Consider Eq. (1.2) when $K \equiv \text{const} > 0$. All the regular solutions $u(r)$ of Eq. (1.2) are monotone decreasing G.S. with fast decay. They correspond to homoclinic trajectories of (2.2), belonging to the closed 4th quadrant and having all the same graph.*

There exist uncountably many S.G.S. $v(r)$ with slow decay. They correspond to the periodic trajectories contained in the open 4th quadrant, therefore $v(r) \sim r^{-\alpha}$, both as $r \rightarrow 0$ and as $r \rightarrow +\infty$. There is a S.G.S. with slow decay that can be explicitly computed, that is $\bar{v}(r) = P_1 r^{-\alpha}$; it corresponds to the critical point P .

Fig. 1. Sketch of the level sets of the function $H(x_1, x_2, T)$, for T fixed. The solid line is the level set H_0 , the dotted and the dashed lines represent some level sets H_b where respectively $b < 0$ and $b > 0$.

In the autonomous case we can also write the exact expression of the G.S., see [13] for example, so we can deduce the exact expression of the homoclinic trajectories. We denote by $U^K(\tau, t) = (U_1^K(\tau, t), U_2^K(\tau, t))$ the homoclinic trajectory of the system (2.2) where $K(r) \equiv K$ and crossing the isocline U^0 at $t = 0$.

$$(2.8) \quad U_1^K(\tau, t) = \left[\frac{1}{D_1(e^{-(t+\tau)} + D_2 e^{(t+\tau)/(p-1)})} \right]^{(n-p)/p} K^{-(n-p)/p^2}$$

where $D_1 = (n-p)^{(p-1)/p} n^{1/p}$ and $D_2 = [(p-1)/d]^{(p-1)/p}$ are constants.

Note that the autonomous system is invariant for translations in t , therefore all the homoclinic trajectories have the same graph.

Now we need a technical Lemma which shows that unbounded trajectories $x(t)$ of (2.2) cannot correspond to positive solutions $u(r)$ of (1.2).

Lemma 2.3. *Consider equation (1.2) and the corresponding system (2.2), where $K(r)$ satisfies H1. Then, if $x(t)$ is unbounded, it rotates clockwise crossing infinitely many times the x_1 and x_2 axes.*

Proof. From H1 it follows that $M = \max\{K(r) \mid r \in \mathbf{R}\}$ and $m = \min\{K(r) \mid r \in \mathbf{R}\}$ are positive and finite. Consider a trajectory $x(t)$ which becomes unbounded as $t \rightarrow c^-$ where c can also be $+\infty$.

Fix t_1 and the corresponding point $P^1 = x(t_1)$ in $\mathbf{R}^2$, assume at first that $P^1 \in U^+$. Consider system (2.2), where $\phi(t) \equiv M$. Recall that the solution of (2.2) which are not homoclinic to $(0,0)$ and which do not coincide with equilibria are periodic, hence the corresponding trajectories define closed curves in $\mathbf{R}^2$. We choose a periodic solution $x^{1,M}(t)$ of (2.2) which crosses the coordinate axes and such that P^1 lies in the exterior of the disc $D^{1,M}$ enclosed by $x^{1,M}(t)$ for $t \in \mathbf{R}$. Such a choice is always possible, since we can choose $\|P^1\|$ as large as we wish. In a similar way, consider system (2.2), where $\phi(t) \equiv m$. We choose a periodic solution $x^{1,m}(t)$ of (2.2) which crosses the coordinate axes and such that P^1 lies in the open disc $D^{1,m}$ enclosed by $x^{1,m}(t)$, for $t \in \mathbf{R}$. We can choose $D^{1,M}$ and $D^{1,m}$ in such a way that $D^{1,m} \supset D^{1,M}$. Let $\partial D^{1,M}$ and $\partial D^{1,m}$ denote the boundary respectively of $D^{1,M}$ and $D^{1,m}$. Let us denote by $R^+ := D^{1,m} \setminus D^{1,M}$ and by ∂R^+ its boundary.

We return to the non-autonomous system (2.2). Obviously $P^1 \in R^+ \cap U^+$; we claim that the flow on $\partial R^+ \cap U^+$ is always going towards the interior of R^+ . In fact choose $Q \in \partial D^{1,m} \cap U^+$; denote by $x^{1,m}(Q, t)$ and $x(Q, t)$ the trajectories passing through Q at $t = 0$ respectively of system (2.2) where $\phi \equiv m$ and of the non autonomous system (2.2). Then we have $\dot{x}_1(Q, 0) = \dot{x}_1^{1,m}(Q, 0)$ and $\dot{x}_2(Q, 0) \leq \dot{x}_2^{1,m}(Q, 0)$. So the flow of the non-autonomous system (2.2) on $\partial D^{1,m} \cap U^+$ points towards the interior of R^+ , for any t . The proof for the case $Q \in \partial D^{1,M} \cap U^+$ is completely analogous, so the claim is proved.

Consider now the unbounded trajectory $x(t)$. Note that it lies in $R^+ \cap U^+$ for $t \geq t_1$, until it crosses the isocline U^0 in a point P^2 . Thus there exists $t_2 > t_1$ such that $x(t_2) = P^2 \in U^0$ and $x(t)$ enters U^- for $t > t_1$.

Once again we consider the autonomous system where $\phi(t) \equiv m$. We choose a periodic solution $x^{2,m}(t)$ of (2.2) which crosses the coordinate axes and such that P^2 lies in the exterior of the disc $D^{2,m}$ enclosed by $x^{2,m}(t)$ for $t \in \mathbf{R}$. In a similar way, we choose a periodic solution $x^{2,M}(t)$ of (2.2) where $\phi(t) \equiv M$ which crosses the coordinate axes, and such that P^2 lies in the open disc $D^{2,M}$ enclosed by $x^{2,M}(t)$, where $t \in \mathbf{R}$. We choose $D^{2,m}$ and $D^{2,M}$ in such a way that $D^{2,M} \supset D^{2,m}$ and define $R_2^- := D^{2,M} \setminus D^{2,m}$. Now we return to the non-autonomous system (2.2). Reasoning as above we find that $x(t) \in R_2^-$, for all $t > t_2$ such that $x(t) \in U^-$. Recalling that $x(t)$ is unbounded we conclude that there exists $t_3 > t_2$ such that $x(t_3) \in U^0$. Therefore $x(t)$ rotates clockwise crossing the x_2 and x_1 negative semi-axes, then it enters U^+ for $t > t_3$.

Iterating the reasoning we obtain that $x(t)$ must cross the coordinate axes infinitely many times. The proof in the case of a trajectory that becomes unbounded going backwards in t is completely analogous. $\square$

With a similar argument we get also the following result.

Lemma 2.4. *Consider a trajectory $x(t)$ of system (2.3) and assume that $\liminf_{t \rightarrow +\infty} H(x(t), t) > 0$. Then, if we follow $x(t)$ forward in t we find that it must cross the negative x_2 semi-axis for some $t = t_1$. Analogously consider a trajectory $\bar{x}(t)$ and assume that $\liminf_{t \rightarrow -\infty} H(\bar{x}(t), t) > 0$. Then, if we follow $\bar{x}(t)$ backwards in t we find that it must cross the positive x_2 semi-axis for some $t = t_2$.*

Proof. Assume that $\liminf_{t \rightarrow +\infty} H(x(t), t) > 0$; then there are $a > 0$ small and $T > 0$ large so that $H(x_1(t), x_2(t), t) > 2a$ for any $t \geq T$. Let us denote by $\bar{D}^a(t) := \{(x_1, x_2) \mid H(x_1, x_2, t) \leq a\}$. Observe that $\bar{D}^a(t)$ is compact and connected for any t , see figure 1. Assume first that $x(0) = Q \in U^+$. Consider the trajectory $x_m(Q^+, t)$ of system (2.2) where $\phi \equiv m = \inf_{t \in \mathbf{R}} \phi(t)$, departing from Q^+ at $t = 0$. We denote by A the bounded subset enclosed by the trajectory $x_m(Q^+, t)$, for $t \in \mathbf{R}$, and by $A^+ = A \cap U^+$. We choose $\|Q^+\|$ large enough so that $\bar{D}^a(t) \subset A$ for any $t \in \mathbf{R}$. Then $x(t)$ is forced to stay in $A^+ \setminus \bar{D}^a(t)$ for $t > 0$, until it reaches the isocline $\dot{x} = 0$ at $t = T_1 \leq \infty$. Note that as long as $x(t) \in \bar{D}^a(t)$, we have that $|\dot{x}(t)| + |\dot{y}(t)|$ is uniformly positive; therefore $T_1 < \infty$.

So we can assume that $Q \in U^-$. Let us denote by $x_M(Q^-, t)$ the trajectory of system (2.2) where $\phi \equiv M = \sup_{t \in \mathbf{R}} \phi(t)$, departing from Q^- at $t = 0$, and denote by B the bounded set enclosed by the trajectory $x_M(Q^-, t)$, for $t \in \mathbf{R}$. We choose $\|Q^-\|$ large enough so that $\bar{D}^a(t) \subset B$ for any $t \in \mathbf{R}$, and denote by $B^- = (B \cap U^-)$. It follows that $x(t)$ is forced to stay in $B^- \setminus \bar{D}^a(t)$ for any $t > T$, until it crosses the x_2 axes. Reasoning as above it follows that $x(t)$ crosses the x_2 axes at a certain $t = T_2 > T$ finite. Hence the first part of the Lemma is proved; the second can be proved with a similar argument so we will skip it. $\square$

Now we discuss in details asymptotic behaviours of positive solutions, both as $r \rightarrow 0$ and as $r \rightarrow \infty$.

Proposition 2.5. *Consider a solution $u(r)$ such that for the corresponding trajectory $x(t)$ of (2.2) we have $\lim_{t \rightarrow -\infty} H(x(t), t) = l < 0$. Then $u(r)$ is a singular solution, $u(r) \sim r^{-\alpha}$ as $r \rightarrow 0$, and there exists a S.G.S. $v(r)$ of the frozen Eq. (1.2) where $K(r) \equiv K(0)$ such that $\lim_{r \rightarrow 0} (u(r) - v(r))r^\alpha = 0$.*

Analogously consider a solution $\bar{u}(r)$ such that for the corresponding trajectory $\bar{x}(t)$ of (2.2) we have $\lim_{t \rightarrow +\infty} H(\bar{x}(t), t) = L < 0$. Then $\bar{u}(r)$ has slow decay, $\bar{u}(r) \sim r^{-\alpha}$ as $r \rightarrow +\infty$ and there is a S.G.S. $\bar{v}(r)$ of the frozen Eq. (1.2) where $K(r) \equiv \lim_{r \rightarrow \infty} K(r)$ such that $\lim_{r \rightarrow \infty} (\bar{u}(r) - \bar{v}(r))r^\alpha = 0$.

Proof. For this proof we follow the ideas developed in [9]. Consider the solution $\bar{u}(r)$ and the corresponding trajectory $\bar{x}(t)$ of (2.2). We denote by

$\bar{X}(t) = (\bar{x}(t), e^{-t})$ the trajectory of system (2.5) where $\xi = -1$. Observe that the ω -limit set of the trajectory $\bar{X}(t)$ must belong to the plane $x_3 = 0$. Furthermore the dynamics in this plane is the one of the autonomous system (2.2) where $\phi(t) \equiv K(\infty)$. It follows that the ω -limit set of $\bar{X}(t)$ is the periodic trajectory of the plane $x_3 = 0$ corresponding to the level set $H = L$ of the function H . This periodic trajectory corresponds to a S.G.S. with slow decay $\bar{v}(r)$ of the autonomous system. Therefore $\bar{u}(r)$ has slow decay and $\lim_{r \rightarrow \infty} (\bar{u}(r) - \bar{v}(r))r^\alpha = 0$. The other claim can be proved setting $\xi = 1$ in (2.5) and reasoning in the same way. $\square$

We assume now that $K(r)$ is monotone for r small and for r large. We consider again a solution $u(r)$ which is positive and decreasing for r small (respectively for r large) and the corresponding trajectory $x(t)$ of (2.2). From (2.7) we easily deduce that the limit $\lim_{t \rightarrow -\infty} H(x_1(t), y_1(t), t) = l$ (respectively $\lim_{t \rightarrow +\infty} H(x_1(t), y_1(t), t) = L$) exists; moreover $l \leq 0$ (resp. $L \leq 0$). In fact from Lemma 2.4 we get that, when $l > 0$, $u(r)$ cannot correspond to a positive solution. So we distinguish two cases: $l < 0$ (resp. $L < 0$), discussed in Proposition 2.5, and $l = 0$ (resp. $L = 0$), analyzed below. In fact, when $l = 0$ (resp. $L = 0$) we have two possibilities: either the α -limit set (resp. the ω -limit set) is the origin, or it is the homoclinic trajectory of the autonomous system (2.2) where $\phi \equiv K(0)$ (resp. of the autonomous system (2.2) where $\phi \equiv \lim_{r \rightarrow \infty} K(\infty)$). The former case is discussed in Proposition 2.6, the latter in Remark 2.7.

Proposition 2.6. *Consider a positive solution $u(r)$ of (1.2) and the corresponding trajectory $x(t)$ of (2.2). Then $u(r)$ is a regular solution if and only if $x(t)$ has the origin as α -limit set. Moreover $u(r)$ has fast decay if and only if $x(t)$ has the origin as ω -limit set.*

Proof. The proof concerning regular solutions follows from the ideas suggested in [21] Theorem 5.2, and developed in details in [10]. From [10] we also get that if $\lim_{t \rightarrow +\infty} x(t) = (0, 0)$ then $u(r)r^{(n-p)/(p-1)}$ is bounded, and viceversa; so we just need to show that $u(r)r^{(n-p)/(p-1)}$ admits a positive limit as $r \rightarrow +\infty$.

Note that $K(r)u^{\sigma-1}(r)r^{n-1} \in \mathcal{L}^1([1, \infty))$. From (1.2) it follows that, for any R , r large enough, $R > r$, we have

$$|u'(R)R^{(n-1)/(p-1)}|^{p-1} = |u'(r)r^{(n-1)/(p-1)}| + \int_r^R K(s)u^{\sigma-1}(s)s^{n-1} ds$$

Since $u(r)$ is positive for r large we have that the right hand side is positive and increasing in R , so there exists the limit $\lim_{R \rightarrow +\infty} |u'(R)R^{(n-1)/(p-1)}| = \lambda > 0$.

Then, using De l'Hospital rule, we get that $\lim_{R \rightarrow +\infty} |u(R)R^{(n-p)/(p-1)}| = \lambda(p-1)/(n-p) > 0$. $\square$

Remark 2.7. Consider a positive solution $u(r)$ of (1.2) and the corresponding trajectory $x(t)$ of (2.2). If $\lim_{t \rightarrow -\infty} H(x_1(t), y_1(t), t) = 0$, but $x(t)$ does not converge to the origin, then $\lim_{r \rightarrow 0} u(r) = +\infty$, and

$$(2.9) \quad 0 = \liminf_{r \rightarrow 0} u(r)r^{(n-p)/p} < \limsup_{r \rightarrow 0} u(r)r^{(n-p)/p} < +\infty.$$

So $u(r)$ is a singular solution but there is a subsequence $r_n \rightarrow 0$, such that $u(r_n)$ grows slower than the singular solutions described in Proposition 2.5.

Analogously if $\lim_{t \rightarrow +\infty} H(x_1(t), y_1(t), t) = 0$, but $x(t)$ does not converge to the origin, then $\lim_{r \rightarrow \infty} u(r)r^{(n-p)/(p-1)} = +\infty$ so $u(r)$ has slow decay, but

$$(2.10) \quad 0 = \liminf_{r \rightarrow +\infty} u(r)r^{(n-p)/p} < \limsup_{r \rightarrow +\infty} u(r)r^{(n-p)/p} < +\infty.$$

So there is a subsequence $r_n \rightarrow +\infty$, such that $u(r_n)$ tends to 0 faster than the singular solutions described in Proposition 2.5.

Proof. Assume that $\lim_{t \rightarrow -\infty} H(x_1(t), y_1(t), t) = 0$, but $x(t) \not\rightarrow (0, 0)$. Then the α -limit set of $x(t)$ is the homoclinic trajectory of the autonomous system (2.2) where $\phi \equiv K(0)$, so (2.9) easily follows. Since $u'(r) < 0$ for r small, the limit $\lim_{r \rightarrow 0} u(r)$ exists; hence from (2.9) we get $\lim_{r \rightarrow 0} u(r) = +\infty$. To prove the second claim observe that the limit $\lim_{r \rightarrow \infty} u(r)r^{(n-p)/(p-1)}$ exists, and conclude reasoning as above. $\square$

Now we want to get some information about the shape of the first branches of the manifolds $W^u(\tau)$ and $W^s(\tau)$. We recall that $M = \sup_{t \in \mathbf{R}} \phi(t)$ and $m = \inf_{t \in \mathbf{R}} \phi(t)$, and that $U^m(\tau; t) = (U_1^m(\tau; t), U_2^m(\tau; t))$ and $U^M(\tau; t) = (U_1^M(\tau; t), U_2^M(\tau; t))$ are the homoclinic trajectories of system (2.3) where $\phi \equiv m$ and $\phi \equiv M$, such that $\dot{U}_1^m(\tau; 0) = \dot{U}_1^M(\tau; 0)$.

Let us consider now the non-autonomous system (2.3): the curves $U^m(\tau, t)$ and $U^M(\tau, t)$ are not anymore trajectories. We call ∂E the union of these two curves and by E the set delimited by the origin and ∂E ; moreover we define $E^+ = E \cap U^+$, $E^- = E \cap U^-$, $\partial E^+ = \partial E \cap U^+$, $\partial E^- = \partial E \cap U^-$. We claim that $W^u(\tau)$ and $W^s(\tau)$ intersect the isocline $\dot{x} = 0$, see Lemma 2.8 below. Follow the manifold $W^u(\tau)$ (respectively $W^s(\tau)$) from the origin towards $\mathbf{R}_+^2$: we denote by $\tilde{Q}^u(\tau)$ (resp. $\tilde{P}^s(\tau)$) the first intersection of $W^u(\tau)$ (resp. $W^s(\tau)$) with the isocline $\dot{x} = 0$. In [11] it is shown that if K is either a regular or a singular perturbation of a constant (that is respectively $K(r) = 1 + \varepsilon k(r)$ and $K(r) = k(r^\varepsilon)$, with $\varepsilon > 0$ small and $k \in C^2$ and bounded), these intersections are transversal: this might not be the case in this setting. We denote by $\tilde{W}^u(\tau)$

Fig. 2. Sketch of $\bar{W}^u(\tau)$ and $\tilde{W}^u(\tau)$.

the branch of $W^u(\tau)$ between the origin and $\tilde{Q}^u(\tau)$ and by $\tilde{W}^s(\tau)$ the branch of $W^s(\tau)$ between the origin and $\tilde{P}^s(\tau)$. Finally we will use the following notation: $x(\tau, \tilde{P}^u(\tau); t) = (x_1^u(\tau; t), x_2^u(\tau; t))$ and $x(\tau, \tilde{P}^s(\tau); t) = (x_1^s(\tau; t), x_2^s(\tau; t))$. We give the following definitions:

$$\xi^u(\tau) := \{Q \in W^u(\tau) \cap U^0 \mid x(\tau, Q; t) \in E^+ \text{ for any } t < 0\}$$

$$\xi^s(\tau) := \{Q \in W^s(\tau) \cap U^0 \mid x(\tau, Q; t) \in E^- \text{ for any } t > 0\}.$$

Now we can state the following result.

Lemma 2.8. *The sets $\xi^u(\tau)$ and $\xi^s(\tau)$ are non empty for any τ . The manifold $\tilde{W}^u(\tau)$ is contained in E^+ and $\tilde{W}^s(\tau)$ is contained in E^- .*

Proof. Let us denote by $a^+ = \{U^M(\tau; t) \mid t \leq 0\}$, $b^+ = \{U^m(\tau; t) \mid t \leq 0\}$, $a^- = \{U^M(\tau; t) \mid t \geq 0\}$, $b^- = \{U^m(\tau; t) \mid t \geq 0\}$, $A = U^M(\tau; 0)$, $B = U^m(\tau; 0)$, and finally by $\bar{c}$ the segment of the isocline U^0 between A and B . Reasoning as in Lemma 2.3 we see that the flow on ∂E^+ points towards the interior of E^+ , for any $t \in \mathbf{R}$. Choose $Q \in \bar{c}$; we denote by $t = T(Q)$ the largest nonpositive value (if it exists) for which the trajectory $x(\tau, Q; t)$ crosses ∂E^+ and leaves E^+ .

Then we construct the sets

$$\mathfrak{A} := \{Q \in \bar{c} \mid x(\tau, Q; T(Q)) \in a^+\} \quad \mathfrak{B} := \{Q \in \bar{c} \mid x(\tau, Q; T(Q)) \in b^+\}.$$

Note that $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$; moreover, from the continuity of the flow of (2.2) it follows that $\mathfrak{A}$ and $\mathfrak{B}$ are open in $\bar{c}$. Let $\mathfrak{C}$ denote the border of $\mathfrak{B}$ in $\bar{c}$, and $Q \in \bar{c}$. Since there are no critical points in $\bar{c}$, using a continuity argument we find that $x(\tau, Q; T(Q))$ lies on the border of b^+ . Hence $T(q) = -\infty$, $x(\tau, Q; T(Q)) = (0, 0)$ and $Q \in \xi^u(\tau)$; thus $\mathfrak{C} \subset \xi^u(\tau)$ and $\xi^u(\tau) \neq \emptyset$. The proof concerning $\xi^s(\tau)$ can be easily obtained observing that the flow on ∂E^- points towards the exterior of E^- for any $t \in \mathbf{R}$ and reasoning as above.

Since the manifold $W^u(\tau)$ and $W^s(\tau)$ are constructed from $W_{loc}^u(\tau)$ and $W_{loc}^s(\tau)$ using the flow of (2.2), it follows that $\tilde{W}^u(\tau)$ and $\tilde{W}^s(\tau)$ are contained respectively in E^+ and in E^- . $\square$

Observe now that for any $Q \in \tilde{W}^u(\tau)$ and any $P \in \tilde{W}^s(\tau)$ we have $\lim_{t \rightarrow -\infty} H(x(\tau, Q; t), \tau) = 0$ and $\lim_{t \rightarrow +\infty} H(x(\tau, P; t), \tau) = 0$; thus

$$(2.11) \quad \begin{aligned} H(Q, \tau) &= \int_{-\infty}^0 \dot{\phi}(t + \tau) \frac{|x_1(\tau, Q; t)|^\sigma}{\sigma} dt \\ H(P, \tau) &= - \int_0^{\infty} \dot{\phi}(t + \tau) \frac{|x_1(\tau, P; t)|^\sigma}{\sigma} dt, \end{aligned}$$

In the following section we need also some exponential estimates for the trajectories $x^u(\tau; t)$ and $x^s(\tau; t)$. We recall that $U^M(0) = A = (A_x, A_y)$. Let us denote by B^+ and B^- the intersections of the line $x = A_x$ respectively with b^+ and b^- . Let L^+ and L^- be the segments delimited respectively by B^+ and A and by A and B^- . It is possible to compute explicitly two values $\mathfrak{T}^+ < 0 < \mathfrak{T}^-$, independent of τ , such that $U^m(\tau; \mathfrak{T}^+) = B^+$ and $U^m(\tau; \mathfrak{T}^-) = B^-$. It follows that $U^m(\tau, B^+; t) := U^m(\tau + \mathfrak{T}^+; t)$ passes through B^+ at $t = 0$, and $U^m(\tau, B^-; t) := U^m(\tau + \mathfrak{T}^-; t)$ passes through B^- at $t = 0$. Therefore, from (2.8) we can find $C > c > 0$ such that

$$(2.12) \quad \begin{aligned} \sqrt[p]{c} e^{(n-p)t/p} &\leq U_1^m(\tau, B^+; t) \leq \sqrt[p]{C} e^{(n-p)t/p} && \text{for } t \leq 0 \\ \sqrt[p]{C} e^{-(n-p)t/(p-1)} &\leq U_1^m(\tau, B^-; t) \leq \sqrt[p]{c} e^{-(n-p)t/(p-1)} && \text{for } t \geq 0 \end{aligned}$$

Follow $\tilde{W}^u(\tau)$ (respectively $\tilde{W}^s(\tau)$) from the origin towards $\mathbf{R}_+^2$ and denote by $\bar{Q}^u(\tau)$ (resp. $\bar{P}^s(\tau)$) the first intersection of $\tilde{W}^u(\tau)$ with L^+ (resp. $\tilde{W}^s(\tau)$ with L^-). Then we denote by $\bar{W}^u(\tau)$ the branch of $W^u(\tau)$ between the origin and $\bar{Q}^u(\tau)$ and by $\bar{W}^s(\tau)$ the branch of $W^s(\tau)$ between the origin and $\bar{P}^s(\tau)$. From Lemma 2.8 we easily get that $\bar{W}^u(\tau) \subset \tilde{W}^u(\tau)$ is contained in E^+ and $\bar{W}^s(\tau) \subset \tilde{W}^s(\tau)$ is contained in E^- , for any τ .

In the next section we need to evaluate $H(Q, \tau)$ for any point Q belonging to $\bar{W}^u(\tau)$ and to $\bar{W}^s(\tau)$. For this purpose we need an estimate on the trajectories $x(\tau, \bar{Q}^u(\tau); t)$ and $x(\tau, \bar{P}^s(\tau); t)$ of the non autonomous system (2.2). We claim that

$$(2.13) \quad \sqrt[p]{ce^{(n-p)t/p}} \leq x_1(\tau, \bar{Q}^u(\tau); t) \leq \sqrt[p]{Ce^{(n-p)t/p}} \quad \text{for } t \leq 0$$

$$\sqrt[p]{ce^{-(n-p)t/(p-1)}} \leq x_1(\tau, \bar{P}^s(\tau); t) \leq \sqrt[p]{Ce^{-(n-p)t/(p-1)}} \quad \text{for } t \geq 0$$

We just prove the first estimate since the second is similar. We want to show that $U_1^m(\tau, B^+; t) \leq x_1(\tau, \bar{Q}^u(\tau); t) \leq U_1^M(\tau; t)$ for any $t \leq 0$: then (2.13) follows from (2.12). Let us denote by $T = \inf\{t \leq 0 \mid U_1^m(\tau, B^+; t) \leq x_1(\tau, \bar{Q}^u(\tau); t)\}$. Assume for contradiction that T is finite. Then $U_1^m(\tau, B^+; T) = x_1(\tau, \bar{Q}^u(\tau); T)$ and $\dot{U}_1^m(\tau, B^+; T) < \dot{x}_1(\tau, \bar{Q}^u(\tau); T)$. Hence, from (2.2) we get $U_2^m(\tau, B^+; T) < x_2(\tau, \bar{Q}^u(\tau); T)$, a contradiction; so $T = -\infty$. Reasoning similarly we find $x_1(\tau, \bar{Q}^u(\tau); t) \leq U_1^M(\tau; t)$ for any $t \leq 0$. So (2.13) follows.

Now fix any point $Q \in \bar{W}^u(\tau)$ and $P \in \bar{W}^s(\tau)$; denote by $t = \bar{T}^u(Q) > 0$ and $t = \bar{T}^s(P) < 0$ respectively the smallest positive value such that $x(\tau, Q; t)$ crosses L^+ and the largest negative value such that $x(\tau, P; t)$ crosses L^- . Then $x(\tau, Q; t + \bar{T}^u(Q)) \equiv x(\tau + \bar{T}^u(Q), \bar{Q}^u(\tau + \bar{T}^u(Q)); t)$ and $x(\tau, P; t + \bar{T}^s(P)) \equiv x(\tau + \bar{T}^s(P), \bar{P}^s(\tau + \bar{T}^s(P)); t)$, for any t . Hence, for any $Q \in \bar{W}^u(\tau)$ and $P \in \bar{W}^s(\tau)$, from (2.13) we get the following result:

$$(2.14) \quad \sqrt[p]{ce^{(n-p)(t-\bar{T}^u(Q))/p}} \leq x_1(\tau, Q; t) \leq \sqrt[p]{Ce^{(n-p)(t-\bar{T}^u(Q))/p}} \quad \text{for } t \leq 0,$$

$$\sqrt[p]{ce^{-(n-p)(t-\bar{T}^s(P))/(p-1)}} \leq x_1(\tau, P; t) \leq \sqrt[p]{Ce^{-(n-p)(t-\bar{T}^s(P))/(p-1)}} \quad \text{for } t \geq 0,$$

where $C > c > 0$ are the constants defined in (2.13).

The estimate $U_1^m(\tau + \mathfrak{T}^+; t) \leq x_1(\tau, \bar{Q}^u(\tau); t) \leq U_1^M(\tau; t)$, helps us to explore further also the relationship between solutions of (1.2) and trajectories of (2.2). First we observe that local uniqueness and continuous dependence on initial data for regular solution of (1.2) is ensured. This has been proved in Appendix A of [12] in the spatial independent case, but the proof can be trivially extended to the spatial dependent case, see also [21]. So we can denote by $u(d; r)$ the regular solution of (1.2) such that $u(d; 0) = d$. Then we easily get the following.

Lemma 2.9. *For any $\tau \in \mathbf{R}$ there is a unique $d(\tau) > 0$ such that the trajectory $x(\tau, \bar{Q}^u(\tau); t)$ of (2.2) correspond to the solution $u(d(\tau), r)$ of (1.2). Moreover the function $d(\tau) : \mathbf{R} \rightarrow (0, \infty)$ is bijective and continuous and $d(\tau) \rightarrow +\infty$ as $\tau \rightarrow -\infty$ and viceversa, and $d(\tau) \rightarrow 0$ as $\tau \rightarrow +\infty$ and viceversa.*

Proof. The fact that the function $d(\tau)$ is well defined, follows from local uniqueness for the problem (1.2) with initial data $u(0) = d$ and $u'(0) = 0$. Since

the trajectory $x(\tau, \bar{Q}^u(\tau); t)$ is uniquely defined for any τ , we see that $d(\tau)$ is bijective. The asymptotic estimate follows from (2.14). $\square$

3. Oscillatory potentials

Now we turn to consider potentials $K(r)$ which are monotone for r large and for r small. As usual we always assume that H1 and H2 are satisfied. First of all we need the following Lemma:

Lemma 3.1. • Assume that hypothesis α^- is satisfied, then there is N_α^- such that $H(P, \tau) > 0$, for any $P \in \bar{W}^s(\tau)$ and any $\tau < N_\alpha^-$.

Analogously assume that hypothesis α^+ is satisfied, then there is N_α^+ such that $H(P, \tau) < 0$, for any $P \in \bar{W}^s(\tau)$ and any $\tau < N_\alpha^+$.

• Assume that hypothesis Ω^- is satisfied, then there is N_ω^- such that $H(Q, \tau) < 0$, for any $Q \in \bar{W}^u(\tau)$ and any $\tau > N_\omega^-$.

Assume that hypothesis Ω^+ is satisfied, then there is N_ω^+ such that $H(Q, \tau) > 0$ for any $Q \in \bar{W}^u(\tau)$ and any $\tau > N_\omega^+$.

Proof. Assume that α^- is satisfied; then $\dot{\phi}(t) < 0$ for $t < t_0 = \ln(\rho)$. Integrating by parts, for any $P \in \bar{W}^s(\tau)$, we get

$$\begin{aligned} H(P, \tau) &= - \int_0^{+\infty} \dot{\phi}(\tau + t) \frac{|x_1(\tau, P; t)|^\sigma}{\sigma} dt \\ &= - \int_{t_0}^{+\infty} \dot{\phi}(\zeta) \frac{|x_1(\tau, P; \zeta - \tau)|^\sigma}{\sigma} d\zeta - \int_\tau^{t_0} \dot{\phi}(\zeta) \frac{|x_1(\tau, P; \zeta - \tau)|^\sigma}{\sigma} d\zeta \\ &= \frac{\phi(t_0)}{\sigma} |x_1(\tau, P; t_0 - \tau)|^\sigma + \int_{t_0}^{+\infty} \phi(\zeta) \frac{d}{d\zeta} \left[\frac{|x_1(\tau, P; \zeta - \tau)|^\sigma}{\sigma} \right] d\zeta \\ &\quad - \int_\tau^{t_0} \frac{\dot{\phi}(\zeta)}{\sigma} |x_1(\tau, P; \zeta - \tau)|^\sigma d\zeta \\ &\geq \frac{\phi(t_0) - M}{\sigma} |x_1(\tau, P; t_0 - \tau)|^\sigma - \int_\tau^{t_0} \frac{\dot{\phi}(\zeta)}{\sigma} |x_1(\tau, P; \zeta - \tau)|^\sigma d\zeta \end{aligned}$$

Then, exploiting (2.14) we get

$$(3.1) \quad \begin{aligned} H(P, \tau) &\geq \frac{e^{np(\tau - \bar{T}^s(P))/(p-1)}}{\sigma} \left[C[K(\rho) - M] \rho^{-np/(p-1)} \right. \\ &\quad \left. - c \int_\tau^{t_0} \dot{\phi}(\zeta) e^{-np\zeta/(p-1)} d\zeta \right] \end{aligned}$$

Since $\int_{-\infty}^0 \dot{\phi}(\zeta) e^{-np\zeta/(p-1)} d\zeta = -\infty$, it follows that there is $N_\alpha^+ < t_0$ such that $H(P, \tau) > 0$, for any $\tau < N_\alpha^+$ and $P \in \bar{W}^s(\tau)$, so the claim is proved.

Now assume that Ω^+ is satisfied. Then $\dot{\phi}(t) > 0$ for any $t > T_0 = \ln(R)$; therefore, reasoning as above we find

$$\begin{aligned}
 (3.2) \quad H(Q, \tau) &= \int_{-\infty}^0 \dot{\phi}(\tau+t) \frac{|x_1(\tau, Q; t)|^\sigma}{\sigma} dt \\
 &\geq \frac{\phi(T_0) - M}{\sigma} |x_1(\tau, Q; T_0 - \tau)|^\sigma + \int_{T_0}^\tau \frac{\dot{\phi}(\zeta)}{\sigma} |x_1(\tau, Q; \zeta - \tau)|^\sigma d\zeta \\
 &\geq \frac{e^{-n(\tau - \tilde{T}^u(Q))}}{\sigma} \left[C(K(R) - M)R^n + c \int_{T_0}^\tau \dot{\phi}(\zeta) e^{n\zeta} d\zeta \right]
 \end{aligned}$$

Once again, since $\dot{\phi}(\zeta) e^{n\zeta} \notin \mathcal{L}^1[[0, \infty))$, we can find $N_\omega^+ > 0$ such that $H(Q, \tau) > 0$ for any $Q \in \tilde{W}^u(\tau)$ and any $\tau > N_\omega^+$, so the claim is proved. The other claims can be proved reasoning in the same way. $\square$

Now we are ready to prove the following key result.

Lemma 3.2. *Let $\tilde{Q}$ be an arbitrary point in $\tilde{W}^u(\tau)$ and $\tilde{P}$ an arbitrary point in $\tilde{W}^s(\tau)$.*

- *Assume that hypothesis α^- is satisfied, then there is T_α^- such that, for any $\tau < T_\alpha^-$, we have $H(\tilde{Q}, \tau) < 0 < H(\tilde{P}, \tau)$.*

- *Assume that hypothesis α^+ is satisfied, then there is T_α^+ such that, for any $\tau < T_\alpha^+$, we have $H(\tilde{P}, \tau) < 0 < H(\tilde{Q}, \tau)$.*

- *Assume that hypothesis Ω^- is satisfied, then there is T_ω^- such that, for any $\tau > T_\omega^-$, we have $H(\tilde{Q}, \tau) < 0 < H(\tilde{P}, \tau)$.*

- *Assume that hypothesis Ω^+ is satisfied, then there is T_ω^+ such that, for any $\tau > T_\omega^+$, we have $H(\tilde{P}, \tau) < 0 < H(\tilde{Q}, \tau)$.*

Proof. Assume that Ω^+ is satisfied, so that $\dot{\phi}(t) > 0$ for any $t > T_0 = \ln(R)$. For any $\tilde{P} \in \tilde{W}^s(\tau)$ where $\tau > T_0$ we have

$$H(\tilde{P}, \tau) = - \int_0^{+\infty} \dot{\phi}(\tau+t) \frac{|x_1(\tau, \tilde{P}; t)|^\sigma}{\sigma} dt < 0.$$

The estimate of the points $H(\tilde{Q}, \tau)$ for $\tilde{Q} \in \tilde{W}^u(\tau)$ is more delicate. For any $\bar{Q} \in \bar{W}^u(\tau)$, we denote by $\tilde{T}^u(\bar{Q})$ the unique positive value such that $x(\tau, \bar{Q}; t) \in E^+$ for any $t < \tilde{T}^u(\bar{Q})$ and $x(\tau, \bar{Q}; \tilde{T}^u(\bar{Q})) \in \xi^u(T^u(\bar{Q}) + \tau)$. A priori we can have some points $\bar{Q}$ such that $x(\tau, \bar{Q}; \tilde{T}^u(\bar{Q})) \in E^+$ for any $t > 0$ and the trajectory converges to the critical point $P(\infty)$ of the autonomous system (2.2) where $\phi(t) \equiv K(\infty)$: in such a case we set $\tilde{T}^u(\bar{Q}) = +\infty$. Let us denote by $\bar{x}^\infty(\tau, \bar{Q}; t)$ the trajectory of the autonomous system (2.2) where $\phi(t) \equiv K(\infty)$ departing from $\bar{Q}$ at $t = \tau$. We can choose N large enough so that $\tilde{T}^u(\bar{Q}) < +\infty$ for any $\tau > N$. This follows from the fact that, for τ large the flow of the

non-autonomous system is close to the flow of the autonomous system (2.2) where $\phi(t) \equiv K(\infty)$, and the fact that the graphs of the trajectories $\bar{x}^\infty(\tau, \bar{Q}, t)$ are uniformly far from the critical point $P(\infty)$.

From Lemma 3.1 we know that there is $N_\omega^+ > T_0$ such that $H(\bar{Q}, \tau) > 0$, for any $\tau \geq N_\omega^+ > T_0$ and any $\bar{Q} \in \bar{W}^u(\tau)$. We can assume without loss of generality that $N_\omega^+ > N$. We denote by $\Phi_\tau(t, \cdot)$ the diffeomorphism defined by the flow of (2.2). For any $\bar{Q} \in \bar{W}^u(\tau)$, where $\tau > N_\omega^+$, and any $t \geq 0$, we have

$$H(\Phi_\tau(t, \bar{Q}), t + \tau) = H(\bar{Q}, \tau) + \int_0^t \dot{\phi}(\tau + s) \frac{|x_1(\tau, \bar{Q}; s)|^\sigma}{\sigma} ds > H(\bar{Q}, \tau) > 0,$$

since $\dot{\phi}(s) > 0$ for $s > \tau > N_\omega^+$. We choose $T_\omega^+ = \inf\{\tilde{T}^u(\bar{Q}) + N_\omega^+ \mid \bar{Q} \in \bar{W}^u(N_\omega^+)\}$: since $N_\omega^+ > N$ this infimum is finite and it is attained in some point $\bar{Q} \in \bar{W}^u(N_\omega^+)$; it follows that $\Phi_{N_\omega^+}[t, \bar{W}^u(N_\omega^+)] \supset \tilde{W}^u(N_\omega^+ + t)$, for any $t \geq T_\omega^+ - N_\omega^+$. Hence $H(\bar{Q}, \tau) > 0$ for any $\bar{Q} \in \tilde{W}^u(\tau)$ for any $\tau > T_\omega^+$, so the claim is proved.

Now assume that α^- is satisfied, so that $\dot{\phi}(t) < 0$ for any $t < t_0$. For any $\bar{Q} \in \tilde{W}^u(\tau)$, from (2.11) we have

$$H(\bar{Q}, \tau) = \int_{-\infty}^0 \dot{\phi}(\tau + t) \frac{|x_1(\tau, \bar{Q}; t)|^\sigma}{\sigma} dt < 0$$

for any $\tau < t_0 = \ln(\rho)$.

From Lemma 3.1 we know that there is $N_\alpha^- < t_0$ such that $H(\bar{P}, \tau) > 0$, for any $\tau \leq N_\alpha^-$ and any $\bar{P} \in \bar{W}^s(\tau)$. Observe that there is a unique $t = \tilde{T}^s(\bar{P}) < 0$ such that $x(\tau, \bar{P}(\tau); t) \in E^-$ for any $t > \tilde{T}^s(\bar{P}(\tau))$ and $x(\tau, \bar{P}(\tau); \tilde{T}^s(\bar{P})) \in \xi^s(\tilde{T}^s(\bar{P}) + \tau)$; reasoning as above we can show that, possibly choosing a larger $|N_\alpha^-|$, $\tilde{T}^s(\bar{P}) > -\infty$ for any $\bar{P} \in \bar{W}^s(\tau)$. Note that for any $\bar{P} \in \bar{W}^s(\tau)$, where $\tau < N_\alpha^-$, and any $t \leq 0$, we have

$$H(\Phi_\tau(t, \bar{P}), t + \tau) = H(\bar{P}, \tau) - \int_0^t \dot{\phi}(\tau + t) \frac{|x_1(\tau, \bar{P}; s)|^\sigma}{\sigma} ds > H(\bar{P}, \tau) > 0.$$

Let us denote by $T_\alpha^- = \max\{\tilde{T}^s(\bar{P}) + N_\alpha^- \mid \bar{P} \in \bar{W}^s(N_\alpha^-)\}$; it follows that $\Phi_{N_\alpha^-}[t, \bar{W}^s(N_\alpha^-)] \supset \tilde{W}^s(N_\alpha^- + t)$, for any $t \leq T_\alpha^- - N_\alpha^-$. Hence $H(\bar{Q}, \tau) > 0$ for any $\bar{Q} \in \tilde{W}^s(\tau)$ for any $\tau < T_\alpha^-$, so the claim is proved.

The other claims can be proved reasoning in the same way. $\square$

We need to introduce the following surface for system (2.4)

$$S := \{(x_1, x_2, x_3) \mid H(x_1, x_2, x_3) = 0 \text{ and } (x_1, x_2) \in \mathbf{R}_+^2\}.$$

Note that, for any τ such that $\phi(\tau) > 0$ we have that $S(\tau) = S \cap \{(x_1, x_2, x_3) \mid x_3 = \tau\}$ is a closed bounded curve and its diameter has a uniform lower bound, see figure 1. Now we are ready to state and prove the first important result of

the paper; even if it will be completed afterwards, it is interesting in itself. In fact the part concerning singular solutions is new even for the Laplacian.

Theorem 3.3. *Consider Eq. (1.2) and assume that H1 and H2 are satisfied.*

- *Assume that hypothesis α^+ is satisfied. Then there are uncountably many S.G.S. with fast decay $w(r)$ such that $w(r) \sim r^{(n-p)/p}$ as $r \rightarrow 0$.*
- *Assume that hypothesis α^- is satisfied. Then there are uncountably many Dirichlet solutions $u(r)$ in the exterior of balls.*
- *Assume that hypothesis Ω^+ is satisfied. Then there are uncountably many crossing solutions.*
- *Assume that hypothesis Ω^- is satisfied. Then there are uncountably many G.S. with slow decay $u(r)$ such that $u(r) \sim r^{(n-p)/p}$ as $r \rightarrow \infty$.*

Proof. Let P and Q be arbitrary points respectively in $\tilde{W}^s(\tau)$ and $\tilde{W}^u(\tau)$. Assume that hypothesis α^+ is satisfied; then, recalling Lemma 3.2 for any $\tau < T_\alpha^+$ we have that $H(P, \tau) < 0$, and $\dot{\phi}(\tau + t) > 0$ for any $t < 0$. Therefore we have

$$H(x(\tau, P; t), t) = \int_0^t \dot{\phi}(\tau + \zeta) \frac{|x_1(\tau, P; \zeta)|^\sigma}{\sigma} d\zeta + H(P, \tau) < H(P, \tau) < 0,$$

for any $\tau < T_\alpha^+$ and for any $t < 0$. Thus the trajectory $(x(\tau, P; t), \tau + t)$ of system (2.4) is forced to stay in the set enclosed by the surface S , so $x(\tau, P; t) \in R_+^2$ for any t . Since $\lim_{t \rightarrow -\infty} H(x(\tau, P; t), t) < 0$, from Proposition 2.5 we deduce that the corresponding solution $v(r)$ of (1.2) is a S.G.S. with slow decay, and $v(r) \sim r^{-\alpha}$ as $r \rightarrow 0$.

Now assume that hypothesis α^- is satisfied. Reasoning as above we can show that $\lim_{t \rightarrow -\infty} H(x(\tau, P; t), t) > 0$ for any $\tau < T_\alpha^-$. Therefore the corresponding solution $v(r)$ of (1.2) is a solution of the Dirichlet problem in the exterior of a ball, see Lemma 2.4. Reasoning in the same way we can see that, if hypothesis Ω^- is satisfied, $\lim_{t \rightarrow +\infty} H(x(\tau, Q; t), t) < 0$ for any $\tau > T_\omega^-$, while if hypothesis Ω^+ is satisfied, $\lim_{t \rightarrow +\infty} H(x(\tau, Q; t), t) > 0$ for any $\tau > T_\omega^-$. In the former case the corresponding solution $u(r)$ of (1.2) is a G.S. with slow decay, while in the latter case $u(r)$ is a crossing solution, see again Lemma 3.2, Proposition 2.5 and Lemma 2.4. $\square$

Remark 3.4. The proof of this and of the other Theorems of this paper work even if $K(r)$ is only locally Lipschitz. In this case we should replace the derivative $\dot{\phi}$ in the integral expressions containing it with the weak derivative. It is also possible to rewrite the expressions, used for example in (3.1), without using the term $\dot{\phi}$, simply integrating by parts, as it is done in [22]. However this computation is beyond the purpose of this analysis so it is left to the interested reader.

We denote by $R_1(d)$ the first positive zero of $u(d, r)$, setting $R_1(d) = +\infty$ if $u(d, r) > 0$ for any $r \geq 0$.

Corollary 3.5. • Assume that Hypothesis α^- is satisfied. Then there is $\rho^- > 0$ such that, for any $0 < R < \rho^-$ there is a solution of the Dirichlet problem in exterior of the ball of radius R .

• Assume that Hypothesis Ω^+ is satisfied. Then there is $\delta^+ > 0$, such that $u(d, r)$ is a crossing solution for each $0 < d < \delta^+$. Moreover $R_1(d)$ is well defined and continuous for $d \in (0, \delta^+)$ and $\lim_{d \rightarrow 0} R_1(d) = +\infty$. Assume that Hypothesis Ω^- is satisfied. Then there is $\delta^- > 0$, such that $u(d, r)$ is a G.S. with slow decay such that $u(d, r) \sim r^{-(n-p)/p}$ as $r \rightarrow \infty$, for each $0 < d < \delta^-$.

Proof. Assume that α^- is satisfied, then from Lemma 3.2 we know that there is T_α^- such that $H(\tilde{P}^s(\tau), \tau) > 0$ for any $\tau < T_\alpha^-$; moreover $\dot{\phi}(\tau + t)$ is decreasing for $t < 0$. It follows that $\lim_{t \rightarrow -\infty} H(x^s(\tau; t), \tau + t) > 0$, for any $\tau < T_\alpha^-$. So from Lemma 2.4, we deduce that there is $T^1(\tau) < 0$ such that $x^s(\tau; t)$ crosses the x_2 positive semi-axis at $t = T^1(\tau)$, it is in $\mathbf{R}_+^2$ for any $t > T^1(\tau)$ and it converges to the origin as $t \rightarrow +\infty$. So the corresponding solutions $u(r)$ of (1.2) are such that $u(e^{\tau+T^1(\tau)}) = 0$, $u(r) > 0$ for $r > e^{\tau+T^1(\tau)}$ and they have fast decay. Let us denote by $\rho^- = \max\{e^{\tau+T^1(\tau)} \mid \tau < T_\alpha^-\}$; from the continuity of the flow of (2.2) it follows that $T^1(\tau)$ varies continuously with respect to τ , so the first claim follows.

Now assume that Ω^+ is satisfied, then from Lemma 3.1 we know that $H(\bar{Q}^u(\tau), \tau) > 0$ for any $\tau > T_\omega^+$. Since $\dot{\phi}(t) > 0$ for $t > T_\omega^+$, we have that $\lim_{t \rightarrow +\infty} H(x(\tau, \bar{Q}^u(\tau); t), \tau + t) > 0$; so, from Lemma 2.4, we deduce that there is $T^2(\tau) > 0$ such that $x(\tau, \bar{Q}^u(\tau); t) \in \mathbf{R}_+^2$ for $t < T^2(\tau)$ and it crosses the x_2 negative semi-axis for $t = T^2(\tau)$. So the corresponding solution $u(d(\tau), r)$ is positive for $0 \leq r < R_1(d) = e^{\tau+T^2(\tau)}$ and becomes null for $r = R_1(d)$.

Using Lemma 2.9 we can find $\delta^+ = \max\{d(\tau) \mid \tau > T_\omega^+\}$ such that $u(d, r)$ is a crossing solution for $0 < d < \delta^+$. Therefore, for any $0 < d < \delta^+$ there is $R_1(d) = e^{\tau+T^2(\tau)} > 0$ such that $u(d, r) > 0$ for $0 \leq r < R_1(d)$. From Lemma 2.9 we know that $\tau(d) \rightarrow +\infty$ as $d \rightarrow 0$, hence $\lim_{d \rightarrow +\infty} R_1(d) = +\infty$, so the claim is proved. The claim concerning hypothesis Ω^- can be proved reasoning in the same way. $\square$

Now we are ready to prove the main Theorem of the paper.

Proof of Theorem 1.2. First we prove the existence of G.S. with fast decay. Assume that α^+ and Ω^- are satisfied; from Lemma 3.2 we know that $H(\hat{Q}^u(\tau), \tau) < 0 < H(\tilde{P}^s(\tau), \tau)$ for any $\tau > T_\omega^+$ and $H(\tilde{P}^s(\tau), \tau) < 0 < H(\bar{Q}^u(\tau), \tau)$ for any $\tau < T_\alpha^-$. We observe that $\hat{Q}^u(\tau)$ and $\tilde{P}^s(\tau)$ may not depend continuously on τ since the crossing between U^0 and $W^u(\tau)$ and

$W^s(\tau)$ respectively may be not transversal. However fix $\tau \in \mathbf{R}$ and consider the functions $T^u : W^u(\tau) \rightarrow \mathbf{R}$, $T^s : W^s(\tau) \rightarrow \mathbf{R}$ defined by

$$T^u(Q) := \sup\{t \mid x(\tau, Q; s) \in U^+ \text{ for any } s < t\}$$

$$T^s(Q) := \inf\{t \mid x(\tau, Q; s) \in U^- \text{ for any } s > t.\}$$

From the continuity of the flow and the fact that $x(\tau, Q^u; t)$ and $x(\tau, Q^s; t)$ cross U^0 transversally at $t = T^u(Q^u)$ and at $t = T^s(Q^s)$, whenever $Q^u \in W^u(\tau)$ and $Q^s \in W^s(\tau)$, it follows that $T^u(Q)$ and $T^s(Q)$ are continuous. Hence $\Gamma^u = \{(x(\tau, Q; T^u(Q)), T^u(Q)) \mid Q \in W^u(\tau)\} \subset U^0 \times \mathbf{R}$ and $\Gamma^s = \{(x(\tau, Q; T^s(Q)), T^s(Q)) \mid Q \in W^s(\tau)\} \subset U^0 \times \mathbf{R}$ are connected. Since for any $\tau \in \mathbf{R}$, $(\tilde{Q}^u(\tau), \tau) \in \Gamma^u$ and $(\tilde{Q}^s(\tau), \tau) \in \Gamma^s$, there is $\tau_0 \in (T_\alpha^-, T_\omega^+)$ such that $(\tilde{Q}^u(\tau_0), \tau_0) = \tilde{Q}^u(\tau_0), \tau_0) \in \Gamma^u \cap \Gamma^s$. So $x^u(\tau_0, \tilde{Q}^u(\tau_0); t)$ coincide with $x^s(\tau_0, \tilde{P}^s(\tau_0); t)$ for any t ; hence, by construction, $x^u(\tau_0, \tilde{Q}^u(\tau_0); t)$ is in E^+ for $t < 0$ and in E^- for $t > 0$. Thus the corresponding $u(r)$ is a decreasing G.S. with fast decay.

Using Corollary 3.5 and Theorem 3.3, we already know the results concerning G.S. with slow decay, crossing solutions, Dirichlet solutions in exterior domains and S.G.S. with fast decay. The only thing left to prove is the existence of S.G.S. with slow decay.

The proof relies on a geometrical analysis of the phase portrait, thus we need to deepen our knowledge of the mutual positions of the manifolds $\tilde{W}^u(\tau)$ and $\tilde{W}^s(\tau)$. Reasoning as done in the proof of Theorem 4.1 in [11], we can construct a bounded subset of the 4th quadrant to which some trajectories of system (2.3) must belong. Since the construction is the same as the one given in [11] we will just sketch it.

Fix τ , we want to construct a closed bounded curve belonging to the 4th quadrant, made up of branches of the manifolds $W^u(\tau)$ and $W^s(\tau)$. We follow $W^s(\tau)$ starting from the origin and we denote by $P_1(\tau)$ its first crossing with $W^u(\tau)$, and by $Z_1(\tau)$ the branch of $W^s(\tau)$ between the origin and $P_1(\tau)$. Then we follow $W^u(\tau)$ towards the origin and we denote by $P_2(\tau)$ its further crossing with $W^s(\tau)$, and by $Z_2(\tau)$ the branch of $W^u(\tau)$ between $P_1(\tau)$ and $P_2(\tau)$. Then we follow $W^s(\tau)$ and we denote by $P_3(\tau)$ its further crossing with $W^u(\tau)$, and by $Z_3(\tau)$ the branch of $W^s(\tau)$ joining $P_2(\tau)$ and $P_3(\tau)$, and so on. Note that we can find also infinitely many intersections. Eventually we will end with a last crossing denoted by $P_\infty(\tau)$. Then we follow $W^u(\tau)$ until we reach the origin. Let us call $Z(\tau)$ the union of all these curves $Z_i(\tau)$ just constructed, and Z the surface of system (2.4) obtained letting τ take values in the whole of $\mathbf{R}$.

Note that, for any τ , $Z(\tau)$ belongs to the 4th quadrant. In fact both $W^u(\tau)$ and $W^s(\tau)$ can cross the axes, but the branches have been chosen in order to have that $Z(\tau)$ belongs to the bounded set delimited by the curve $U^m(t)$ for $t \in \mathbf{R}$, see fig. 3 and 4. A detailed proof of this fact is given in [11];

Fig. 3. Sketch of the curve $Z(\tau)$ when $K'(r)$ admits exactly one critical point which is a maximum.

however it depends on two facts. First $\tilde{W}^u(\tau)$ and $\tilde{W}^s(\tau)$ are contained in E for any τ . Second, choose an intersection $P_k(\tau)$. There is a unique value $\bar{\tau}_k$ such that $P_k(\bar{\tau}_k)$ belongs to the isocline $\dot{x} = 0$. Moreover the sequence $\bar{\tau}_k$ is increasing in k . If $\tau \in (\bar{\tau}_k, \bar{\tau}_{k+1})$ and k is odd or $\tau > \tau_\infty$, $\tilde{Q}^u(\tau)$ is on the left of $\tilde{P}^s(\tau)$, while if k is even or $\tau < \tau_1$ we have the opposite situation: we think of the x_1 axis as horizontal, and of the x_2 axis as vertical. Exploiting this fact we can choose each time between $W^u(\tau)$ and $W^s(\tau)$ the branch which lies in the interior of the 4th quadrant.

Let us denote by $D(\tau)$ the bounded subset delimited by $Z(\tau)$ and by $D = \bigcup_{\tau \in \mathbb{R}} D(\tau) \times \{\tau\}$. Note that there are uncountably many points $Q(\tau) \in D(\tau) \setminus (W^u(\tau) \cup W^s(\tau))$. Let us consider a trajectory $X(\tau + t) = (x(\tau, Q(\tau); t), \tau + t)$ of the extended system (2.4), departing from one of these points.

Observe that it is forced to stay inside D for any t and that $x(\tau, Q(\tau); t)$ cannot converge to the origin, nor as $t \rightarrow -\infty$ neither as $t \rightarrow +\infty$. Thus the corresponding solution $v(r)$ of Eq. (1.2) is a S.G.S. with slow decay.

Observe also that if $\tau > T_\omega^-$ and $Q(\tau)$ is such that $H(Q(\tau), \tau) < 0$, then $\lim_{t \rightarrow +\infty} H(x(\tau, Q(\tau); t), t + \tau) < 0$, so $v(r) \sim r^{-\alpha}$ as $r \rightarrow +\infty$; but we do not know whether $H(x(\tau, Q(\tau); t), t + \tau)$ converges to a negative value or to 0 as $t \rightarrow -\infty$, so we can have either the asymptotic behavior described in Proposition 2.5 or the one described in Remark 2.7 respectively. Analogously if

Fig. 4. Sketch of the curve $Z(\tau)$ when there are 9 intersections between stable and unstable manifolds.

$\tau < T_\alpha^+$ and $Q(\tau)$ is such that $H(Q(\tau), \tau) < 0$, then $\lim_{t \rightarrow -\infty} H(x(\tau, Q(\tau); t), t + \tau) < 0$, so $v(r) \sim r^{-\alpha}$ as $r \rightarrow 0$; but we do not know whether $H(x(\tau, Q(\tau); t), t + \tau)$ converges to a negative value or to 0 as $t \rightarrow +\infty$, so we can have either the asymptotic behavior described in Proposition 2.5 or the one described in Remark 2.7 respectively.

We wish to remark that, exploiting the curve $Z(\tau)$ and this kind of analysis, we could give a different proof of Theorems 3.3 and 1.2, as it is done in [11]. $\square$

We observe now that putting together the results of Theorem 3.3 and Corollary 3.5 we can also prove easily Theorem 1.3.

Let us introduce the function $J^+(r) = \int_0^r K'(s)s^n ds$ defined in [21] and [22], and the following analogous function $J^-(r) = \int_r^\infty K'(s)s^{-np/(p-1)} ds$. Combining the ideas of Theorem 1.2 with the results of Theorem 1 and Proposition 4.2 in [22] we find the following.

Theorem 3.6. *Consider Eq. (1.2) and assume that H1 and H2 hold; moreover assume that the hypotheses α^+ and Ω^- are satisfied. Furthermore*

assume that there exists $R > 0$ such that one of the following hypotheses is satisfied

- $J^+(r) \geq 0$ for any $r < R$ and $K'(r) \leq 0$ for any $r > R$.
- $J^-(r) \leq 0$ for any $r > R$ and $K'(r) \geq 0$ for any $r < R$.

Then positive solutions have a structure of type A. Furthermore we have that there exists $D > 0$ such that $u(D, r)$ is a G.S. with fast decay, each $u(d, r)$ is a crossing solutions for any $d > D$, while $u(d, r)$ is a G.S. with slow decay if $0 < d < D$. Therefore the G.S. with fast decay is unique.

The structure result follows from Theorem 1.2. The uniqueness of the G.S. with fast decay has been proved in [22] assuming that the Hypothesis concerning $J^+(r)$ is satisfied. However, when the second hypothesis concerning $J^-(r)$ holds, the result can be proved reasoning similarly, but we need the following Lemma.

Lemma 3.7. *If $u'(r) < 0 < u(r)$ for $r > R_0$, then $u(r)r^{(n-p)/(p-1)}$ is increasing for $r > R_0$.*

Proof. In the classical case $p = 2$ this is a trivial consequence of the existence of the Kelvin transformation, see e.g. [24]. When $p \neq 2$ we have to modify the change of coordinates (2.1) as follows: set $\alpha^* = (n - p)/(p - 1)$ and $\beta^* = n - 1$ and $x_1^* = u(e^t)e^{\alpha^* t}$ and $x_2^*(t) = u'|u'|^{p-2}(e^t)e^{\beta^* t}$. Then (1.2) is changed into

$$(3.3) \quad \begin{pmatrix} \dot{x}_1^* \\ \dot{x}_2^* \end{pmatrix} = \begin{pmatrix} \alpha^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} + \begin{pmatrix} x_2^* |x_2^*|^{(2-p)/(p-1)} \\ -\phi(t)e^{pt/(p-1)}x_1^*|x_1^*|^{\sigma-2} \end{pmatrix}$$

So if $x^*(t)$ is the solution corresponding to $u(r)$ we have that $x_1^*(t) > 0$ and $\dot{x}_2^*(t) < 0$ for $t > T_0 = \ln(R_0)$. From Proposition 2.6 we know that if $u(r)$ has fast decay, then $x_1^*(t)$ converges to a positive finite value, while if it has slow decay it diverges to $+\infty$, so $x_1^*(t) \rightarrow 0$. From an elementary analysis on the phase portrait it follows that, either $\dot{x}_1^*(t) > 0$ for $t > T_0$ or there are $T_2 > T_1 > T_0$ such that $\dot{x}_1^*(T_1) = 0$ and $x_1^*(T_2) = 0$. But the latter possibility is in contradiction with the fact that $u(r) > 0$ for $r > R$, so the Lemma is proved. $\square$

Now to prove Theorem 3.6, when the Hypothesis concerning $J^-(r)$ holds, we just have to consider the solutions $v(r)$ with fast decay of (1.2) and to observe that the function $P(v(r), v'(r), r)$ defined in (2.6) is such that

$$P(v(r), v'(r), r) = -J^-(r) \frac{|u|^\sigma(r)}{\sigma} - \int_r^{+\infty} J^-(s) \frac{d}{ds} |v(s)s^{(n-p)/(p-1)}|^\sigma ds.$$

So $P(v(r), v'(r), r) > 0$ for $r > R$ and it is decreasing for $r < R$. Following [22], we find that if $v(r)$ is a G.S. with fast decay, then $\lim_{r \rightarrow 0} P(v(r), v'(r), r) = 0$; therefore $P(v(r), v'(r), r) > 0$ for any $r > 0$. Repeating all the reasonings developed in [22], then we obtain the same structure result as if the first hypothesis is satisfied; so the uniqueness of the G.S. with fast decay follows.

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