Instability of Standing Waves for a System of Nonlinear Schrödinger Equations with Three-Wave Interaction

By

Mathieu COLIN¹, Thierry COLIN¹ and Masahito OHTA²

(¹Université de Bordeaux, France and ²Saitama University, Japan)

Abstract. We consider a three-component system of nonlinear Schrödinger equations related to the Raman amplification in a plasma. In dimension $N \leq 3$, we study the orbital instability of standing wave solution of the form $(0, 0, e^{2i\omega t}\varphi)$, where φ is a ground state of scalar nonlinear Schrödinger equation. Using time derivative instead of space derivatives to estimate nonlinear terms, we improve an instability result in our previous paper [4], and also give a simpler proof.

Key Words and Phrases. Nonlinear Schrödinger equations, Standing waves, Orbital instability.

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1. Introduction

This paper is a sequel to our previous paper [4]. We continue the study of instability of standing wave solutions for the following system of nonlinear Schrödinger equations with three-wave interaction

$$(1) \quad \begin{cases} i\partial_t u_1 + \Delta u_1 + |u_1|^{p-1}u_1 = -\gamma u_3 \overline{u_2}, \\ i\partial_t u_2 + \Delta u_2 + |u_2|^{p-1}u_2 = -\gamma u_3 \overline{u_1}, \\ i\partial_t u_3 + \Delta u_3 + |u_3|^{p-1}u_3 = -\gamma u_1 u_2 \end{cases}$$

for $(t, x) \in \mathbf{R} \times \mathbf{R}^N$, where $N \leq 3$, $1 < p < 1 + 4/N$, $\gamma > 0$ is a constant, and u_1 , u_2 and u_3 are complex valued functions of $(t, x) \in \mathbf{R} \times \mathbf{R}^N$. The aim in this paper is to extend the instability result presented in [4] to the three dimensional case by using a different approach which is also valid for the cases $N = 1, 2$. We can notice that the method we develop in the present paper is easier to handle than that of [4].

System (1) was introduced in [4] as a simplified version of a quasilinear Zakharov type system related to the Raman amplification in a plasma studied in [2, 3]. In [4], we studied the orbital stability and instability of a semitrivial standing wave solution $(0, 0, e^{2i\omega t}\varphi)$ of (1), where $\omega > 0$ is a constant, and $\varphi \in H^1(\mathbf{R}^N)$ is a positive radial solution of

$$(2) \quad -\Delta v + 2\omega v - |v|^{p-1}v = 0, \quad x \in \mathbf{R}^N.$$

It is well-known that $\varphi \in H^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. This fact will be used later.

Definition. We say that the standing wave solution $(0, 0, e^{2i\omega t}\varphi)$ of (1) is *orbitally stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\vec{u}_0 \in H^1(\mathbf{R}^N, \mathbf{C}^3)$ and $\|\vec{u}_0 - (0, 0, \varphi)\|_{H^1} < \delta$, then the solution $\vec{u}(t)$ of (1) with $\vec{u}(0) = \vec{u}_0$ satisfies

$$\sup_{t \geq 0} \inf_{\theta \in \mathbf{R}, y \in \mathbf{R}^N} \|\vec{u}(t) - (0, 0, e^{i\theta}\varphi(\cdot + y))\|_{H^1} < \varepsilon.$$

Otherwise, $(0, 0, e^{2i\omega t}\varphi)$ is called *orbitally unstable*.

In [4], we proved the following.

Theorem 1 (Theorem 3 in [4]). *Let $1 \leq N \leq 3$, $1 < p < 1 + 4/N$, $\omega > 0$, and let φ be the positive radial solution of (2). Then, there exists a positive constant $\gamma^* = \gamma^*(N, p, \omega)$ satisfying the following.*

- (i) *If $0 < \gamma < \gamma^*$, then the standing wave solution $(0, 0, e^{2i\omega t}\varphi)$ of (1) is orbitally stable.*
- (ii) *Assume further that $N \leq 2$ and $p > 2$. If $\gamma > \gamma^*$, then the standing wave solution $(0, 0, e^{2i\omega t}\varphi)$ of (1) is orbitally unstable.*

Remark that the additional assumption $N \leq 2$ and $p > 2$ in Theorem 1 (ii) is related to the regularity of the nonlinearity $|u|^{p-1}u$, and it was needed to estimate the nonlinear terms of the linearized equation (4) around $(0, 0, e^{2i\omega t}\varphi)$ (see Lemmas 12 and 13 in [4]). Note that the function $z \mapsto |z|^{p-1}z$ is not smooth enough under our assumption $p < 1 + 4/N$. The purpose in the present paper is to remove the technical assumption $N \leq 2$ and $p > 2$ from Theorem 1 (ii). In [4], the nonlinearity F of Equation (4) is estimated in H^1 (resp. $H^{1+\varepsilon}$) in dimension 1 (resp. dimension 2). The main idea here is to use time derivative instead of space derivatives so that F is estimated only in L^2 as well as its time derivative $\partial_t F$ (see Lemmas 5 and 8 below). This kind of idea has been used for construction of H^2 solutions to nonlinear Schrödinger equations by Kato [7] and Tsutsumi [8] (see also Section 4.8 in [1]). As a consequence, we do not need to introduce a Besov space framework to handle the following extended instability result.

Theorem 2. *Let $1 \leq N \leq 3$, $1 < p < 1 + 4/N$, $\omega > 0$, and let φ be the positive radial solution of (2). Let $\gamma^* = \gamma^*(N, p, \omega)$ be the same positive constant as in Theorem 1. If $\gamma > \gamma^*$, then the standing wave solution $(0, 0, e^{2i\omega t}\varphi)$ of (1) is orbitally unstable.*

The rest of this paper is organized as follows. In Section 2, we first prove some basic estimates on nonlinearities, and then recall some results proved in

[4]. In Section 3, we give the proof of Theorem 2 along the lines of Section 6 in Grillakis, Shatah and Strauss [6] together with the estimates proved in Section 2. In the sequel, $C, C_1, C_2, \dots$ denote positive constants which depend only on N, p, γ, ω and $\|\varphi\|_{L^\infty}$.

2. Preliminaries

In what follows, we put $f(z) = |z|^{p-1}z$ for $z \in \mathbf{C}$. Since $p > 1$, the function $f : \mathbf{C} \rightarrow \mathbf{C}$ is of class C^1 in the real sense. For $z \in \mathbf{C}$, the $\mathbf{R}$ -linear map $f'(z) : \mathbf{C} \rightarrow \mathbf{C}$ is given by

$$f'(z)w = \frac{\partial f}{\partial z}(z)w + \frac{\partial f}{\partial \bar{z}}(z)\bar{w} = \frac{p+1}{2}|z|^{p-1}w + \frac{p-1}{2}|z|^{p-3}z^2\bar{w}$$

for $w \in \mathbf{C}$, and f' satisfies

$$(3) \quad |f'(z) - f'(w)| \leq C \begin{cases} |z - w|^{p-1} & \text{if } 1 < p \leq 2, \\ (|z|^{p-2} + |w|^{p-2})|z - w| & \text{if } p > 2 \end{cases}$$

for all $z, w \in \mathbf{C}$ (see Lemma 2.4 in [5] for the case $p \leq 2$).

Lemma 3. *Let $N \leq 3$. There exists a constant C such that*

$$\|f'(\varphi + v) - f'(\varphi)\|_{L^\infty} \leq C(\|v\|_{H^2} + \|v\|_{H^2}^{p-1})$$

for all $v \in H^2(\mathbf{R}^N, \mathbf{C})$.

Proof. By (3), we have

$$\|f'(\varphi + v) - f'(\varphi)\|_{L^\infty} \leq C \begin{cases} \|v\|_{L^\infty}^{p-1} & \text{if } 1 < p \leq 2, \\ (\|\varphi\|_{L^\infty}^{p-2} + \|v\|_{L^\infty}^{p-2})\|v\|_{L^\infty} & \text{if } p > 2. \end{cases}$$

Since $N \leq 3$, by the embedding $H^2(\mathbf{R}^N) \hookrightarrow L^\infty(\mathbf{R}^N)$, we have the desired estimate. $\square$

As in [4], to prove the orbital instability of $(0, 0, e^{2i\omega t}\varphi)$, we make the following change of variables

$$\vec{u}(t) = (e^{i\omega t}v_1(t), e^{i\omega t}v_1(t), e^{2i\omega t}(\varphi + v_2(t)))$$

in (1). Note that (1) is symmetric with respect to u_1 and u_2 . Then, the equations for (v_1, v_2) read

$$(4) \quad \partial_t v_1 = \mathcal{L}_1 v_1 + F_1(v_1, v_2), \quad \partial_t v_2 = \mathcal{L}_2 v_2 + F_2(v_1, v_2),$$

where the linear terms $\mathcal{L}_1 v_1$ and $\mathcal{L}_2 v_2$ are given by

$$\begin{aligned}\mathcal{L}_1 v &= -i(-\Delta v + \omega v - \gamma \varphi \bar{v}), \\ \mathcal{L}_2 v &= -i\{-\Delta v + 2\omega v - f'(\varphi)v\},\end{aligned}$$

and the nonlinear terms $F_1(v_1, v_2)$ and $F_2(v_1, v_2)$ are given by

$$\begin{aligned}F_1(v_1, v_2) &= i(\gamma \bar{v}_1 v_2 + |v_1|^{p-1} v_1), \\ F_2(v_1, v_2) &= i\{\gamma v_1^2 + f(\varphi + v_2) - f(\varphi) - f'(\varphi)v_2\}.\end{aligned}$$

We write (4) as

$$\begin{aligned}(5) \quad \partial_t \mathbf{v} &= \mathcal{L} \mathbf{v} + F(\mathbf{v}), \\ \mathbf{v} &= (v_1, v_2), \quad \mathcal{L} \mathbf{v} = (\mathcal{L}_1 v_1, \mathcal{L}_2 v_2), \quad F(\mathbf{v}) = (F_1(\mathbf{v}), F_2(\mathbf{v})).\end{aligned}$$

We begin with basic estimates on $\mathcal{L}$ and F .

Lemma 4. *There exists a constant C such that*

$$\|\mathbf{v}\|_{H^2} \leq \|\mathcal{L} \mathbf{v}\|_{L^2} + C\|\mathbf{v}\|_{L^2}, \quad \|\mathcal{L} \mathbf{v}\|_{L^2} \leq C\|\mathbf{v}\|_{H^2}$$

for all $\mathbf{v} \in H^2(\mathbf{R}^N, \mathbf{C}^2)$.

Proof. Since $\varphi \in H^1(\mathbf{R}^N)$ is a solution of (2), it is well-known that $\varphi \in L^\infty(\mathbf{R}^N)$. Thus, by the definition of $\mathcal{L}$, we see that

$$|\|\mathcal{L} \mathbf{v}\|_{L^2} - \|\Delta \mathbf{v}\|_{L^2}| \leq \|\mathcal{L} \mathbf{v} - i\Delta \mathbf{v}\|_{L^2} \leq C\|\mathbf{v}\|_{L^2}$$

for all $\mathbf{v} \in H^2(\mathbf{R}^N, \mathbf{C}^2)$, where the constant C depends on $\|\varphi\|_{L^\infty}$. This implies the desired inequalities. $\square$

Lemma 5. *Let $N \leq 3$. There exists a constant C such that*

$$\|F(\mathbf{v})\|_{L^2} \leq C(\|\mathbf{v}\|_{H^2}^2 + \|\mathbf{v}\|_{H^2}^p)$$

for all $\mathbf{v} \in H^2(\mathbf{R}^N, \mathbf{C}^2)$.

Proof. We put $g(v) = f(\varphi + v) - f(\varphi) - f'(\varphi)v$. Then, by Lemma 3, we have

$$\begin{aligned}\|g(v)\|_{L^2} &= \left\| \int_0^1 \{f'(\varphi + \theta v) - f'(\varphi)\}v \, d\theta \right\|_{L^2} \\ &\leq \int_0^1 \|f'(\varphi + \theta v) - f'(\varphi)\|_{L^\infty} \|v\|_{L^2} d\theta \leq C(\|v\|_{H^2}^2 + \|v\|_{H^2}^p).\end{aligned}$$

Thus, we have

$$\begin{aligned}\|F(\mathbf{v})\|_{L^2} &\leq \gamma \|v_1\|_{L^\infty} (\|v_1\|_{L^2} + \|v_2\|_{L^2}) + \|v_1\|_{L^\infty}^{p-1} \|v_1\|_{L^2} + \|g(v_2)\|_{L^2} \\ &\leq C(\|\mathbf{v}\|_{H^2}^2 + \|\mathbf{v}\|_{H^2}^p)\end{aligned}$$

for all $\mathbf{v} \in H^2(\mathbf{R}^N, \mathbf{C}^2)$. $\square$

Let $\gamma^* = \gamma^*(N, p, \omega)$ be the positive constant defined in Lemma 7 of [4]. The following lemmas are proved in [4].

Lemma 6 (Proposition 11 in [4]). *Let $N \geq 1$, $1 < p < 1 + 4/N$ and $\gamma > \gamma^*$. Then, there exist a constant $\lambda_\gamma > 0$ and $\zeta_\gamma \in H^2(\mathbf{R}^N, \mathbf{C})$ such that*

$$\|\zeta_\gamma\|_{H^2} = 1, \quad \mathcal{L}_1 \zeta_\gamma = \lambda_\gamma \zeta_\gamma, \quad \sup\{\Re \lambda : \lambda \in \sigma(\mathcal{L}_1)\} = \lambda_\gamma.$$

Moreover, $\mathbf{z}_\gamma := (\zeta_\gamma, 0)$ satisfies

$$\mathcal{L} \mathbf{z}_\gamma = \lambda_\gamma \mathbf{z}_\gamma, \quad \sup\{\Re \lambda : \lambda \in \sigma(\mathcal{L})\} = \lambda_\gamma.$$

Lemma 7 (Lemma 14 in [4]). *Let $N \geq 1$, $1 < p < 1 + 4/N$ and $\gamma > \gamma^*$. Let λ_γ be the constant as in Lemma 6, and*

$$\alpha := \frac{1}{2} \min\{1, p - 1\}.$$

Then, there exists a constant C such that

$$\|e^{t\mathcal{L}} \mathbf{v}\|_{L^2} \leq C e^{(1+\alpha)\lambda_\gamma t} \|\mathbf{v}\|_{L^2}$$

for all $t \geq 0$ and $\mathbf{v} \in L^2(\mathbf{R}^N, \mathbf{C}^2)$.

3. Proof of Theorem 2

For $\delta > 0$, let $\vec{u}_\delta(t)$ be the solution of (1) with $\vec{u}_\delta(0) = (0, 0, \varphi) + \delta(\zeta_\gamma, \zeta_\gamma, 0)$. Since $\varphi, \zeta_\gamma \in H^2(\mathbf{R}^N)$, we have $\vec{u}_\delta(0) \in H^2(\mathbf{R}^N, \mathbf{C}^3)$. Since $N \leq 3$ and $1 < p < 1 + 4/N$, we know that $\vec{u}_\delta \in C(\mathbf{R}, H^1(\mathbf{R}^N, \mathbf{C}^3))$ (see Proposition 1 in [4]). Thus, by the H^2 regularity theorem for (1), we see that $\vec{u}_\delta \in C(\mathbf{R}, H^2(\mathbf{R}^N, \mathbf{C}^3)) \cap C^1(\mathbf{R}, L^2(\mathbf{R}^N, \mathbf{C}^3))$ (see, e.g., Section 5.3 in [1]). Moreover, by symmetry between u_1 and u_2 in (1), we see that $\vec{u}_\delta(t)$ has the form

$$\vec{u}_\delta(t) = (e^{i\omega t} v_{\delta,1}(t), e^{i\omega t} v_{\delta,1}(t), e^{2i\omega t}(\varphi + v_{\delta,2}(t))),$$

where $\mathbf{v}_\delta(t) := (v_{\delta,1}(t), v_{\delta,2}(t))$ is a solution of (5) with $\mathbf{v}_\delta(0) = \delta \mathbf{z}_\gamma$. Remark that $\mathbf{v}_\delta \in C(\mathbf{R}, H^2(\mathbf{R}^N, \mathbf{C}^2)) \cap C^1(\mathbf{R}, L^2(\mathbf{R}^N, \mathbf{C}^2))$, and $\mathbf{v}_\delta(t)$ satisfies the integral equation

$$(6) \quad \mathbf{v}_\delta(t) = \delta e^{\lambda_\gamma t} \mathbf{z}_\gamma + \int_0^t e^{(t-\tau)\mathcal{L}} F(\mathbf{v}_\delta(\tau)) d\tau$$

for all $t \in \mathbf{R}$. Since

$$\begin{aligned} \partial_t \int_0^t e^{(t-\tau)\mathcal{L}} F(\mathbf{v}_\delta(\tau)) d\tau &= \partial_t \int_0^t e^{\tau\mathcal{L}} F(\mathbf{v}_\delta(t-\tau)) d\tau \\ &= e^{t\mathcal{L}} F(\mathbf{v}_\delta(0)) + \int_0^t e^{\tau\mathcal{L}} \partial_t F(\mathbf{v}_\delta(t-\tau)) d\tau \\ &= e^{t\mathcal{L}} F(\delta \mathbf{z}_\gamma) + \int_0^t e^{(t-\tau)\mathcal{L}} \partial_\tau F(\mathbf{v}_\delta(\tau)) d\tau, \end{aligned}$$

we have

$$(7) \quad \partial_t \mathbf{v}_\delta(t) = \lambda_\gamma \delta e^{\lambda_\gamma t} \mathbf{z}_\gamma + e^{t\mathcal{L}} F(\delta \mathbf{z}_\gamma) + \int_0^t e^{(t-\tau)\mathcal{L}} \partial_\tau F(\mathbf{v}_\delta(\tau)) d\tau.$$

Lemma 8. *There exists a constant C such that*

$$\|\partial_t F(\mathbf{v}_\delta(t))\|_{L^2} \leq C(\|\mathbf{v}_\delta(t)\|_{H^2} + \|\mathbf{v}_\delta(t)\|_{H^2}^{p-1}) \|\partial_t \mathbf{v}_\delta(t)\|_{L^2}$$

for all $t \in \mathbf{R}$.

Proof. We put

$$g(v) = f(\varphi + v) - f(\varphi) - f'(\varphi)v \quad \text{and} \quad \mathbf{v}_\delta(t) = (v_{\delta,1}(t), v_{\delta,2}(t)).$$

Since φ does not depend on t , we have

$$\partial_t g(v_{\delta,2}(t)) = \{f'(\varphi + v_{\delta,2}(t)) - f'(\varphi)\} \partial_t v_{\delta,2}(t).$$

Thus, by Lemma 3, we have

$$\begin{aligned} \|\partial_t g(v_{\delta,2}(t))\|_{L^2} &\leq \|f'(\varphi + v_{\delta,2}(t)) - f'(\varphi)\|_{L^\infty} \|\partial_t v_{\delta,2}(t)\|_{L^2} \\ &\leq C(\|v_{\delta,2}(t)\|_{H^2} + \|v_{\delta,2}(t)\|_{H^2}^{p-1}) \|\partial_t v_{\delta,2}(t)\|_{L^2}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|\partial_t F_2(\mathbf{v}_\delta(t))\|_{L^2} &\leq 2\gamma \|v_{\delta,1}(t)\|_{L^\infty} \|\partial_t v_{\delta,1}(t)\|_{L^2} \\ &\quad + C(\|v_{\delta,2}(t)\|_{H^2} + \|v_{\delta,2}(t)\|_{H^2}^{p-1}) \|\partial_t v_{\delta,2}(t)\|_{L^2} \\ &\leq C(\|\mathbf{v}_\delta(t)\|_{H^2} + \|\mathbf{v}_\delta(t)\|_{H^2}^{p-1}) \|\partial_t \mathbf{v}_\delta(t)\|_{L^2}. \end{aligned}$$

The estimate for $\|\partial_t F_1(\mathbf{v}_\delta(t))\|_{L^2}$ can be obtained in the same way. □

Lemma 9. *There exist positive constants $C_1, \dots, C_5$ such that*

$$\begin{aligned} \|\mathbf{v}_\delta(t)\|_{H^2} &\leq \|\partial_t \mathbf{v}_\delta(t)\|_{L^2} + C_1 \|\mathbf{v}_\delta(t)\|_{L^2} + C_2 (\|\mathbf{v}_\delta(t)\|_{H^2}^2 + \|\mathbf{v}_\delta(t)\|_{H^2}^p), \\ \|\mathbf{v}_\delta(t) - \delta e^{\lambda_\gamma t} \mathbf{z}_\gamma\|_{L^2} &\leq C_3 \int_0^t e^{(1+\alpha)\lambda_\gamma(t-\tau)} (\|\mathbf{v}_\delta(\tau)\|_{H^2}^2 + \|\mathbf{v}_\delta(\tau)\|_{H^2}^p) d\tau, \\ \|\partial_t \mathbf{v}_\delta(t)\|_{L^2} &\leq \lambda_\gamma \delta e^{\lambda_\gamma t} + C_4 (\delta^2 + \delta^p) e^{(1+\alpha)\lambda_\gamma t} \\ &\quad + C_5 \int_0^t e^{(1+\alpha)\lambda_\gamma(t-\tau)} (\|\mathbf{v}_\delta(\tau)\|_{H^2} + \|\mathbf{v}_\delta(\tau)\|_{H^2}^{p-1}) \|\partial_t \mathbf{v}_\delta(\tau)\|_{L^2} d\tau \end{aligned}$$

for all $t \geq 0$.

Proof. By Lemmas 4 and 5 and by (5), we have

$$\begin{aligned} \|\mathbf{v}_\delta(t)\|_{H^2} &\leq \|\mathcal{L} \mathbf{v}_\delta(t)\|_{L^2} + C_1 \|\mathbf{v}_\delta(t)\|_{L^2} \\ &\leq \|\partial_t \mathbf{v}_\delta(t)\|_{L^2} + \|F(\mathbf{v}_\delta(t))\|_{L^2} + C_1 \|\mathbf{v}_\delta(t)\|_{L^2} \\ &\leq \|\partial_t \mathbf{v}_\delta(t)\|_{L^2} + C_1 \|\mathbf{v}_\delta(t)\|_{L^2} + C_2 (\|\mathbf{v}_\delta(t)\|_{H^2}^2 + \|\mathbf{v}_\delta(t)\|_{H^2}^p) \end{aligned}$$

for all $t \geq 0$. Next, by (6) and by Lemmas 7 and 5, we have

$$\begin{aligned} \|\mathbf{v}_\delta(t) - \delta e^{\lambda_\gamma t} \mathbf{z}_\gamma\|_{L^2} &\leq C \int_0^t e^{(1+\alpha)\lambda_\gamma(t-\tau)} \|F(\mathbf{v}_\delta(\tau))\|_{L^2} d\tau \\ &\leq C_3 \int_0^t e^{(1+\alpha)\lambda_\gamma(t-\tau)} (\|\mathbf{v}_\delta(\tau)\|_{H^2}^2 + \|\mathbf{v}_\delta(\tau)\|_{H^2}^p) d\tau \end{aligned}$$

for all $t \geq 0$. Finally, recall that $\|\mathbf{z}_\gamma\|_{H^2} = \|\zeta_\gamma\|_{H^2} = 1$. By (7) and by Lemmas 5, 7 and 8, we have

$$\begin{aligned} \|\partial_t \mathbf{v}_\delta(t)\|_{L^2} &\leq \lambda_\gamma \delta e^{\lambda_\gamma t} \|\mathbf{z}_\gamma\|_{L^2} + C e^{(1+\alpha)\lambda_\gamma t} \|F(\delta \mathbf{z}_\gamma)\|_{L^2} \\ &\quad + C \int_0^t e^{(1+\alpha)\lambda_\gamma(t-\tau)} \|\partial_\tau F(\mathbf{v}_\delta(\tau))\|_{L^2} d\tau \\ &\leq \lambda_\gamma \delta e^{\lambda_\gamma t} + C_4 (\delta^2 + \delta^p) e^{(1+\alpha)\lambda_\gamma t} \\ &\quad + C_5 \int_0^t e^{(1+\alpha)\lambda_\gamma(t-\tau)} (\|\mathbf{v}_\delta(\tau)\|_{H^2} + \|\mathbf{v}_\delta(\tau)\|_{H^2}^{p-1}) \|\partial_t \mathbf{v}_\delta(\tau)\|_{L^2} d\tau \end{aligned}$$

for all $t \geq 0$. This completes the proof. $\square$

Proof of Theorem 2. Let $C_1, \dots, C_5$ be the positive constants given in Lemma 9, and put

$$C_0 = C_1 + 2\lambda_\gamma + 4C_4 + 1.$$

Let $0 < \varepsilon_0 < 1$ be a constant satisfying

$$(8) \quad 2C_0\varepsilon_0 < 1, \quad 8\left(C_2 + \frac{C_1C_3 + 2C_5}{\alpha\lambda_\gamma}\right)(2C_0\varepsilon_0)^{2\alpha} < 1,$$

$$(9) \quad \frac{8C_0C_3}{\alpha\lambda_\gamma}(2C_0\varepsilon_0)^{2\alpha} < \|\zeta_\gamma\|_{L^2}.$$

For $\delta \in (0, \varepsilon_0)$, we define

$$T_\delta = \frac{1}{\lambda_\gamma} \log \frac{\varepsilon_0}{\delta}, \quad V_\delta(t) = \|\mathbf{v}_\delta(t)\|_{H^2} + \|\partial_t \mathbf{v}_\delta(t)\|_{L^2}.$$

Remark that $\varepsilon_0 = \delta e^{\lambda_\gamma T_\delta}$ and $1 + 2\alpha = \min\{2, p\}$. First, we prove that

$$(10) \quad V_\delta(t) \leq 2C_0\delta e^{\lambda_\gamma t}$$

for all $0 \leq t \leq T_\delta$. Let $\tilde{T}_\delta$ be the supremum of T such that (10) holds for all $0 \leq t \leq T$. Suppose that $\tilde{T}_\delta < T_\delta$. Then, by (8), we have

$$(11) \quad V_\delta(t) \leq 2C_0\delta e^{\lambda_\gamma t} \leq 2C_0\delta e^{\lambda_\gamma T_\delta} = 2C_0\varepsilon_0 < 1$$

for all $0 \leq t \leq \tilde{T}_\delta$. Thus, by Lemma 9 and (11) and recalling $\|\mathbf{z}_\gamma\|_{L^2} \leq 1$, we have

$$\begin{aligned} V_\delta(t) &\leq 2\|\partial_t \mathbf{v}_\delta(t)\|_{L^2} + C_1\|\mathbf{v}_\delta(t)\|_{L^2} + 2C_2V_\delta(t)^{1+2\alpha} \\ &\leq (C_1 + 2\lambda_\gamma)\delta e^{\lambda_\gamma t} + 4C_4\delta^{1+2\alpha}e^{(1+\alpha)\lambda_\gamma t} + 2C_2V_\delta(t)^{1+2\alpha} \\ &\quad + 2(C_1C_3 + 2C_5) \int_0^t e^{(1+\alpha)\lambda_\gamma(t-\tau)} V_\delta(\tau)^{1+2\alpha} d\tau. \end{aligned}$$

Using (11), we have

$$\begin{aligned} (12) \quad \int_0^t e^{(1+\alpha)\lambda_\gamma(t-\tau)} V_\delta(\tau)^{1+2\alpha} d\tau &\leq (2C_0\delta)^{1+2\alpha} \int_0^t e^{(1+\alpha)\lambda_\gamma(t-\tau)} e^{(1+2\alpha)\lambda_\gamma\tau} d\tau \\ &= (2C_0\delta)^{1+2\alpha} \frac{e^{(1+2\alpha)\lambda_\gamma t}}{\alpha\lambda_\gamma} \leq (2C_0\varepsilon_0)^{2\alpha} 2C_0\delta \frac{e^{\lambda_\gamma t}}{\alpha\lambda_\gamma}. \end{aligned}$$

Then, by (12) and (8) and by the fact that $V_\delta(t)^{1+2\alpha} \leq (2C_0\varepsilon_0)^{2\alpha} 2C_0\delta e^{\lambda_\gamma t}$, we obtain

$$\begin{aligned} V_\delta(t) &\leq (C_1 + 2\lambda_\gamma + 4C_4\delta^\alpha\varepsilon_0^\alpha)\delta e^{\lambda_\gamma t} + 4\left(C_2 + \frac{C_1C_3 + 2C_5}{\alpha\lambda_\gamma}\right)(2C_0\varepsilon_0)^{2\alpha}C_0\delta e^{\lambda_\gamma t} \\ &\leq \frac{3}{2}C_0\delta e^{\lambda_\gamma t} \end{aligned}$$

for all $0 \leq t \leq \tilde{T}_\delta$. This contradicts the definition of $\tilde{T}_\delta$. Thus, we have $T_\delta \leq \tilde{T}_\delta$, so (10) holds for all $0 \leq t \leq T_\delta$. Next, by Lemma 9 and by (9) and (12), we have

$$\begin{aligned} \|\mathbf{v}_\delta(t) - \delta e^{\lambda_\gamma t} \mathbf{z}_\gamma\|_{L^2} &\leq 2C_3 \int_0^t e^{(1+\alpha)\lambda_\gamma(t-\tau)} V_\delta(\tau)^{1+2\alpha} d\tau \\ &\leq \frac{4C_0C_3}{\alpha\lambda_\gamma} (2C_0\varepsilon_0)^{2\alpha} \delta e^{\lambda_\gamma t} \leq \frac{\delta}{2} e^{\lambda_\gamma t} \|\zeta_\gamma\|_{L^2} \end{aligned}$$

for all $0 \leq t \leq T_\delta$. In particular, since $\delta e^{\lambda_\gamma T_\delta} = \varepsilon_0$ and $\mathbf{z}_\gamma = (\zeta_\gamma, 0)$, we have

$$\begin{aligned} \varepsilon_0 \|\zeta_\gamma\|_{L^2} - \|v_{\delta,1}(T_\delta)\|_{L^2} &\leq \|v_{\delta,1}(T_\delta) - \varepsilon_0 \zeta_\gamma\|_{L^2} \\ &\leq \|\mathbf{v}_\delta(T_\delta) - \varepsilon_0 \mathbf{z}_\gamma\|_{L^2} \leq \frac{\varepsilon_0}{2} \|\zeta_\gamma\|_{L^2}. \end{aligned}$$

Therefore, we have

$$\inf_{\theta \in \mathbf{R}, y \in \mathbf{R}^N} \|\tilde{\mathbf{u}}_\delta(T_\delta) - (0, 0, e^{i\theta} \varphi(\cdot + y))\|_{H^1} \geq \|v_{\delta,1}(T_\delta)\|_{L^2} \geq \frac{\varepsilon_0}{2} \|\zeta_\gamma\|_{L^2}.$$

Since $\|\tilde{\mathbf{u}}_\delta(0) - (0, 0, \varphi)\|_{H^1} = \delta \|(\zeta_\gamma, \zeta_\gamma, 0)\|_{H^1} \rightarrow 0$ as $\delta \rightarrow 0$, this means that $(0, 0, e^{2i\omega t} \varphi)$ is orbitally unstable. $\square$

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nuna adreso:

Mathieu Colin
Institut Mathématiques de Bordeaux
Université de Bordeaux
351 cours de la libération
33405 Talence Cedex
France
E-mail: mcolin@math.u-bordeaux1.fr

Thierry Colin
Institut Mathématiques de Bordeaux
Université de Bordeaux
351 cours de la libération
33405 Talence Cedex
France
E-mail: colin@math.u-bordeaux1.fr

Masahito Ohta
Department of Mathematics
Faculty of Science
Saitama University
Saitama 338-8570
Japan
E-mail: mohta@mail.saitama-u.ac.jp

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